THE THEORY
OF
ELECTRICITY

BY

G. H. LIVENS, M.A.
SOMETIME FELLOW OF JESUS COLLEGE, CAMBRIDGE
LECTURER IN MATHEMATICS IN THE UNIVERSITY OF SHEFFIELD

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PREFACE

THE following work is offered as a general text book on the mathem-aspects of modern electrical theory, and incidentally also as an at to present the complete subject in a consistent form. There seems for a comprehensive work of this kind, for in the standard text book this subject the treatment, besides being incomplete, is often far convincing and at times not free from error.

The treatment is based mainly on the original Faraday-Maxwell theory generalised and extended to the case of moving systems by Sir Joseph Larmor. This form of the theory has been almost entirely abandoned in recent accounts of the subject, but it remains the only one which appears to be completely satisfactory from the point of view of mathematical and physical consistency, and in its generality it is unapproached by any other form.

Although the present exposition is essentially a mathematical one, much of the purely analytical mathematics usually associated with the subject has been omitted. Particular attention has however been given to the rigorous formulation of underlying physical principles and to their translation into a mathematical theory. The dynamical aspects of the subject have been specially emphasised throughout.

In the development of the general plan and of the details of the book I have derived great assistance from my notes of lectures delivered by Sir Joseph Larmor at Cambridge during the academical year 1909–10, afterwards supplemented from his various published works, more particularly the papers, 'On a dynamical theory of the electric and luminiferous medium' [*Phil. Trans.* 1894–1897] and the book, *Aether and Matter* [Cambridge, 1900]. I am extremely grateful to Professor Larmor for his kind permission to make free use of these notes.

For assistance in the preparation of the work I am indebted also to the various other standard works on this subject. In addition to the essential *Treatise on Electricity and Magnetism* of Maxwell I may mention especially: *Recent Researches on Electricity and Magnetism*, Oxford, 1893, and *Elements of the Mathematical Theory of Electricity and Magnetism*, 3rd ed., Cambridge, 1904, by Sir J. J. Thomson; *The Theory of Electricity and Magnetism*, Cambridge, 1907, by J. H. Jeans; *Modern Electrical Theory*, 1st ed., Cambridge, 1907, by N. R. Campbell; *Electrical and Optical Wave Motion*, Cambridge, H. Bateman; *The Electron Theory of Matter*, Cambridge, 1915, by chardson; *Das elektromagnetische Feld*, Leipzig, 1900, by E. Cohn; *rie der Elektrizität*, 2nd ed., Leipzig, 1907, by M. Abraham and ; *The Theory of Electrons*, Leipzig, 1910, by H. A. Lorentz; and ie various appropriate articles in the *Encyclopaedia Britannica* and *yklopädie der mathematischen Wissenschaften*, Bd. v.

friend Mr H. Spencer Jones, Chief Assistant at the Royal Observatory, vich, has laid me under the deepest obligation by his generous help production of this book. At a time of great pressure in his own work adly offered to read all the proofs and his criticisms and suggestions thereon have been of the greatest service to me.

I have lastly to offer my thanks to the officials of the University Press for their kindness and courtesy in all matters concerning the printing.

G. H. L.

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CONTENTS

CORRECTIONS AND ADDITIONS.

pp. 39, 158, 661. Until quite recently I have not had an opportunity of examining Einstein's generalised theory of relativity and was in consequence unable to take up any other point of view as regards gravitational phenomena.

p. 84, § 88. The argument of the exponential function under the integral sign in the formulae for ϕ should read $(-kz)$ not $(-kr)$.

p. 338. The values for the electronic charge given on this page are those first obtained by the method under discussion: subsequent improvements and corrections have increased this value by about 50 % to that given on p. 341 and in other parts of the work.

p. 345, § 388. In the expression for the potential of a magnetic shell the solid angle is measured by that area of the unit sphere round the field point which is such that if it were coated with a double sheet with a similar disposition to that on the given shell, then the direction of the outward drawn radius at any point would correspond to the positive direction of magnetisation in this sheet.

MATHEMATICAL INTRODUCTION

1. **Scalars and Vectors.** We shall find it most convenient and certainly ducive to greater mathematical precision to adopt the system of notation analysis which is now associated with the word '*vector-analysis* *.' ng however to the fact that there is as yet no suitable work of reference this subject accessible to the average English student it was deemed isable to outline in this introduction the main propositions of this analysis ar as it is needed for our future work. We may also take the opportunity leveloping in full certain complex mathematical formulae, the derivation which in the text would hinder the progress of the reader through the train physical reasoning to be there exposed.

In physics we distinguish between two kinds of quantities which we now ally designate as *scalars* and *vectors*. A scalar quantity is completely ermined by the number expressing the ratio of its magnitude to that of efinitely chosen unit. A vector, on the other hand, has in addition to a ierical value, also a definite direction, so that vectors are distinguished n one another not only by their numerical values but also by their direction; numerical value is described as the magnitude of the vector.

Following the usual methods we shall always find it convenient to represent hically a vector by a straight line, whose length on a definite scale is equal he magnitude of the vector and whose direction and sense (considered as vn from one end, the origin, to the other) are the same as of the vector.

Throughout this book we shall always use thick letters of the Clarendon † e, e.g. **A**, **P**, **c**, **n**, etc. to represent vector quantities, scalar quantities being ressed by letters of the ordinary Latin or Greek types. The magnitude vector may occasionally be denoted by the corresponding Latin letter, A, P, c, n, etc., or in the usual notation of function theory by enclosing vector thus $|\mathbf{A}|$; $|\mathbf{v}|$. We shall always represent a line or curve by s,

* First systematically expounded by Hamilton, *Elements of Quaternions*. See also Kelland Tait, *Introduction to Quaternions*; Tait, *Elementary Treatise on Quaternions*; E. B. Wilson. r *Analysis founded on the lectures of J. W. Gibbs*. The treatment given in the text is on the followed by W. von Ignatowsky, *Die Vektoranalysis* (Leipzig, 1910). A very illuminating al exposition is given by Burali-Forti and Marcolongo, *Éléments de calcul vectorielle* (Paris,).

† This is the notation employed by Heaviside, *Electromagnetic Theory* (London, 1891–93) subsequently by Gibbs, Lorentz and others.

a surface by f and a volume by v, differential elements of these quantities being respectively denoted by ds, df and dv; it is to be noticed that the element ds of a curve may also be required with its direction, it will then be denoted in the vector sense by $d\mathbf{s}$. In the case of a surface we shall often have to consider the normal to it; this is denoted by n. It is always to be drawn towards a definite side and we shall agree to draw it towards the outside if we are dealing with a closed surface.

The geometrical space with which we deal will generally be referred to a system of rectangular coordinate axes with the right-handed screw conventions more usual in such expositions. The coordinates (x, y, z) of any point of space are then the distances measured parallel to the perpendicular axes from the respective coordinate planes. In certain special problems however we shall find it convenient to adopt other types of coordinates. We shall in fact use both the spherical polar and cylindrical polar coordinate systems. In the former system the coordinates of a point are its distance r from a fixed pole and the declination and azimuth angles θ and ϕ of this distance referred to fixed polar axis and azimuth plane through the pole. In the cylindrical or columnar system the coordinates of a point are its distances (z, r) from a fixed plane and from a fixed axis perpendicular to the plane and the azimuth angle θ of the latter distance.

2. Elementary vector operations. (i) *Addition and subtraction of vectors:* suppose we are given two vectors $\mathbf{A}$ and $\mathbf{B}$; at the end of the line representing $\mathbf{A}$ draw a line representing $\mathbf{B}$. We then say that the sum of the two vectors $\mathbf{A}$ and $\mathbf{B}$ is the vector $\mathbf{C}$ which is represented by the line joining the origin of $\mathbf{A}$ to the end of $\mathbf{B}$, and in this sense. We express this by the equation

$$\mathbf{A} + \mathbf{B} = \mathbf{C}.$$

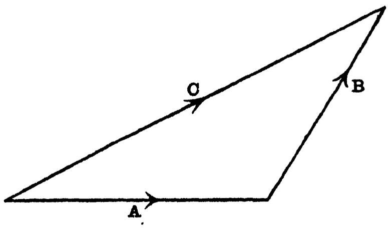


Fig. 1

The difference $(\mathbf{A} - \mathbf{B})$ between two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as the vector $\mathbf{D}$ which is such that

$$\mathbf{D} + \mathbf{B} = \mathbf{A}.$$

This involves the conclusion that the sum of the vectors $\mathbf{B}$ and $(-\mathbf{B})$ is zero and also that

$$(\mathbf{A} - \mathbf{B}) = \mathbf{A} + (-\mathbf{B})$$

since each of these is equal to the vector

$$(\mathbf{A} - \mathbf{B}) + \mathbf{B} + (-\mathbf{B}).$$

We have thus sufficient rules for the addition and subtraction of vectors; we see at once that the ordinary commutative and associative laws of algebra are true for vector quantities as well as for scalar.

It is also quite clear that we may regard a vector as the sum of any number of other vectors. In particular we can represent a vector **A** as the sum of three vectors whose directions are not all parallel to one plane; such a method of resolution is of importance because the knowledge of three such vectors is sufficient to completely determine the vector **A**: the magnitudes of three such vectors are called the *components* of the vector **A** along the three directions respectively. If we take the three directions mutually perpendicular to one another and respectively parallel to the axes of a rectangular coordinate system, then the three components are denoted by $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{A}_z$ respectively.

All this will be quite familiar to the student who is acquainted with the elementary ideas of statics and kinetics and it is not necessary to dwell at any greater length on the significance of the definitions thus involved.

We now introduce the fundamental conception of the *unit vector*. The unit vector in any direction is that vector whose magnitude is unity, so that if we denote it by **u**, any other vector **A** is equivalent to the vector $A\mathbf{u}$;

$$\mathbf{A} = A\mathbf{u}.$$

If we use **x**, **y**, **z** for the unit vectors along the directions of the coordinate axes, we get from the rule for the addition of vectors

$$\mathbf{A} = \mathbf{A}_x\mathbf{x} + \mathbf{A}_y\mathbf{y} + \mathbf{A}_z\mathbf{z};$$

and similarly for any other vector **B**

$$\mathbf{B} = \mathbf{B}_x\mathbf{x} + \mathbf{B}_y\mathbf{y} + \mathbf{B}_z\mathbf{z},$$

so that

$$\mathbf{A} + \mathbf{B} = (\mathbf{A}_x + \mathbf{B}_x)\,\mathbf{x} + (\mathbf{A}_y + \mathbf{B}_y)\,\mathbf{y} + (\mathbf{A}_z + \mathbf{B}_z)\,\mathbf{z}$$

and generally the component of the geometric sum of any number of vectors in any definite direction is equal to the algebraic sum of the components of the separate vectors in this direction.

3. (ii) *Multiplication of vectors; scalar products.* From considerations of the rules of the previous paragraph and of the fundamental concepts underlying the idea of a vector it is easy to deduce the general rule for the multiplication of a vector **A** by a scalar quantity a, the resulting vector **B** defined by

$$\mathbf{B} = a\mathbf{A}$$

being such that its direction is the same as that of **A**, but its magnitude a times as large. Thus also

$$ab\,.\,\mathbf{A} = a\,.\,b\mathbf{A} = b\,.\,a\mathbf{A}$$

and also

$$a\,(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B},$$

$$(a + b)\,\mathbf{A} = a\mathbf{A} + b\mathbf{A},$$

so that the ordinary rules of algebra are still applicable.

4. *The scalar product.* We must next define the *scalar product* of two vectors **A** and **B**. This is defined as the scalar quantity whose value is equal to the product of the magnitudes of the two given vectors multiplied by the cosine of the angle included between their directions. We usually denote this quantity by (**A**, **B**) so that

$$(\mathbf{A},\ \mathbf{B}) = A\,.\,B \cos \widehat{AB},$$

where $\widehat{AB}$ denotes the angle between the positive directions of the vectors **A** and **B**. We deduce then that

$$(\mathbf{A},\ \mathbf{B}) = (\mathbf{B},\ \mathbf{A}),$$
$$(\mathbf{A} + \mathbf{B},\ \mathbf{C}) = (\mathbf{A},\ \mathbf{C}) + (\mathbf{B},\ \mathbf{C}),$$
$$(\mathbf{A},\ \mathbf{A}) = A^2 = (\mathbf{A}^2).$$

If **A**, **B** are at right angles, then $\cos \widehat{AB} = 0$ and thus

$$(\mathbf{A},\ \mathbf{B}) = 0.$$

From this we deduce the important properties of the fundamental unit vectors **x**, **y**, **z**, viz.

$$(\mathbf{xy}) = (\mathbf{yz}) = (\mathbf{zx}) = 0$$

and

$$(\mathbf{x}^2) = (\mathbf{y}^2) = (\mathbf{z}^2) = 1.$$

Thus
$$(\mathbf{A},\ \mathbf{B}) = (\mathbf{A}_x\mathbf{x} + \mathbf{A}_y\mathbf{y} + \mathbf{A}_z\mathbf{z},\quad \mathbf{B}_x\mathbf{x} + \mathbf{B}_y\mathbf{y} + \mathbf{B}_z\mathbf{z})$$
$$= \mathbf{A}_x\mathbf{B}_x + \mathbf{A}_y\mathbf{B}_y + \mathbf{A}_z\mathbf{B}_z,$$

expressing the vector products of two vectors in terms of their components.

5. (iii) *The vector product.* Suppose we are given a plane surface element df: call one side of it the positive side and the other side the negative. On the positive side draw a unit vector $\mathbf{n}_1$ normal to the surface through its mean centre. This surface element will be bounded by a curve s and we choose the sense in this curve so that the positive direction of its description corresponds to the direction of the vector $\mathbf{n}_1$ in the same way as translation to rotation in a left-handed screw (Fig. 2). We can now specify this surface element as a vector $d\mathbf{f}$ whose magnitude is df and direction that of $\mathbf{n}_1$; thus

$$d\mathbf{f} = \mathbf{n}_1 df.$$

Such a vector is called an axial vector as it has associated with it a definite sense round its direction as axis.

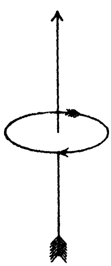


Fig. 2

Since the sum of any number of vectors is still a vector we may have associated with any unclosed surface f a vector **C** defined by

$$\mathbf{C} = \int_f d\mathbf{f},$$

the integral being extended over the whole of the surface f. In the particular case in which the surface f is the parallelogram whose sides are the two

vectors **A** and **B**, this third vector **C** is called the vector product of the two vectors **A** and **B**. The implied direction about the axis of **C** is given by the order in which the vectors **A** and **B** are taken and in the standard case exhibited in the figure and in which **A** is taken first

$$\mathbf{C} = \mathbf{n}AB \sin \hat{AB},$$

the angle AB being that described in rotation round the axis C from the plane (CA) to the plane (CB) in the positive direction. This is usually expressed in the form

$$\mathbf{C} = [\mathbf{A}, \mathbf{B}].$$

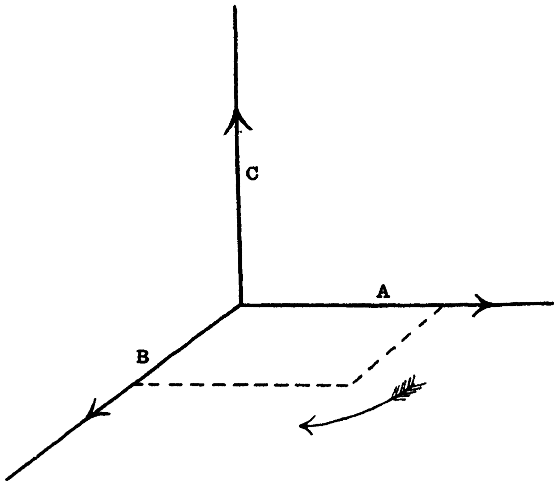


Fig. 3

We see at once that

$$[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}], \quad [\mathbf{A}, \mathbf{A}] = 0$$

and

$$[\mathbf{A} + \mathbf{B}, \mathbf{D}] = [\mathbf{A}, \mathbf{D}] + [\mathbf{B}, \mathbf{D}].$$

And thus also

$$[\mathbf{x}, \mathbf{y}] = \mathbf{z}, \quad [\mathbf{y}, \mathbf{z}] = \mathbf{x}, \quad [\mathbf{z}, \mathbf{x}] = \mathbf{y},$$
$$[\mathbf{x}, \mathbf{x}] = [\mathbf{y}, \mathbf{y}] = [\mathbf{z}, \mathbf{z}] = 0.$$

Therefore also

$$\begin{aligned}[\mathbf{A}, \mathbf{B}] &= [\mathbf{A}_x\mathbf{x} + \mathbf{A}_y\mathbf{y} + \mathbf{A}_z\mathbf{z}, \quad \mathbf{B}_x\mathbf{x} + \mathbf{B}_y\mathbf{y} + \mathbf{B}_z\mathbf{z}] \\ &= \mathbf{x}(\mathbf{A}_y\mathbf{B}_z - \mathbf{A}_z\mathbf{B}_y) + \mathbf{y}(\mathbf{A}_z\mathbf{B}_x - \mathbf{A}_x\mathbf{B}_z) + \mathbf{z}(\mathbf{A}_x\mathbf{B}_y - \mathbf{A}_y\mathbf{B}_x),\end{aligned}$$

or in determinant form

$$[\mathbf{A}, \mathbf{B}] = \begin{vmatrix} \mathbf{x}, & \mathbf{y}, & \mathbf{z} \\ \mathbf{A}_x, & \mathbf{A}_y, & \mathbf{A}_z \\ \mathbf{B}_x, & \mathbf{B}_y, & \mathbf{B}_z \end{vmatrix}.$$

The components of the vector product are the minors of **x**, **y**, **z** and are denoted by $[\mathbf{A}, \mathbf{B}]_x$, etc.

6. (iv) *Products of three vectors.* There are several important results involving three vectors which it would be as well to set out in detail here, for future reference.

(*a*) The product of a vector and the scalar product of two other vectors; $\mathbf{A}(\mathbf{B},\ \mathbf{C})$.

Since $(\mathbf{B},\ \mathbf{C})$ is a scalar quantity the product is a vector parallel to $\mathbf{A}$ whose magnitude is equal to $ABC \cos \widehat{BC}$.

(*b*) The scalar product of a vector and the vectorial product of two other vectors; $(\mathbf{A},\ [\mathbf{B},\ \mathbf{C}])$.

This is equal to

$$\mathbf{A}_x [\mathbf{B},\ \mathbf{C}]_x + \mathbf{A}_y [\mathbf{B},\ \mathbf{C}]_y + \mathbf{A}_z [\mathbf{B},\ \mathbf{C}]_z = \mathbf{A}_x (\mathbf{B}_y \mathbf{C}_z - \mathbf{B}_z \mathbf{C}_y) + \ldots + \ldots$$
$$= \begin{vmatrix} \mathbf{A}_x, & \mathbf{A}_y, & \mathbf{A}_z \\ \mathbf{B}_x, & \mathbf{B}_y, & \mathbf{B}_z \\ \mathbf{C}_x, & \mathbf{C}_y, & \mathbf{C}_z \end{vmatrix},$$

so that
$$(\mathbf{A},\ [\mathbf{B},\ \mathbf{C}]) = (\mathbf{B},\ [\mathbf{C},\ \mathbf{A}]) = (\mathbf{C},\ [\mathbf{A},\ \mathbf{B}])$$
and each is equal to the volume of the pa·allelepiped whose three edges are $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$.

(*c*) The vector product of a vector and the vector product of two other vectors; $[\mathbf{A},\ [\mathbf{B},\ \mathbf{C}]]$. Call this the vector $\mathbf{D}$, then its x-component is

$$\begin{aligned} \mathbf{D}_x &= \mathbf{A}_y (\mathbf{B}_x \mathbf{C}_y - \mathbf{B}_y \mathbf{C}_x) - \mathbf{A}_z (\mathbf{B}_z \mathbf{C}_x - \mathbf{B}_x \mathbf{C}_z) \\ &= \mathbf{B}_x (\mathbf{A}_x \mathbf{C}_x + \mathbf{A}_y \mathbf{C}_y + \mathbf{A}_z \mathbf{C}_z) - \mathbf{C}_x (\mathbf{A}_x \mathbf{B}_x + \mathbf{A}_y \mathbf{B}_y + \mathbf{A}_z \mathbf{B}_z) \\ &= \mathbf{B}_x (\mathbf{A},\ \mathbf{C}) - \mathbf{C}_x (\mathbf{A},\ \mathbf{B}). \end{aligned}$$

Thus
$$\mathbf{D} = \mathbf{B}(\mathbf{C},\ \mathbf{A}) - \mathbf{C}(\mathbf{A},\ \mathbf{B}).$$

7. Differentiation of scalars and vectors. In the general application of the present analysis we shall have continually to deal with what are called *scalar* and *vector fields*. That is instead of single isolated scalars and vectors we have to discuss infinite groups of these quantities inasmuch as each point of a finite or infinite space has associated with it certain scalar and vector quantities the magnitudes, and in the latter case also the directions of which in general vary continuously not only from point to point in the field but also in time. In such cases the rates of variation of the magnitudes of the scalar quantities and both the magnitudes and directions of the vector quantities are matters of fundamental importance. We must therefore establish rules for the differentiation of scalar and vector quantities with respect to their scalar (time) or vector (space position) arguments.

The differentiation of a scalar quantity with respect to a scalar argument follows the usual rules of the calculus and need not now be discussed.

The rules for the differentiation of a vector function with respect to a scalar argument are easily established. Suppose that a vector $\mathbf{A}$ regarded

as a vector function of the scalar quantity t undergoes a small variation and changes to the slightly different vector $\mathbf{A}'$ corresponding to the value $t + \delta t$ of the argument t; the vector

$$\delta\mathbf{A} = \mathbf{A}' - \mathbf{A}$$

is the differential variation of the vector $\mathbf{A}$ with respect to the argument t; $\delta\mathbf{A}$ of course refers to a variation not only of the magnitude of $\mathbf{A}$ but also of its direction. The differential quotient $\frac{d\mathbf{A}}{dt}$ of the vector $\mathbf{A}$ with respect to the scalar quantity t is now defined as the vector represented by the limiting value of the quotient

$$\frac{\delta\mathbf{A}}{\delta t},$$

when δt is made indefinitely small. The implied existence of a limit to this quotient is involved in the definition.

Our rules for vector operation are identical with those of ordinary algebraic analysis so that it is evident that we may employ the elementary rules of the differential calculus for the differentiation of vectors and functions of vectors. Thus if b is any scalar function of the variable t

$$d\frac{b\mathbf{A}}{dt} = b\frac{d\mathbf{A}}{dt} + \mathbf{A}\frac{db}{dt},$$

and also

$$\frac{d\mathbf{A}}{da} = \frac{d\mathbf{A}}{dt}\cdot\frac{dt}{da}.$$

We can similarly deduce that

$$\frac{d}{dt}(\mathbf{A}\,.\,\mathbf{B}) = \left(\mathbf{A}\,.\,\frac{d\mathbf{B}}{dt}\right) + \left(\mathbf{B}\,.\,\frac{d\mathbf{A}}{dt}\right),$$

and also

$$\frac{d}{dt}[\mathbf{A}\,.\,\mathbf{B}] = \left[\frac{d\mathbf{A}}{dt}\,.\,\mathbf{B}\right] + \left[\mathbf{A}\,.\,\frac{d\mathbf{B}}{dt}\right],$$

and many other results are directly deducible.

8. We have next to discuss the differentiation of scalar and vector quantities with respect to other vectors: the most frequent operation of this nature and the only one with which we shall now deal involves the differentiation of a scalar or vector quantity along a line element $d\mathbf{s}$, the result depending essentially on the direction of this element.

The rate of increase of any scalar quantity ϕ along the line element $d\mathbf{s}$ is as usual

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds},$$

and this may be regarded as the component along $d\mathbf{s}$ of the vector

$$\mathbf{A} = \mathbf{x}\frac{\partial\phi}{\partial x} + \mathbf{y}\frac{\partial\phi}{\partial y} + \mathbf{z}\frac{\partial\phi}{\partial z}.$$

The components of the vector **A** in the three principal directions are

$$(\mathbf{A}_x,\ \mathbf{A}_y,\ \mathbf{A}_z) = \left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\phi,$$

so that

$$\mathbf{A} = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2},$$

and its direction is parallel to the direction

$$\frac{\partial\phi}{\partial x} : \frac{\partial\phi}{\partial y} : \frac{\partial\phi}{\partial z}.$$

We call this vector **A** the gradient of the scalar ϕ and express the relation in the form

$$\mathbf{A} = \operatorname{grad}\phi,$$

or

$$\mathbf{A} = \nabla\phi,$$

where ∇ is the Hamiltonian vector operator

$$\nabla \equiv \mathbf{x}\frac{\partial}{\partial x} + \mathbf{y}\frac{\partial}{\partial y} + \mathbf{z}\frac{\partial}{\partial z}.$$

As we shall see later the great importance of this latter form of operator lies in the fact that ∇ may be treated by the ordinary rules of this analysis just as if it were a vector with components

$$\nabla_x = \frac{\partial}{\partial x},\quad \nabla_y = \frac{\partial}{\partial y},\quad \nabla_z = \frac{\partial}{\partial z}.$$

The gradient of a vector quantity **B** may also be expressed in the same way but the result is rather different and is better approached in the indirect way through the important analytical theorem of the next section.

✓ **9. Green's Lemma.** Let A_x be any function of the three rectangular coordinates (x, y, z) which is continuous inside any finite space v. The region v is bounded by the closed surface f which has at each point of it a definite normal n (or if we wish to imply direction we use a vector $\mathbf{n}$). Now consider the integral

$$\int \frac{\partial A_x}{\partial x}\,dv = \iiint \frac{\partial A_x}{\partial x}\,dx\,dy\,dz$$

taken throughout the whole space v. If we assume that a line parallel to the x-axis cuts the surface f in two points only, then integration with respect to x gives

$$dy\,dz\int_{x'}^{x''}\frac{\partial A_x}{\partial x} = dy\,dz\,(A_x'' - A_x'),$$

where A_x'', A_x' are the values of the function A_x at the points (x', y, z), (x'', y, z) on the surface f. The surface element $dy\,dz$ in the Oyz plane is the common projection of the surface elements df', df'' which a small prism with its axis parallel to the x-axis cuts on the surface. Since the normal n' forms an obtuse angle with Ox and n'' an acute angle, we have

$$dy\,dz = -\,\mathbf{n}_{1x}'\,df' = +\,\mathbf{n}_{1x}''\,df''$$

$\mathbf{n}_1'$ and $\mathbf{n}_1''$ denoting respectively the unit vectors along the normals n' and n''. Thus

$$dy\,dz \int_{x'}^{x''} \frac{\partial A_x}{\partial x}\,dx = +\,A_x'\mathbf{n}_{1x}'df' + A_x''\mathbf{n}_{1x}''df''.$$

If the surface f is more complicated so that a line parallel to the x-axis cuts it in more than two points $x', x'', x''', \ldots$, these points will be alternately points of entry and exit from the space v. The number is even and the values of $\mathbf{n}_{1x}', \mathbf{n}_{1x}'', \mathbf{n}_{1x}''', \ldots$ are alternately negative and positive and thus

$$dy\,dz \int_{x'}^{x''} \frac{\partial A_x}{\partial x}\,dx = dy\,dz\,(-\,A_x' + A_x'' - A_x''' + \ldots)$$
$$= \Sigma \mathbf{n}_{1x} A_x\,df.$$

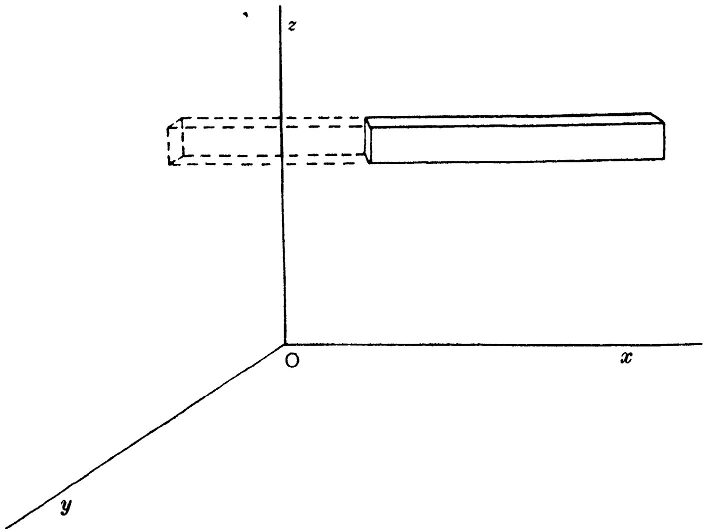


Fig. 4

If now we take the sum over all the elements $dy\,dz$ every element df of the surface occurs once in the total sum and thus we get

$$\int_v \frac{\partial A_x}{\partial x}\,dv = \int_f \mathbf{n}_{1x} A_x\,df,$$

the second integral being taken over the whole of the closed surface f. Similar formulae can be obtained by changing x into y and z respectively: combining them together we have Green's Lemma*

$$\int_v \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dv = \int_f (\mathbf{n}_{1x} A_x + \mathbf{n}_{1y} A_y + \mathbf{n}_{1z} A_z)\,df$$

* 'An Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism' (Nottingham, 1828) reprinted in *Mathematical Papers* (Paris, 1903).

expressing a surface integral taken over a closed surface as a volume integral taken through the volume enclosed. A_y, A_z are any other functions of (x, y, z).

10. Let us firstly suppose that

$$A_x = A_y = A_z = \phi$$

we then see that our theorem proves that

$$\int_v \nabla\phi \, dv = \int_f \mathbf{n}_1 \phi \, df,$$

thus if all the functions concerned are continuous we shall have in the limit when the volume v is reduced indefinitely in every direction that

$$\nabla\phi = \operatorname*{Lt}_{v \to 0} \frac{1}{v} \int_f \mathbf{n}_1 \phi \, df,$$

and this property can be used to define the operator ∇.

11. If as is usually the case in our theory A_x, A_y, A_z are the components of some vector $\mathbf{A}$, the integral on the right can be expressed in the form

$$\int_f \mathbf{A}_n df \quad \text{or} \quad \int_f (\mathbf{A} \cdot \mathbf{n}_1) \, df,$$

where $\mathbf{A}_n$ is used to denote the normal component of $\mathbf{A}$ at the surface element df.

The expression forming the integrand on the left, viz.

$$\frac{\partial \mathbf{A}_x}{\partial x} + \frac{\partial \mathbf{A}_y}{\partial y} + \frac{\partial \mathbf{A}_z}{\partial z},$$

is called the divergence of $\mathbf{A}$ and is usually written in the form $\operatorname{div} \mathbf{A}$; so that Green's lemma is expressible in the form

$$\int_v \operatorname{div} \mathbf{A} \, dv = \int_f \mathbf{A}_n df.$$

It follows immediately that if $\operatorname{div} \mathbf{A}$ is a continuous function of the space coordinates (x, y, z) that

$$\operatorname{div} \mathbf{A} = \operatorname*{Lt}_{v \to 0} \frac{1}{v} \int_f (\mathbf{A}\mathbf{n}_1) \, df,$$

and again this property may be used to define the operator $\operatorname{div} \mathbf{A}$. Moreover if we now notice that the scalar quantity $\operatorname{div} \mathbf{A}$ may be regarded as obtained as the scalar product of the vector operator ∇ and the vector $\mathbf{A}$, or that

$$\operatorname{div} \mathbf{A} = (\nabla \mathbf{A})$$

then the analogy with the result obtained above for the definition of ∇ is obvious.

12. Suppose finally that we put

$$A_x = 0, \quad A_y = \mathbf{A}_z, \quad A_z = -\mathbf{A}_y$$

then we see that

$$\int_v \left(\frac{\partial \mathbf{A}_z}{\partial y} - \frac{\partial \mathbf{A}_y}{\partial z}\right) dv = \int_f (\mathbf{n}_{1y}\mathbf{A}_z - \mathbf{n}_{1z}\mathbf{A}_y)\, df.$$

Generalising this result we see that if **B** is the vector whose components are

$$\frac{\partial \mathbf{A}_z}{\partial y} - \frac{\partial \mathbf{A}_y}{\partial z},\quad \frac{\partial \mathbf{A}_x}{\partial z} - \frac{\partial \mathbf{A}_z}{\partial x},\quad \frac{\partial \mathbf{A}_y}{\partial x} - \frac{\partial \mathbf{A}_x}{\partial y},$$

this vector **B** so derived from the vector **A** is called the *curl* of the vector **A** and is written

$$\text{Curl } \mathbf{A}.$$

On the basis of the ideas sketched above in the two former cases we see that this vector Curl **A** may be defined as the limiting value of the function

$$\frac{1}{v}\int [\mathbf{n}_1 \mathbf{A}]\, df,$$

as the volume v is diminished indefinitely.

It is important to notice that

$$\text{Curl } \mathbf{A} = [\nabla \mathbf{A}]$$

where the vector product on the right is formed in the usual way.

The real significance of these derived scalar and vector quantities will appear in the physical theories to be expounded in the sequel; their analytical import is to some extent elucidated in the general theorems developed immediately below. Before however concluding the present discussion we must present another aspect of the present definitions.

13. The fundamental differential operations just defined have a meaning only when the field is referred to an ordinary frame of reference. It is however sometimes more convenient to use other types of coordinates and we must therefore extend our definitions to apply to these cases. Let us first consider the case where the field is defined by a spherical polar coordinate system; there are then three principal perpendicular directions at any point (r_0, θ_0, ϕ_0) in the field to which it is convenient to refer any vector associated with the point in the ordinary way. These directions are the directions of the radius vector and of the tangents to the meridian and latitude circles defined by the intersection of the sphere $r = r_0$ with the cone $\theta = \theta_0$ and the plane $\phi = \phi_0$ respectively.

The differential elements of length in the principal directions just defined are easily seen to be

$$dr, \quad r\,d\theta, \quad r \sin\theta\, d\phi$$

respectively. Thus if we place the cartesian axes parallel to the polar coordinates at the point we see at once that the vector operator ∇ has components

$$\frac{\partial}{\partial r}, \quad \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi},$$

parallel to the three principal polar directions. The vector $\nabla\Phi$ has components

$$\frac{\partial\Phi}{\partial r}, \quad \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \quad \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi},$$

in the same directions.

Again if a vector $\mathbf{A}$ has components $\mathbf{A}_r$, $\mathbf{A}_\theta$, $\mathbf{A}_\phi$ parallel to the respective polar directions and if df is an element of surface normal to the unit vector $\mathbf{n}_1$, then

$$\begin{aligned}(\mathbf{A}\mathbf{n}_1)\, df &= \mathbf{A}_x dy\, dz + \mathbf{A}_y dz\, dx + \mathbf{A}_z dx\, dy \\ &= \mathbf{A}_r r\, d\theta \,.\, r \sin d\phi + \mathbf{A}_\theta r \sin\theta\, d\phi \,.\, dr + \mathbf{A}_\phi dr \,.\, r\, d\theta\end{aligned}$$

and thus if we now treat (r, θ, ϕ) as analytical coordinates in an ordinary cartesian space we can apply Green's lemma in its simple form and then find that

$$\begin{aligned}\int (\mathbf{A}\mathbf{n}_1)\, df &= \int_f [\mathbf{A}_r r^2 \sin\theta\, d\theta\, d\phi + \mathbf{A}_\theta r \sin\theta\, dr\, d\phi + \mathbf{A}_\phi r\, dr\, d\theta] \\ &= \int_v \left[\frac{\partial}{\partial r}(r^2 \sin\theta \mathbf{A}_r) + \frac{\partial}{\partial\theta}(r \sin\theta \mathbf{A}_\theta) + \frac{\partial}{\partial\phi}(r\mathbf{A}_\phi)\right] dr\, d\theta\, d\phi.\end{aligned}$$

We can now deduce a definition of the operator *div* in these coordinates by an application of the form derived above. In fact, if we make the surface f diminish indefinitely in all directions we derive at once, assuming continuity

$$\operatorname*{Lt}_{v=0} \frac{1}{v}\int_f (\mathbf{A}\mathbf{n}_1)\, df = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta \mathbf{A}_r) + \frac{\partial}{\partial\theta}(r \sin\theta \mathbf{A}_\theta) + \frac{\partial}{\partial\phi}(r\mathbf{A}_\phi)\right],$$

and thus

$$\operatorname{div} \mathbf{A} \equiv \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta \mathbf{A}_r) + \frac{\partial}{\partial\theta}(r \sin\theta \mathbf{A}_\theta) + \frac{\partial}{\partial\phi}(r\mathbf{A}_\phi)\right].$$

The operator *curl* may be similarly interpreted but it is most convenient to derive it from the analytical theorem of the next section.

14. If the coordinate frame of reference had been that suited to columnar fields the results could be equally readily obtained. The elements of length in the three principal directions at any point in the field are

$$dr, \quad r\, d\theta, \quad dz$$

and the operator ∇ has components

$$\frac{\partial}{\partial r}, \quad \frac{1}{r}\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial z},$$

and the operator *div* is such that

$$\operatorname{div} \mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\mathbf{A}_r) + \frac{\partial}{\partial\theta}(\mathbf{A}_\theta) + \frac{\partial}{\partial z}(r\mathbf{A}_z)\right],$$

$\mathbf{A}_r$, $\mathbf{A}_\theta$, $\mathbf{A}_z$ denoting the principal components of $\mathbf{A}$ at the point under consideration.

√15. **Miscellaneous Rules for Vector Calculations.** Before concluding this preliminary account of the analytical foundations of the formulae we will collect a few miscellaneous rules for the calculation and transformation of certain vector functions which will be of subsequent use.

(i)
$$\operatorname{div}[\mathbf{AB}] = \frac{\partial}{\partial x}(\mathbf{B}_y\mathbf{C}_z - \mathbf{B}_z\mathbf{C}_y) + \ldots + \ldots$$
$$= (\mathbf{C},\ \text{Curl}\,\mathbf{B}) - (\mathbf{B},\ \text{Curl}\,\mathbf{C}),$$

and thus

$$\int_f [\mathbf{B},\mathbf{C}]_n\, df = \int (\mathbf{C},\ \text{Curl}\,\mathbf{B})\, dv - \int (\mathbf{B},\ \text{Curl}\,\mathbf{C})\, dv,$$

the former integral being taken over any closed surface and the two latter over the interior space.

(ii)
$$\text{Curl}\, a\mathbf{A} = a\,\text{Curl}\,\mathbf{A} + [\nabla a,\ \mathbf{A}],$$

a being any scalar function.

If b is another scalar function and

$$\mathbf{A} = \nabla b,$$

then
$$\text{Curl}\,\mathbf{A} = \text{Curl}\,(\nabla b) = 0,$$

so that
$$\text{Curl}\, a\nabla b = [\nabla a,\ \nabla b].$$

(iii) We often want to use the operation

$$(\mathbf{A},\ \nabla)$$

on other scalar and vector functions. $(\mathbf{A}, \nabla)\,\phi$ denotes in general the rate of increase of ϕ per unit length in the direction of the vector $\mathbf{A}$. We may even use the operator on a second vector $\mathbf{B}$ and the interpretation of the result is similar; thus

$$(\mathbf{A},\ \nabla)\,\mathbf{B}_x = \mathbf{A}_x\frac{\partial \mathbf{B}_x}{\partial x} + \mathbf{A}_y\frac{\partial \mathbf{B}_x}{\partial y} + \mathbf{A}_z\frac{\partial \mathbf{B}_x}{\partial z},$$

$$(\mathbf{A},\ \nabla)\,\mathbf{B}_y = \mathbf{A}_x\frac{\partial \mathbf{B}_y}{\partial x} + \ldots + \ldots,$$

$$(\mathbf{A},\ \nabla)\,\mathbf{B}_z = \mathbf{A}_x\frac{\partial \mathbf{B}_z}{\partial x} + \ldots + \ldots,$$

and many results can be deduced in this way. We need not however enter into this subject any farther at present.

√16. **Stokes' Theorem.** The definition of the differential operator *curl* obtained above hardly emphasises the most important property of this operator. This is however easily obtained from a theorem given by Stokes* and which expresses the line integral of a vector $\mathbf{A}$ taken along a closed curve by a surface integ al taken over any surface bounded by the curve, viz.

$$\int_s (\mathbf{A}\,d\mathbf{s}) = \int_f \left\{\mathbf{n}_{1x}\left(\frac{\partial \mathbf{A}_z}{\partial y} - \frac{\partial \mathbf{A}_y}{\partial z}\right) + \mathbf{n}_{1y}\left(\frac{\partial \mathbf{A}_x}{\partial z} - \frac{\partial \mathbf{A}_z}{\partial x}\right) + \mathbf{n}_{1z}\left(\frac{\partial \mathbf{A}_y}{\partial x} - \frac{\partial \mathbf{A}_x}{\partial y}\right)\right\} df,$$

the direction of the normal $\mathbf{n}_1$ in the second integral being generally related

* Smith's Prize Examination (1854). *Math. and Phys. Papers*, v, p. 320.

to the sense in which the first integral is taken round the circuit in the same manner as translation to rotation in a left-handed screw relation.

In order to prove this theorem it is first necessary to notice that it will be merely sufficient to prove it for a small elementary plane circuit; because if we consider any finite barrier surface divided up into a large number of such small surfaces by means of a network of lines, then the total sum of the line integrals taken for each little elementary circuit separately is equivalent to the integral taken round the outer circuit alone, any interior element of a circuit being counted on the whole twice over with opposite signs in each case.

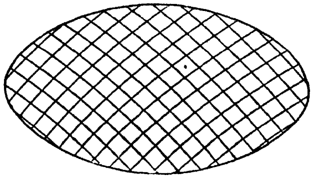


Fig. 5

We shall therefore content ourselves with proving the theorem for a small plane area.

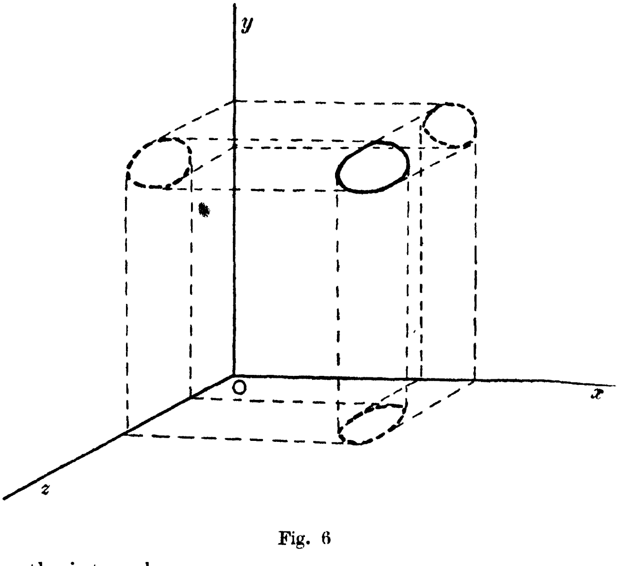


Fig. 6

Consider the integral

$$\int \mathbf{A}_x dx$$

taken round the boundary of such an element. If $\mathbf{A}_x$ is the value of this quantity at the mean centre of the element, the value at a point on the boundary whose small coordinates relative to this centre are (x', y', z') is

$$\mathbf{A}_x + x' \frac{\partial \mathbf{A}_x}{\partial x} + y' \frac{\partial \mathbf{A}_x}{\partial y} + z' \frac{\partial \mathbf{A}_x}{\partial z},$$

and thus

$$\int_s \mathbf{A}_x dx = \int_s \left(\mathbf{A}_x + x' \frac{\partial \mathbf{A}_x}{\partial x} + y' \frac{\partial \mathbf{A}_x}{\partial y} + z' \frac{\partial \mathbf{A}_x}{\partial z}\right) dx',$$

which is practically

$$= \frac{\partial \mathbf{A}_x}{\partial y} \int y' dx' + \frac{\partial \mathbf{A}_x}{\partial z} \int z' dx',$$

but taking account of the signs

$$\int y' dx' = -\mathbf{n}_{1z} df, \qquad \int z' dx' = \mathbf{n}_{1y} df,$$

so that

$$\int_s \mathbf{A}_x dx = \left(\mathbf{n}_{1y} \frac{\partial \mathbf{A}_x}{\partial z} - \frac{\partial \mathbf{A}_z}{\partial x} \mathbf{n}_{1z}\right) df.$$

Similar expressions are obtained for the other parts of the line integral

$$\int_f \mathbf{A}_y dy \text{ and } \int_f \mathbf{A}_z dz,$$

and adding them together we have

$$\int_s \mathbf{A}_s ds = \left[\mathbf{n}_{1x} \left(\frac{\partial \mathbf{A}_z}{\partial y} - \frac{\partial \mathbf{A}_y}{\partial z}\right) + \ldots + \ldots\right] df,$$

and summation over the whole area establishes the theorem for any circuit with the corresponding barrier surface.

We usually write this relation in the form

$$\int_s (\mathbf{A} d\mathbf{s}) = \int_f \mathbf{B}_n df,$$

where $\mathbf{B}_n$ is the normal component of the vector $\mathbf{B} = \text{curl } \mathbf{A}$ so that Stokes' theorem expresses that

$$\int_s \mathbf{A}_s ds = \int_f \text{curl}_n \, \mathbf{A} df,$$

where $\mathbf{A}_s$ denotes the component of $\mathbf{A}$ along the direction of ds and $\text{curl}_n \, \mathbf{A}$ the normal component of the vector curl $\mathbf{A}$.

17. We shall find it frequently necessary to quote this theorem of Stokes in spherical polar coordinates instead of the ordinary cartesian coordinates as above. If the axis Ox is taken as polar axis with the origin still the same, and θ denotes the angular distance of the radius vector r from this axis and ϕ azimuth round it measured from the plane Ozx, the transformation may be effected by the substitution

$$x = r \cos \theta, \quad y = r \sin \theta \cos \phi, \quad z = r \sin \theta \sin \phi,$$

with the appropriate transformation for the other vector components. It is however more easily obtained if we notice that the orthogonal elements of length in polar coordinates are

$$dr, \quad r d\theta, \quad r \sin \theta d\phi,$$

for then

$$\int (\mathbf{A}\, d\mathbf{s}) = \int_s (\mathbf{A}_x dx + \mathbf{A}_y dy + \mathbf{A}_z dz)$$

$$= \int_s (\mathbf{A}_r dr + \mathbf{A}_\theta r\, d\theta + \mathbf{A}_\phi r \sin\theta\, d\phi),$$

where $\mathbf{A}_r$, $\mathbf{A}_\theta$, $\mathbf{A}_\phi$ are used to denote the components of the vector $\mathbf{A}$ in the directions of the respective small displacements. Moreover

$$\int_f \mathbf{B}_n df = \int_f (\mathbf{B}_x dy\, dz + \mathbf{B}_y dz\, dx + \mathbf{B}_z dx\, dy)$$

$$= \int_f (\mathbf{B}_r r^2 \sin\theta\, d\theta\, d\phi + \mathbf{B}_\theta r \sin\theta\, dr\, d\phi + \mathbf{B}_\phi r\, dr\, d\theta),$$

and if therefore we regard (r, θ, ϕ) as coordinates of the (x, y, z) type we may apply the analytical transformation of Stokes and we thus get

$$r^2 \sin\theta \mathbf{B}_r = -\frac{\partial}{\partial\phi}(r\mathbf{A}_\theta) + \frac{\partial}{\partial\theta}(r \sin\theta \mathbf{A}_\phi),$$

$$r \sin\theta \mathbf{B}_\theta = -\frac{\partial}{\partial r}(r \sin\theta \mathbf{A}_\phi) + \frac{\partial}{\partial\phi}(\mathbf{A}_r),$$

$$r\mathbf{B}_\phi = -\frac{\partial}{\partial\theta}(\mathbf{A}_r) + \frac{\partial}{\partial r}(r\mathbf{A}_\theta),$$

and the transformation is thus effected: the vector $\mathbf{B}$ is completely defined as regards its components in the three principal spherical directions.

18. Green's Lemma and Stokes' Theorems for moving circuits*. In our treatment of the electrodynamic phenomena in moving systems we shall find it convenient to be able to apply the general integral theorems of Green and Stokes to non-stationary surfaces or curves. The results for such cases are easily obtained.

(1) The time rate of variation of the line integral

$$\int_1^2 \mathbf{A}_s ds$$

taken along any unclosed curve between the points 1 and 2 is

$$\int_1^2 \left(\frac{d\mathbf{A}}{dt} + \nabla(\mathbf{u}, \mathbf{A}) - [\mathbf{u} \,.\, \operatorname{curl} \mathbf{A}]\right)_s ds,$$

when the points of the line are in motion so that the velocity of the 's' point on it is $\mathbf{u}$.

The time variation of the integral consists of two parts the first of which would exist if the curve remained at rest and this amounts to

$$\int_1^2 \left(\frac{d\mathbf{A}}{dt} \,.\, d\mathbf{s}\right),$$

and the second part, resulting from the motion of the curve, which may be

* Cf. Abraham u. Föppl, *Theorie der Elektrizität*, Bd. I. §§ 33, 34.

calculated as if the field itself were stationary. We denote the position of the curve at the time $t + dt$ by its end points $(1', 2')$: the variation of the line integral during dt so far as it arises from the motion of the curve is then

$$\int_{1'}^{2'} (\mathbf{A}d\mathbf{s}) - \int_{1}^{2} (\mathbf{A}d\mathbf{s}) = \int_{1'}^{2'} (\mathbf{A}d\mathbf{s}) + \int_{2}^{1} (\mathbf{A}d\mathbf{s}).$$

Now consider the integrals of the tangential component of the vector **A** along the two small elements of curves $(1, 1')$ and $(2, 2')$ described by the end points of the given curve during dt: these two integrals are

$$\int_{1}^{1'} (\mathbf{A}d\mathbf{s}) = (\mathbf{Au})_1\, dt, \qquad \int_{2'}^{2} (\mathbf{A}d\mathbf{s}) = -(\mathbf{Au})_2\, dt,$$

and when they are added to the above two integrals we get a single integral

$$\int (\mathbf{A}d\mathbf{s}),$$

taken round the closed curve $(1, 2, 2', 1', 1)$: this integral transforms by Stokes' theorem as above into the integral of the normal component of curl **A** over a surface bounded by this curve. If we take the surface to be that actually described by the elements of the line in its motion it will consist simply of elements

$$[\mathbf{u}d\mathbf{s}]\, dt,$$

described by the separate elements of the curve. Each element of this type thus contributes a part

$$dt\,(\text{curl}\,\mathbf{A},\ [\mathbf{u}d\mathbf{s}]) = dt\,(d\mathbf{s}\,[\text{curl}\,\mathbf{A}, \mathbf{u}])$$

to the integral considered. We thus find that

$$\int_{1'}^{2'} (\mathbf{A}d\mathbf{s}) + \int_{2}^{1} (\mathbf{A}d\mathbf{s}) + (\mathbf{Au})_1\, dt - (\mathbf{Au})_2\, dt = dt \int_{1}^{2} (d\mathbf{s}\,[\text{curl}\,\mathbf{A}, \mathbf{u}]),$$

and if we write

$$(\mathbf{Au})_1 - (\mathbf{Au})_2 = -\int_{1}^{2} (d\mathbf{s}\nabla)\,(\mathbf{Au}),$$

we get

$$\int_{1'}^{2'} (\mathbf{A}d\mathbf{s}) - \int_{1}^{2} (\mathbf{A}d\mathbf{s}) = dt \int_{1}^{2} (d\mathbf{s},\ \nabla\,(\mathbf{u}\,.\,\mathbf{A}) - [\mathbf{u}\,.\,\text{curl}\,\mathbf{A}]),$$

and this is the part of the variation of the line integral which arises from the motion of the curve. The result quoted is obtained by adding this part to the former one due to the variation of the field.

19. (2) The time rate of variation of the surface integral

$$\int \mathbf{B}_n df$$

taken over any part of a surface in motion as above is

$$\int df \left(\frac{d\mathbf{B}}{dt} + \text{curl}\,[\mathbf{B}, \mathbf{u}] + \mathbf{u}\,\text{div}\,\mathbf{B}\right)_n.$$

As before the total variation consists of the part

$$\int \frac{d\mathbf{B}_n}{dt}\, df,$$

due to the variation of the field and which can be calculated as if the surface were at rest, and a part which arises entirely from the motion of the surface and in the calculation of which the time variation of the field itself can be neglected. We denote the position of the surface at time $(t + dt)$ by f', that at the time t being f. The variation of the integral on account of the motion is then

$$\int_{f'} \mathbf{B}_n df - \int_f \mathbf{B}_n df.$$

Now close up the two unclosed surfaces f and f' by adding the small band of surface described by the bounding curve during the motion between the times t and $t + dt$. In this motion each element ds of the curve describes a small area $[ds, \mathbf{u}]\, dt$ so that we have by Green's theorem

$$\int_{f'} \mathbf{B}_n df - \int_f \mathbf{B}_n df + dt \int (\mathbf{B} \,.\, [ds, \mathbf{u}]) = \int \operatorname{div} \mathbf{B} dv,$$

where dv is the element of the space enclosed by the surfaces f and f' and the bounding strip: since this volume element is described by the element df during the time dt we must have

$$dv = dt\, df \mathbf{u}_n.$$

If we also write

$$(\mathbf{B}\, [ds, \mathbf{u}]) = -\, (ds\, [\mathbf{B}, \mathbf{u}]),$$

we find that

$$\int_{f'} \mathbf{B}_n df - \int_f \mathbf{B}_n df = dt \left\{ \int df \mathbf{u}_n \operatorname{div} \mathbf{B} + \int (ds, [\mathbf{B}, \mathbf{u}]) \right\}.$$

The last integral on the right can be transformed by Stokes' theorem and we get

$$\int_{f'} \mathbf{B}_n df' - \int_f \mathbf{B}_n df = dt \int df\, (\mathbf{u} \operatorname{div} \mathbf{B} + \operatorname{curl}\, [\mathbf{B}, \mathbf{u}])_n,$$

whence by addition the result is obtained as stated.

20. Green's Theorem. The derivation of the characteristic analytical properties of electrostatic fields is facilitated by the use of an important theorem due to Green.

Let ϕ, ψ be any two scalar functions. From them we can deduce a vector $\mathbf{A}$ of the form

$$\mathbf{A} = \phi \nabla \psi - \psi \nabla \phi$$

and thus

$$\operatorname{div} \mathbf{A} = \phi \nabla^2 \psi - \psi \nabla^2 \phi,$$

where, as always, we understand by ∇^2 the differential operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2};$$

it is to be noticed that it is precisely the square of the Hamiltonian operator treated according to the ordinary rules.

Now suppose that the functions ϕ and ψ are continuous and have continuous derivatives inside the space v enclosed by the surface f. A simple application of Green's lemma then gives

$$\int_v (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dv = +\int_f \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) df,$$

where we use $\dfrac{\partial\phi}{\partial n}$ as the component of the gradient of ϕ along the outward normal at the element df of the surface. This is Green's Theorem.

The more important forms of this theorem are however obtained by adopting a special form for one of the functions.

If P is a variable point in the region v with coordinates (x, y, z) and P_1 any fixed point with coordinates (x_1, y_1, z_1), then if we use r_1 for the distance PP_1 we have

$$r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

and consequently

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = -\frac{(x - x_1, \; y - y_1, \; z - z_1)}{r^3},$$

and therefore also

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(x - x_1)^2}{r^5},$$

and similarly for $\dfrac{\partial^2}{\partial y^2}\left(\dfrac{1}{r}\right)$ and $\dfrac{\partial^2}{\partial z^2}\left(\dfrac{1}{r}\right)$. We therefore see that

$$\nabla^2\left(\frac{1}{r}\right) = 0.$$

If now the point P_1 lies outside the region v we can put

$$\psi = \frac{1}{r},$$

and the above theorem takes the form

$$\int_v \nabla^2\phi \frac{dv}{r} + \int_f \left\{\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right\} df = 0.$$

21. If however P_1 lies inside the region v then $\dfrac{1}{r}$ regarded as a function of the position of P will be infinite at P_1, and if we wish to use our formula with $\psi = \dfrac{1}{r}$ we must exclude P_1 by putting a small surface round it. We shall do this by taking a small sphere of radius r_1 round the point P_1 as centre. The space between this and the surface f we call v'. We can now apply our theorem to this region v', so that

$$\int_{v'} \nabla^2\phi \frac{dv}{r} + \int_{f'} \left\{\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right\} df' = 0,$$

where f' includes, in addition to the surface f, also the surface of the small sphere.

If now we denote by $d\omega$ the element of solid angle, the element of spherical surface of radius r is $r^2 d\omega$ and the element of volume is $r^2 dr d\omega$. It is also to be noticed that on the surface of the small sphere the normal n coincides with the direction of r (but is in the opposite sense) and so

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = + \frac{1}{r_1^2},$$

and therefore the part of the surface integral due to the surface of the sphere is

$$\int \phi \, d\omega - r_1 \int \frac{\partial \phi}{\partial r} d\omega,$$

where the integrations are over the surface of the unit sphere. If at the point P_1 the function ϕ and its differential coefficients are continuous, then $\frac{\partial \phi}{\partial r}$ is finite and thus if we make the sphere infinitely small the second integral

$$r_1 \int \frac{\partial \phi}{\partial r} d\omega$$

tends to zero, and if the value of ϕ has the value ϕ_1 at P_1 the first integral tends to

$$4\pi\phi_1.$$

In the volume integral the part due to the inside of the sphere is excluded : but this is

$$\iint \nabla^2 \phi \, r \, dr \, d\omega,$$

and obviously vanishes with r_1 if $\nabla^2\phi$ is finite. We have thus in all

$$4\pi\phi_1 = -\int_v \nabla^2\phi \frac{dv}{r} + \int_f \left\{\frac{1}{r}\frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right\} df',$$

where the integral with respect to v is now over the whole region inside f and the surface integral is over f only; all traces of the cavity drawn about the point P_1 have disappeared.

22. The analysis so far is limited to the case in which the functions involved are continuous over the whole region v. We can however immediately extend it to include the most important cases involving discontinuity. Supposing that ϕ and its first differential coefficients are discontinuous over the surface f' lying in this region, otherwise having determinate continuous values throughout the region. Draw a normal n at each point of the surface f' and regard directions in it as positive when in some definite chosen sense. The side of the surface f' on the side of increasing n we call the positive side and the other the negative side and we distinguish the values of functions on the two sides by suffices $+$ and $-$, e.g. ϕ_+ and ϕ_-.

We can then apply our previous formula if we include in the boundary of v the two sides of the surface f' as well as the surface f. On the positive

side of f' dn is negative and on the negative side it is positive. The point P_1 is assumed not to lie on the surface f'. We thus get

$$4\pi\phi_1 = -\int_v \nabla^2\phi \frac{dv}{r} + \int_f \left\{\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right\} df'$$
$$-\int_{f'}\left[\left(\frac{\partial\phi}{\partial n}\right)_+ - \left(\frac{\partial\phi}{\partial n}\right)_-\right]\frac{df'}{r} + \int_{f'}(\phi_+ - \phi_-)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) df',$$

a formula which will hold even if f' consists of several separate surfaces.

This is the general result. In applications however one often has to apply it to indefinitely extended fields from which certain finite spaces are excluded.. In such cases a detailed discussion of the behaviour of the infinite integrals becomes necessary and each case must be treated on its merits. The following general result is however easily deduced if the theorem is applied as though the field were bounded by a very large enclosing surface which is ultimately extended indefinitely in all directions, and it provides a sufficient criterion in most cases.

✓ **23.** Suppose the function ϕ, now given throughout all space, is such that

$$\operatorname*{Lt}_{r\to\infty} \phi = 0$$

and

$$\operatorname{Lt} r^2 \frac{\partial\phi}{\partial n} \text{ is finite,}$$

then the part of the surface integral corresponding to the infinite boundary will be zero in the limit and the formula can be written as

$$4\pi\phi_1 = -\int_v \nabla^2\phi \frac{dv}{r} + \int_f \left\{\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right\} df'$$
$$-\int_{f'}\left[\left(\frac{\partial\phi}{\partial n}\right)_+ - \left(\frac{\partial\phi}{\partial n}\right)_-\right]\frac{df'}{r} + \int_{f'}(\phi_+ - \phi_-)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) df',$$

where the volume integral is extended over the whole of space outside certain specified regions, the first surface integral over the boundaries of these regions and the second over all discontinuity surfaces in the region investigated.

A function ϕ limited by the usual conditions of continuity as well as the above conditions at infinity will be said to be regular in the space investigated.

This general result shows that a function ϕ which fulfils the conditions and which apart from the surfaces f' is with its derivatives continuous in the whole of space, is uniquely determined in the whole region if the values of $\nabla^2\phi$ are given at each point and also the discontinuities in ϕ and its normal gradient on all the surfaces f'.

The continuity of $\nabla^2\phi$ is not involved.

24. If we write

$$\nabla^2\phi = -4\pi\rho,$$

$$\left(\frac{\partial\phi}{\partial n}\right)_+ - \left(\frac{\partial\phi}{\partial n}\right)_- = -4\pi$$

and $$\phi_+ - \phi_- = 4\pi\tau,$$

then our formula shows that

$$\phi = \int \frac{\rho dv}{r} + \int_{f'} \frac{\sigma df'}{r} + \int_{f'} \tau \frac{\partial}{\partial n}\left(\frac{1}{r}\right) df',$$

where we now consider the whole of space without any excluded regions.

If we consider the values of ρ, σ and τ to be those as defined above, then this formula is merely the expression of an identity. It contains an expression by definite integrals of a general function ϕ subject merely to the specified continuity conditions.

If however we regard the question from the other point of view and consider the quantities ρ, σ and τ as given *a priori*, then we want to know whether the function ϕ defined in the same way satisfies the same conditions. It is easily proved that it does.

We define ϕ at any point P by the relation

$$\phi = \int \frac{\rho dv}{r} + \int \frac{\sigma df'}{r} + \int \tau \frac{\partial}{\partial n}\left(\frac{1}{r}\right) df',$$

the first integral being taken over the whole of space and the second and third over those surfaces on which σ and τ have finite values.

If the point P is at an external point, i.e. at a point in space in the immediate neighbourhood of which $\rho = 0$, then we can differentiate each of these integrals with respect to the coordinates of P under the sign of integration. We get

$$\nabla^2\phi = \int \rho \nabla^2\left(\frac{1}{r}\right) dv + \int \sigma \nabla^2\left(\frac{1}{r}\right) df + \int \tau \frac{\partial}{\partial n} \nabla^2\left(\frac{1}{r}\right) df',$$

and since $\nabla^2\left(\frac{1}{r}\right) = 0$ we have

$$\nabla^2\phi = 0.$$

If however P is at any other point where the value of ρ is not zero we must proceed in a different manner. Notice that it is only possible for discontinuities in ϕ or its derivatives to occur near one of the surfaces f'; we may therefore from our previous theorem write

$$4\pi\phi = -\int \nabla^2\phi \frac{dv}{r} - \int_{f'} \left[\left(\frac{\partial\phi}{\partial n}\right)_+ - \left(\frac{\partial\phi}{\partial n}\right)_-\right] \frac{df'}{r} + \int_{f'} (\phi_+ - \phi_-) \frac{\partial}{\partial n}\left(\frac{1}{r}\right) df',$$

and thus on elimination of ϕ we get

$$\int (\nabla^2\phi + 4\pi\rho) \frac{dv}{r} - \int \left\{\left(\frac{\partial\phi}{\partial n}\right)_+ - \left(\frac{\partial\phi}{\partial n}\right)_- + 4\pi\sigma\right\} \frac{df'}{r}$$
$$+ \int (\phi_+ - \phi_- - 4\pi\tau) \frac{\partial}{\partial n}\left(\frac{1}{r}\right) \frac{df'}{r} = 0.$$

Thus we see that, on account of the arbitrariness of the position of the point P, we must have*

$$\nabla^2\phi + 4\pi\rho = 0,$$

$$\left(\frac{\partial\phi}{\partial n}\right)_+ - \left(\frac{\partial\phi}{\partial n}\right)_- + 4\pi\sigma = 0,$$

and
$$\phi_+ - \phi_- - 4\pi\tau = 0,$$

which are precisely the same as the previous conditions.

25. Kirchhoff's Theorem. In the previous section we obtained a solution of the fundamental equation

$$\nabla^2\phi = -4\pi\rho,$$

subject to certain continuity conditions in the form of a volume integral throughout the whole of space together with appropriate surface integrals over the surfaces of discontinuity. We now proceed to the more general case of the fundamental wave equation

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - 4\pi\rho.$$

A general method of solution given by Kirchhoff† in a paper on the theory of rays of light is based on Green's theorem and on the proposition that if r is the distance from a fixed point, and F an arbitrary function, the expression

$$\chi = \frac{1}{r} F\left(t \pm \frac{r}{c}\right),$$

has the property expressed by

$$\nabla^2\chi = \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} \qquad \text{(i)}.$$

This follows at once from the formula

$$\nabla^2\chi = \frac{\partial^2\chi}{\partial r^2} + \frac{2}{r}\frac{\partial\chi}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\chi),$$

which is true for any function of r, not explicitly containing the coordinates, and in virtue of which (i) assumes the form

$$\frac{\partial^2(r\chi)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2(r\chi)}{\partial t^2}.$$

It is well known that

$$r\chi = F\left(t + \frac{r}{c}\right) \quad \text{and} \quad r\chi = F\left(t - \frac{r}{c}\right)$$

are solutions of this equation.

* At any point of the field we can choose an infinite number of variations of the position of P such that along them any two of the integrals together are constant.

† *Berlin. Ber.* (1882), p. 641; *Wied. Ann.* XVIII. (1883); *Ges. Abh.* II. p. 22. The present proof is given by Lorentz, *The Theory of Electrons*, p. 233.

26. Let now f be the bounding surface of a space v throughout which ψ is subjected to the equation (i), P the point of v for which we want to determine the function, dv an element of volume situated at the distance r from P, f' a small spherical surface having P' as centre, n and n' the normals to f and f', both drawn towards the outside*.

Introducing the auxiliary expression

$$\chi = \frac{1}{r} F\left(t + \frac{r}{c}\right),$$

where F is a function to be specified later on, we shall consider the integral

$$I = \int (\psi \nabla^2 \chi - \chi \nabla^2 \psi)\, dv,$$

extended to the space between f and f'.

In the first place we have by Green's theorem

$$I = \int_f \left(\psi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \psi}{\partial n}\right) df - \int_{f'} \left(\psi \frac{\partial \chi}{\partial n'} - \chi \frac{\partial \psi}{\partial n}\right) df',$$

and in the second place on account of the equations satisfied by ψ and χ

$$I = \frac{1}{c^2} \int \left(\psi \frac{\partial^2 \chi}{\partial t^2} - \chi \frac{\partial^2 \psi}{\partial t^2}\right) dv + 4\pi \int \rho \chi dv$$
$$= \frac{1}{c^2} \frac{d}{dt} \int \left(\psi \frac{\partial \chi}{\partial t} - \chi \frac{\partial \psi}{\partial t}\right) dv + 4\pi \int \rho \chi dv.$$

Hence, combining the two results

$$- \int_{f'} \left(\psi \frac{\partial \chi}{\partial n'} - \chi \frac{\partial \psi}{\partial n'}\right) df' = - \int_f \left(\psi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \psi}{\partial n}\right) df + 4\pi \int \rho \chi dv$$
$$+ \frac{1}{c^2} \frac{d}{dt} \int \left(\psi \frac{\partial \chi}{\partial t} - \chi \frac{\partial \psi}{\partial t}\right) dv.$$

This equation must hold for all values of t. After being multiplied by dt, it may therefore be integrated between arbitrary limits t_1 and t_2, giving

$$- \int_{t_1}^{t_2} dt \int_{f'} \left(\psi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \psi}{\partial n'}\right) df' = - \int_{t_1}^{t_2} dt \left(\psi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \psi}{\partial n}\right) df + 4\pi \int_{t_1}^{t_2} dt \int \rho \chi dv$$
$$+ \frac{1}{c^2} \left| \int \left(\psi \frac{\partial \chi}{\partial t} - \chi \frac{\partial \psi}{\partial t}\right) dv \right|_{t_1}^{t_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From this equation we can derive the solution of our problem by means of a proper choice of the function F, which has thus far been left indeterminate.

27. We shall now suppose that $F(\epsilon)$ differs from zero only for values of ϵ lying between 0 and a certain positive quantity δ, this latter being so small that we may neglect the change which any of the other quantities occurring

* We need not here attend to all the saving restrictions that would be necessary in formal pure mathematics: such limitations are, in the main, sufficiently obvious, and are satisfied by the nature of the case, in continuous physical analysis.

in the problem undergoes during an interval of time equal to ϵ. As to the function F itself we shall suppose its values between $\epsilon = 0$ and $\epsilon = d$ to be so great that

$$\int_0^\delta F(\epsilon)\, d\epsilon = 1.$$

Since for a fixed value of r

$$\int_{t_1}^{t_2} F\left(t + \frac{r}{c}\right) dt = \int_{t_1+\frac{r}{c}}^{t_2+\frac{r}{c}} F(\epsilon)\, d\epsilon,$$

it is clear that on the above assumptions

$$\int_{t_1}^{t_2} F\left(t + \frac{r}{c}\right) = 1 \quad \text{and} \quad \int_{t_1}^{t_2} \kappa F\left(t + \frac{r}{c}\right) dt = \kappa_{t=-\frac{r}{c}},$$

provided t_1 and t_2 are such values of t that

$$t_1 + \frac{r}{c} < 0 \quad \text{and} \quad \left(t_2 + \frac{r}{c}\right) > \delta,$$

and κ is one of the functions of t with which we are concerned. This latter formula enables us to select as it were the values of ψ corresponding to definite moments.

Let t_2 have a fixed positive value and t_1 a negative one, so great that even for the points of f most distant from P, $t_1 + \frac{r}{c} < 0$. Then all values of χ occur in the last term of (2). Indeed

$$\frac{\partial \chi}{\partial t} = \frac{1}{r} F'\left(t + \frac{r}{c}\right),$$

and this vanishes for $t = t_1$ and $t = t_2$, because $F'(\epsilon)$, like $F(\epsilon)$ itself vanishes for all values of ϵ outside the interval $(0, \delta)$. The last term on the right-hand side of (2) is thus seen to be zero.

The term involving ρ may be written

$$4\pi \int \frac{1}{r} dv \int_{t_1}^{t_2} \rho F\left(t + \frac{r}{c}\right) dt$$

and this is

$$= 4\pi \int \frac{1}{r} \rho_{t=-\frac{r}{c}}\, dv.$$

Again

$$\int_{t_1}^{t_2} dt \int_f \chi \frac{\partial \psi}{\partial n} df = \int_{t_1}^{t_2} dt \int_f \frac{1}{r} F\left(t + \frac{r}{c}\right) \frac{\partial \psi}{\partial n} df$$

$$= \int_f \frac{df}{r} \int_{t_1}^{t_2} \frac{\partial \psi}{\partial n} F\left(t + \frac{r}{c}\right) dt$$

$$= \int_f \left(\frac{\partial \psi}{\partial n}\right)_{t=-\frac{r}{c}} \frac{df}{r}.$$

28. We have next to consider the integral containing $\frac{\partial \chi}{\partial n}$. This differential coefficient being equal to

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) F\left(t+\frac{r}{c}\right)+\frac{1}{rc}\frac{\partial r}{\partial n} F'\left(t+\frac{r}{c}\right),$$

we have
$$\int_{t_1}^{t_2} dt \int_f \psi \frac{\partial \chi}{\partial n} df = \int_{t_1}^{t_2} dt \int \frac{\partial}{\partial n}\left(\frac{1}{r}\right) \psi F\left(t+\frac{r}{c}\right) df$$
$$+\frac{1}{c}\int_{t_1}^{t_2} dt \int \frac{1}{r}\frac{\partial r}{\partial n} \psi F'\left(t+\frac{r}{c}\right) df.$$

The first integral is

$$\int \frac{\partial}{\partial n}\left(\frac{1}{r}\right) df \int_{t_1}^{t_2} \psi F\left(t+\frac{r}{c}\right) dt = \int \frac{\partial}{\partial n}\left(\frac{1}{r}\right) \psi_{t=-\frac{r}{c}}\, df,$$

and the second expression may be integrated by parts

$$\int_{t_1}^{t_2} dt \int \frac{1}{r}\frac{\partial r}{\partial n} \psi F'\left(t+\frac{r}{c}\right) df$$
$$= \int \frac{1}{r}\frac{\partial r}{\partial n} df \int_{t_1}^{t_2} \psi F'\left(t+\frac{r}{c}\right) dt$$
$$= \int \frac{1}{r}\frac{\partial r}{\partial n} df \left|\psi F\left(t+\frac{r}{c}\right)\right|_{t_1}^{t_2} - \int \frac{1}{r}\frac{\partial r}{\partial n} df \int_{t_1}^{t_2} \frac{\partial \psi}{\partial t} F\left(t+\frac{r}{c}\right) df$$
$$= -\int \frac{1}{r}\frac{\partial r}{\partial n}\left(\frac{\partial \psi}{\partial t}\right)_{t=-\frac{r}{c}} df,$$

because both $F\left(t_1+\frac{r}{c}\right)$ and $F\left(t_2+\frac{r}{c}\right)$ vanish. Combining these results we find for the right-hand side of (2)

$$4\pi \int \frac{1}{r}\, \rho_{t=-\frac{r}{c}}\, dv + \int \frac{1}{r}\left\{\left(\frac{\partial \psi}{\partial n}\right)_{t=-\frac{r}{c}} - \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\psi_{t=-\frac{r}{c}} + \frac{1}{cr}\frac{\partial r}{\partial n}\left(\frac{\partial \psi}{\partial t}\right)_{t=-\frac{r}{c}}\right\} df.$$

29. We now suppose the radius R of the sphere f' to diminish indefinitely. By this process the above terms are unchanged but we must now calculate the limiting value of the terms on the left-hand side of (2). As the integral over the sphere has the same form as that over the surface f just considered, we may write

$$-\int_{t_1}^{t_2} dt \int \left(\psi \frac{\partial \chi}{\partial n'} - \chi \frac{\partial \psi}{\partial n'}\right) df'$$
$$= \int \left\{\frac{1}{r}\left(\frac{\partial \psi}{\partial n'}\right)_{t=-\frac{r}{c}} - \frac{\partial}{\partial n'}\left(\frac{1}{r}\right)\psi_{t=-\frac{r}{c}} + \frac{1}{cr}\frac{\partial r}{\partial n'}\left(\frac{\partial \psi}{\partial t}\right)_{t=-\frac{r}{c}}\right\} df',$$

or since the normal n' has the direction of r and since at the sphere $r = R$ this is

$$\int \left\{\frac{1}{R}\left(\frac{\partial \psi}{\partial n'}\right)_{(t=-\frac{R}{c})} + \frac{1}{R^2}\psi_{(t=-\frac{r}{c})} + \frac{1}{CR}\left(\frac{\partial \psi}{\partial t}\right)_{t=-\frac{R}{c}}\right\} df'.$$

Now when R tends towards 0, the integrals with $\frac{1}{R}$ vanish, so that the expression reduces to

$$\frac{1}{R^2}\int \psi_{t=-\frac{R}{c}}\, df',$$

which in the limit becomes $\quad 4\pi\psi_{P(t=0)}$,

$\psi_{P(t=0)}$ being the value of ψ at P for the instant $t=0$. We thus have

$$\psi_{P(t=0)} = \frac{1}{4\pi}\int\left\{\frac{1}{r}\left(\frac{\partial\psi}{\partial n}\right)_{t=-\frac{r}{c}} - \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\psi_{t=-\frac{r}{c}} + \frac{1}{cr}\frac{\partial r}{\partial n}\left(\frac{\partial\psi}{\partial t}\right)_{t=-\frac{r}{c}}\right\} df + \int \frac{1}{r}\rho_{t=-\frac{r}{c}}\, dv,$$

which determines the value of ψ at the chosen point P for the instant $t=0$. We are, however, free in the choice of this instant, and therefore the formula may serve to calculate the value of ψ_P for any instant t; for this we have only to replace the values of ψ, $\frac{\partial\psi}{\partial n}$ and $\frac{\partial\psi}{\partial t}$ on the right-hand side by those relating to the time $t-\frac{r}{c}$.

30. On the convergence and differentiability of potential integrals*. The analyses of the preceding paragraphs and certain considerations which subsequently present themselves in our physical discussions necessitate a rather close investigation of the validity of the usual processes of the calculus as applied to certain integrals of the type †

$$I = \int\frac{\rho\, dv}{r}, \quad X_1 = \int \rho\, dv \frac{\partial}{\partial x_1}\left(\frac{1}{r}\right), \quad X_2 = \int \rho\, dv \frac{\partial^2}{\partial x_1^2}\left(\frac{1}{r}\right),$$

the notation being the same as before; ρ denoting any function of (x, y, z); dv being the element of volume $dx\,dy\,dz$ and r the distance of the point (x, y, z) defining the position of this element from the fixed point (x_1, y_1, z_1).

When the point (x_1, y_1, z_1) is at a finite distance from all the points at which ρ is not zero there is no question as to convergence but it is necessary to prove that

$$X_1 = \frac{\partial I}{\partial x_1}, \quad X_2 = \frac{\partial X_1}{\partial x_1}.$$

We consider the general integral

$$\phi = \int \rho f(x_1)\, dv,$$

the function $f(x_1)$ involving also the parameters (x, y, z) of integration as well as y_1 and z_1. We have then

$$\frac{\partial\phi}{\partial x_1} = \lim_{\Delta x_1 \to 0}\frac{1}{\Delta x_1}\left\{\int \rho f(x_1+\Delta x_1)\, dv - \int \rho f(x_1)\, dv\right\}.$$

* The considerations of the present paragraph are derived mainly from Poincaré's treatise *Théorie du potentiel newtonien* (Paris, 1899), to which the reader is referred for a more detailed exposition of many points which have been slurred in the present discussion.

† The types here chosen are those that specifically occur in the physical discussions. Otherwise they are probably of little or no import.

If the second differential of $f(x_1)$ exists at and near the point (x_1, y_1, z_1) we may write this in the form

$$\frac{\partial \phi}{\partial x_1} = \operatorname*{Lim}_{\Delta x_1 \to 0} \int \rho \left(\frac{\partial f}{\partial x_1} + \frac{\Delta x}{2} \frac{\partial^2 f}{\partial x_1^2} \right) dv,$$

where however we write $x_1 + \theta \Delta x_1$ $(0 < \theta < 1)$ in $\frac{\partial^2 f_1}{\partial x_1^2}$ after the differentiation has been carried out.

We may proceed at once to the limit when $\Delta x_1 = 0$ and get

$$\frac{\partial \phi}{\partial x_1} = \int \rho \frac{\partial f}{\partial x_1} dv,$$

provided only that the function $\frac{\partial^2 f_2}{\partial x_1^2}$ is limited at all points in the neighbourhood of the point (x_1, y_1, z_1) and for every combination of values (x, y, z) for which the function ρ is different from zero.

Now the function

$$f = \frac{\partial^s}{\partial x_1^s} \left(\frac{1}{r} \right)$$

satisfies the condition stated if (x_1, y_1, z_1) is outside the region of integration so that we have

$$X_1 = \frac{\partial I}{\partial x_1}, \quad X_2 = \frac{\partial X_1}{\partial x_1}, \text{ etc.}$$

When however the point (x_1, y_1, z_1) coincides with one point of the field of integration the integrands become infinite and the question at once arises as to whether the expressions have any meaning at all. We can approach the matter by the consideration of certain simple cases.

***31.** Let us consider the definite integral

$$I = \int_a^b f(x)\, dx. \qquad (a < b)$$

If the function f becomes infinite for $x = a$ the integral has in itself no meaning: to give it one we consider the integral

$$I_\epsilon = \int_{a+\epsilon}^b f(x)\, dx,$$

which is quite definite. If then I_ϵ tends to a limit I when ϵ tends to zero the integral above is said to be convergent and its value is represented by I. If on the contrary I_ϵ increases indefinitely as ϵ diminishes to zero the integral is said to be divergent and the symbol has no meaning.

E.g. if $|f(x)| < \frac{1}{(x-a)^s}$ where $s < 1$ the integral is convergent.

Next consider the double integral

$$\int \phi(x, y)\, df$$

extended over the area f enclosed by a given curve S in the (x, y) plane. If $\phi(x, y)$ becomes infinite at the point $O(x_1, y_1)$ in the area, the integral has no meaning: to give it one surround the point O by a small closed curve S_0 and let f_0 be the area between the two curves S and S_0. The integral

$$I_0 = \int_{f_0} \phi\, df$$

has then a definite meaning, but its value changes as the curve S_0 alters. Now suppose that I_0 has a limit I when the curve S_0 diminishes in all directions to the point O: we then say that the integral is convergent and its value is I. If on the other hand I_0 increases indefinitely as this curve is diminished the integral is divergent.

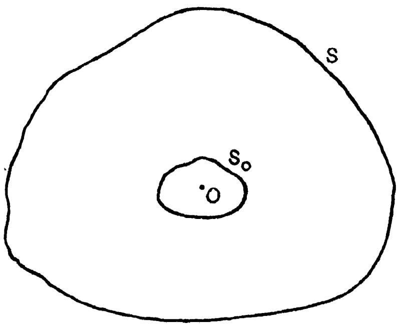


Fig. 7

32. If at each point of the surface f

$$|\phi| < \frac{M}{r^a} \quad \text{and} \quad a < 2,$$

the integral is convergent. To prove this we first suppose the function ϕ to be positive at all points in f. Now draw two circles S_1 and S_2 round O as

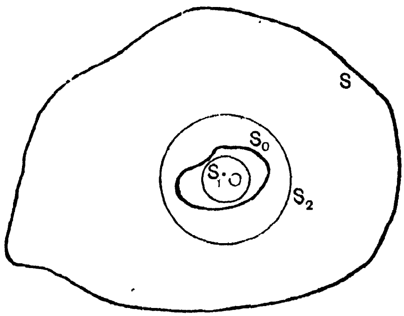


Fig. 8

centre, radii r_1 and r_2, and denote the areas between these circles by f_{12} and between them and the outer curve S by f_1 and f_2 respectively.

The integral

$$\int_{f_1} \phi \, df$$

has a sense; it is > 0 and increases when r_1 diminishes. Moreover

$$\int_{f_1} \phi \, df < \int_{f_1} \frac{M}{r^a} df$$
$$< \int_{f_2} \frac{M}{r^a} df + \int_{f_{12}} \frac{M}{r^a} df.$$

If r_2 is constant the first integral here remains fixed while the second is equal to

$$\frac{2\pi M}{2-a}(r_2^{2-a} - r_1^{2-a}),$$

and also tends to a limit if $a < 2$.

Thus the integral

$$\int_{f_1} \phi \, df$$

is always increasing as r_1 diminishes but always remains less than a certain fixed quantity so that it tends to a finite limit.

This result is not changed if we replace the circle S_1 by any curve S_0 surrounding O and gradually decreasing to the point O. We can see this easily by tracing round O as centre two circles S_1' and S_2' comprising the curve S_0 between them and diminishing with it to the point O.

33. Suppose now that the function f has any sign whatever; the preceding conclusions still apply, for if we put

$$\phi = \phi_1 - \phi_2$$

and choose ϕ_1 and ϕ_2 so that

$$\phi = \phi_1 \quad \text{and} \quad \phi_2 = 0$$

for all points where $\phi > 0$ and

$$\phi = \phi_2 \quad \text{and} \quad \phi_1 = 0$$

where $\phi < 0$, and then apply the theorem to the functions ϕ_1 and ϕ_2, we find that the integrals

$$I_1 = \int_{f_0} \phi_1 df, \quad I_2 = \int_{f_0} \phi_2 df$$

are both convergent: their difference is also convergent and the proposition is proved.

We can go further: in fact the integral

$$\int_{f_0} |\phi| \, df$$

is also convergent for it is equal to $I_1 + I_2$. For this reason the integral I is said to be absolutely convergent. In this case the limits I are independent

of the sequence of forms which the curve f_0 takes when it vanishes into the point O.

It may happen that by particular choice of the curves S_0 it may be possible to prove that the integral

$$I_0 = \int_{f_0} \phi \, df$$

is convergent but that the integral

$$\int_{f_0} |\phi| \, df$$

is divergent. In this case the integral I_0 is said to be semi-convergent and it can be proved that its value depends fundamentally on the sequence of curves S_0 chosen to define the limit; and in fact if it is possible to divide the region near the point O into a finite number of constituent parts in each of which the function ϕ has the same sign, it can be proved that a choice of sequence for the curves S_0 can always be made so as to make the integral I_0 equal to any assignable quantity positive or negative.

34. Let us now define the convergence of triple integrals: this can be done just as for the case of double integrals. The integral

$$\int_v \psi(x, y, z) \, dv$$

extended over any finite volume has no meaning if ψ becomes infinite at the point $O(x_1, y_1, z_1)$ in the field of integration: to give it a meaning consider the integral

$$I_0 = \int_{v_0} \psi \, dv$$

extended over the same volume but excluding a small volume around the point O. If this integral tends to a limit I when this small excluded volume is diminished indefinitely in all directions to the point O, the integral is said to be convergent and its value is I.

The test of convergence corresponding to that given above for double integrals is that

$$|\psi(x, y, z)| < \frac{M}{r^a} \quad \text{and} \quad a < 3$$

at all points of the surface: r is of course the distance from the point O to the point (x, y, z). In this case the integral is absolutely convergent and the limit is independent of the sequence of forms assumed by the excluded volume as it diminishes to the point O.

We can have analogously semi-convergent volume integrals which are such that

$$\int |\psi| \, dv$$

is divergent but such that a definite limit can be obtained for the integral

$$\int_{v_0} \psi dv$$

by a suitable choice of the diminishing sequence of small excluded volumes. This limit will depend fundamentally on the particular sequence of volumes taken.

The integral

$$\int \frac{\partial^2}{\partial x_1^2}\left(\frac{1}{r}\right) dv$$

provides the simplest example of a semi-convergent integral, and it is a fundamentally important one in the physical theory.

35. The question of the integration and differentiation of functions represented by improper integrals of the type under consideration is not one that we can here enter into in any detail but we may state the following results which will be fairly obvious if the corresponding theorems concerning functions represented by uniformly convergent series are borne in mind.

It is permissible under all circumstances to integrate an improper integral with respect to a parameter under the sign of integration, provided that the range of values taken for the parameter is such that the given integral is absolutely convergent for all values of the parameter within it.

It is permissible to differentiate an absolutely convergent improper integral under the sign of integration provided that the new integral obtained is itself absolutely convergent.

If follows therefore that in the above integrals

$$X_1 = \frac{\partial I}{\partial x_1},$$

but it does not follow, and is not necessarily true, that

$$X_2 = \frac{\partial X_1}{\partial x},$$

mainly because the integral representing X_2 has no definite meaning unless the sequence of diminishing excluded volumes is specified in its definition.

Proofs of most of these theorems will be found in Poincaré's book already mentioned. Reference may also be made to the small tract by Mr Leathem on *Volume and Surface Integrals used in Physics* (C.U. Press).

CHAPTER I

ON THE PRODUCTION AND DEFINITION OF THE ELECTROSTATIC FIELD

36. On electrification by friction*. If a piece of glass and a piece of resin be rubbed together they will be found on separation to attract one another. Also if a second piece of glass be rubbed by a second piece of resin and if the pieces be then separated and suspended in the neighbourhood of the former pieces of glass and resin it will be observed that (i) the two pieces of glass repel each other, (ii) each piece of glass attracts each piece of resin, and (iii) the two pieces of resin repel one another.

These phenomena of attraction and repulsion are called *electrical phenomena* and the bodies which exhibit them are said to be *electrified* or *charged with electricity*. Bodies may, as we shall soon see, be electrified in many other ways than by friction.

The electrical properties of two pieces of glass are similar to one another but opposite to those of the two pieces of resin. If a body electrified in any manner behaves as glass does, i.e. it repels the glass and attracts the resin, that body is said to be *positively* electrified, and if it attracts the glass and repels the resin it is said to be *negatively* electrified. The exactly opposite properties of the two kinds of electrification justify us in thus indicating them by opposite signs, but the application of the positive sign to one rather than the other kind must be considered as a matter of arbitrary convention.

No force either of attraction or repulsion can be observed between an electrified body and a body not electrified. When in any case bodies not previously electrified are observed to be acted on by an electrified body it is because they have been electrified by induction, a process which will be more fully explained in the next paragraph.

This property of electrified bodies may be used to examine roughly the degree of electrification of a body and also to discover whether it is positively or negatively electrified. In fact if a glass ball of small dimensions is suspended on a long silk thread and then electrified by rubbing with a piece of resin it will become positively electrified and thus if it is brought into the

* The experiments here described are quoted with slight modifications by Maxwell from Faraday's *Experimental Researches*. It is not pretended that they can be carried out under modern conditions suitably accurately enough to form the basis of a theory; but their description serves to illustrate in a remarkable manner the properties of electricity.

neighbourhood of any other electrified body it will be attracted or repelled by that body according as it is negatively or positively charged and with a force depending on the degree of electrification of the body.

Two similar electrified bodies placed in the same position relative to the suspended ball would produce the same deflection if their electrifications are the same.

37. Electrification by induction. Let us suspend a metallic rod by silk threads and suppose that it is originally uncharged. If then we bring up a small body charged with electricity near to one end of the rod, without however allowing it to touch the rod, it will be found that this end of the rod has become charged with electricity opposite in sign to that on the small body, whilst the other end is found to be charged with electricity of the same sign as that on the given body. On the removal of the small electrified body it will be found that it has itself lost none of its charge and that the rod has lost all signs of electrification. If the rod is arranged so that it can be divided into two parts, we can separate the two parts before removing the inducing charge, and it will then be found on examination of each part separately that the one is positively charged and the other negatively charged.

This electrification of the metal rod which depends on the presence in its neighbourhood of an electrified body and which vanishes when the body is removed, is called *electrification by induction.*

Now let a hollow metallic vessel be hung up by silk threads and let a similar thread be attached to the lid of the vessel so that it may be opened or closed without touching it. Let also pieces of glass and resin be similarly suspended and electrified by rubbing together.

Suppose the vessel to be originally unelectrified: then if the electrified piece of glass is hung up within it and the lid closed the outside of the vessel will be found to be positively electrified by induction, the interior being negatively electrified. *Now it may be shown that the electrification on the outside of the vessel is exactly the same in whatever part of the interior space the glass is suspended, and that it is also not changed by altering the shape of the piece of glass, as might, for instance, be accomplished by having it in detachable portions.* It follows that the electrification of the outside of the vessel in this experiment must measure some definite property of the electrified body which remains constant under similar circumstances. This property is called the *electric* charge of the body.

The experiment thus provides us with a method of comparing the electric charges on different bodies without altering the electrification on them. In fact two bodies will have equal charges if they produce the same electrification outside the vessel when inserted separately into it. A body will have a charge equal to twice that on a given body, if when introduced into the vessel it

produces on the outside an electrification equal to that produced by two bodies whose charges are equal to that of the given body, when inserted together. Finally two bodies will have equal and opposite charges; if when introduced simultaneously into the metal vessel they produce no electrification outside the metal vessel. Proceeding in this way we can test what multiple the charge on any given electrified body is of the charge on another body, so that if we take the latter charge as the unit charge we can express any charge in terms of this unit.

The charge of a body is therefore a physical quantity capable of measurement and two or more charges can be experimentally combined with a result of the same kind as when two quantities are added algebraically. We are therefore entitled to use language fitted to deal with electrification as a quantity as well as a quality and to speak of an electrified body as charged with a certain quantity of positive or negative electricity.

It can now be proved that the charges on the two portions of the separated rod electrified by induction as in the first experiment of this paragraph are equal and opposite by inserting them together in the metal vessel as described above and examining the electrification produced outside.

38. Electrification by conduction. Let the metal vessel of the last experiment be electrified by induction as there explained and let a second metallic body be suspended by silk threads near it, and let a metal wire similarly suspended be brought up so as to touch simultaneously the electrified vessel and the second body.

The second body will now be found to be positively electrified and the positive electrification of the vessel will have diminished. The electrical condition has been transferred from the vessel to the second body by means of the wire. The wire is on this account called a *conductor* of electricity and the second body is said to be *electrified by conduction.*

If a glass rod, a stick or resin or gutta percha, or even a silk thread had been used instead of the metal wire, no transfer of electricity would have taken place. These latter substances are therefore called *non-conductors* of electricity. Such substances are used in electrical experiments to support electrified bodies without carrying off their electricity: they are then called *insulators.* Although it is convenient for the present to draw this distinction between conductors and non-conductors, we shall find that in reality all substances resist the passage of electricity and all substances allow it to pass, though in exceedingly different degrees.

Of all substances the metals are by very much the best conductors. Next come solutions of salts and acids and lastly as very bad conductors (and therefore good insulators) come oils, waxes, silk, glass and such substances as sealing wax, resin, shellac, indiarubber, etc. Gases under ordinary conditions

are good insulators. Flames however conduct well, and, for reasons which will be explained later, all gases become good conductors when in the presence of radium or so-called radio-active substances. Distilled water is an almost perfect insulator, but any other sample of water will contain impurities which generally cause it to conduct tolerably well, and hence a wet body is generally a bad insulator.

When a body is in contact with insulators only, it is said to be *insulated*.

In the second experiment of the previous paragraph an electrified body produced electrification in the metal vessel while separated from it by air, a non-conductor medium. Such a medium, considered as transmitting the electrical effects without conduction, has been called by Faraday a *dielectric medium*, or simply a *dielectric*.

39. On charging a body with electricity. A combination of the results of the experiments of the previous paragraphs provides us with a ready means of charging any body with any desired quantity of electricity.

Let A and B be two metallic vessels of the type of those used above: let pieces of glass and resin be suspended as before and electrified by rubbing with one another. Now let the electrified piece of glass be put into the vessel A and the resin in the vessel B. Let the two vessels be then put into communication by a metal wire. All signs of electrification outside the vessels A and B will disappear.

Next let the wire be removed and let the pieces of glass and of resin be taken out of the vessels without touching them. It will then be found that A is electrified negatively and B positively. If now the glass and the vessel A be introduced into a large insulated metallic vessel C, it will be found that there is no electrification outside C. This shows that there is a charge on the vessel A exactly equal and opposite to that on the piece of glass; it is found similarly that the charge on the vessel B is exactly equal and opposite to that on the piece of resin.

We have thus a method of producing on any conducting body a charge exactly equal and opposite to that of an electrified body without altering the electrification of the latter, and we may in this way charge any number of such bodies with exactly equal quantities of electricity of either kind, which we may take for provisional units.

Now let the vessel B, charged with a quantity of positive electricity, which we shall call, for the present, unity, be introduced into the larger insulated vessel C without touching it. It will produce a positive electrification on the outside of C. Now let B be made to touch the inside of C. No change of external electrification will be observed. If B is now taken out of C without touching it, and removed to a sufficient distance, it will be found that B is completely discharged and that C has become charged with a unit of positive electricity.

We have thus a method of transferring the charge from B to C.

Let B be now charged again with another unit of positive electricity, introduced into C, already charged, made to touch the inside of C and removed. It will be found again that B is completely discharged and that the charge of C is doubled.

If this process is repeated, it will be found that however highly C is previously charged, and in whatever way B is charged, when B is first entirely enclosed in C, then made to touch C, and finally removed without touching it, the charge of B is entirely transferred to C. This experiment thus provides a method of charging a body with any number of units of electricity. We shall find in discussing the mathematical theory that the result of this experiment affords an accurate test of the truth of the theory.

40. The laws of electrical phenomena. In the experiments of § 37 it was shown that if a piece of glass, electrified by rubbing it with resin, is being hung up in an insulated metal vessel, the electrification observed outside does not depend on the position of the piece of glass. If we now introduce the piece of resin, with which the glass was rubbed into the same vessel, without touching either the glass or the vessel, it will be found that there is no electrification outside the vessel. From this we may conclude that the electric charge imparted to the resin by rubbing it with the glass is exactly equal, but opposite in sign, to that imparted to the glass by the same process. *We may conclude that in generating electricity by friction equal quantities of both positive and negative electricity are generated.*

Now suppose that two bodies, both charged with a certain quantity of electricity, are inserted together into a metallic vessel in the manner described above. Electrification will be induced on the outside of the vessel equal to that produced by a single body charged with a quantity of electricity equal to the algebraic sum of the quantities on the two bodies. Suppose now that while thus inside the vessel the two bodies are connected by a wire so that a transfer of electricity takes place between them by conduction through the wire. It will be found that the electrification on the outside of the vessel remains unaltered; we may therefore conclude that the total quantity of electricity on the two bodies remains the same. Thus

When one body electrifies another by conduction the total charge on the two remains the same, that is, the one loses as much positive or gains as much negative electricity as the other gains of positive or loses of negative electricity.

We have also seen that when a body electrifies another one by induction the amount of positive charge induced on the one part of the second body is equal to the amount of negative charge induced on the other part, so that the total charge of the second body remains constantly zero. Thus

The total charge of a body, or a system of bodies, always remains the same, except in so far as it receives electricity from or gives electricity to other bodies.

These are the fundamental laws of electrostatic processes as formulated by Faraday from the result of a long series of experiments of the type of those described above. To render the results deduced from them susceptible of mathematical specification, one further fact is required and that is the law of action between electrified bodies.

41. The law of action in electrical theory. The actual law for the action between electrified bodies was first discovered experimentally by Coulomb (1785)*, who measured the force by means of a torsion balance. This apparatus consists essentially of two light balls A, C fixed at the two ends of a rod which is suspended at its middle point B by a very fine thread of silver quartz or other material. The upper end of the thread is fastened to a moveable head D, so that the thread and rod can be made to rotate by screwing the head. If the rod is acted on only by its weight the condition for equilibrium is that there shall be no torsion in the thread. If however we fix a third small ball E in the same horizonal plane as the other two, and if the three balls are electrified, the forces between the fixed ball and the moveable ones will exert a couple on the moving rod, and the condition for equilibrium is that this couple shall balance that due to the torsion. Coulomb found that the couple exerted by the torsion of the thread was exactly proportional to the angle through which one end of the thread had been turned relative to the other, and in this way he was enabled to measure his electric forces. In Coulomb's experiments only one of the two moveable balls was electrified, the second serving merely as a counterpoise, and the fixed ball was at the same distance from the torsion thread as the two moveable balls.

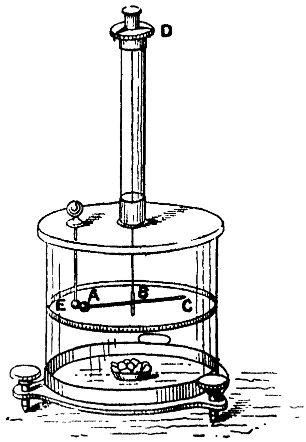


Fig. 9

A similar instrument had previously been used by Cavendish to measure the gravitational force between two small bodies.

The result obtained by Coulomb may be stated in the following terms.

If we suppose the dimensions of the two bodies on which charges† q and q' are placed to be so small compared with the distance between them that the result is not much affected by any inequality of distribution of electrification on either body and if also the bodies be supposed to be suspended in

* *Histoire et Mémoires de l'Académie Royale.* Paris, 1785–1787.

† Measured in any provisional units as suggested above.

air at a considerable distance from other bodies, then the force between them is radial and of amount

$$F = \gamma \frac{qq'}{r^2},$$

when they are at a distance r apart. The constant γ is a physical constant depending on the unit of distance chosen and also on the provisional unit adopted for measuring the charges q and q'.

We now define the *absolute unit* of electric charge in Gauss' manner so as to make γ numerically equal to unity. It is that quantity of electricity which if condensed at unit distance from a similar charge would exert unit force on it. This is the absolute electrostatic unit of charge.

The idea of the concentration of a charge at a point involved in this definition is of course merely a theoretical device introduced to simplify the mathematical expression of the law of action. If we wish to be precise we must speak of the point charge as above as the charge on a small body whose dimensions are infinitesimal compared with the distance at which we investigate its action. In this sense we regard a point charge in our mathematical theory in the same way as we do a mass-particle in ordinary dynamics, and the proved real existence of the electron justifies such a procedure.

42. This law of force is essentially an empirical one and no absolute proof is therefore possible, although very strong theoretical evidence can be advanced in its favour. We may therefore very well ask whether the law is absolutely true or is it merely true within the limits of experimental error?

The experimental proofs of the law originated by Laplace and Cavendish, which will be soon discussed, have established the fact that the force between two point charges q_1 and q_2 must be

$$\frac{q_1 q_2}{r^{2+p}},$$

where p is certainly less than 10^{-5}. Of course we may now assert that if p is so exceedingly small, the chances are that it is zero rigorously.

There is however some powerful but indirect evidence in favour of the exactness of this law: this law of action as the inverse square of the distance is precisely that which holds in gravitational theory; and in this case it must certainly be true to an enormous degree of refinement as it explains the motions of the heavenly bodies so well.

In any case the law for both electrical and gravitational theories is always exact as far as direct experiment can follow it and the conclusion seems therefore to be that it is exactly true. The evidence in favour of its exactness in electrical theory is less complete than the evidence provided by astronomical facts in gravitational theory; but this is counterbalanced by the

fact that our knowledge of the mechanism underlying electrical actions is much more precise.

The real fundamental reason of the exactness of this law is not yet evident. We can show that transmission of force by an elastic medium involves a law like this, but this does not provide us with a reason for the validity of the law in the present case.

43. We have said that the constant γ which enters into our relation

$$F = \gamma \frac{q_1 q_2}{r^2}$$

is a physical constant depending on the units adopted. We may therefore enquire as to the way in which it depends on these units. To answer this it is necessary to find the dimensions* of γ.

The expression of a physical law must be independent of the units of measurement of the quantities involved. The physical law expressing the force between two electric point charges q_1 and q_2, viz. that the force is equal to

$$F = \gamma \frac{q_1 q_2}{r^2},$$

is of this type: and we therefore conclude that the dimensions of the quantities equated in this relation must be the same.

If we use $[m]$, $[l]$, $[t]$, $[q]$ to denote the respective dimensions of the chosen arbitrary units of mass, length, time and quantity of electricity the dimensional equation for the above law is

$$\left[\frac{ml}{t^2}\right] = [\gamma]\left[\frac{q^2}{l^2}\right],$$

so that

$$[\gamma] = \left[\frac{l^3 m}{q^2 t^2}\right]$$

defines the dimensions of γ. Now the definition of dimensions implies that the unit of any quantity is increased in the ratio of its dimensions when these are increased, and the numerical value of the quantity is consequently reduced in the same ratio. Thus if we had chosen different arbitrary units for the fundamental quantities, which have respectively measures M, L, T, Q in terms of the former units the new value Γ of the constant of the physical law of action between electric charges would be given by

$$\Gamma = \frac{\gamma}{\dfrac{L^3 M}{Q^2 T^2}},$$

and this relation defines completely the way in which this constant depends on the fundamental units.

* The doctrine of dimensions was really first started by the section of Newton's *Principia* entitled 'Principle of Dynamical Similarity'; but the theory was not very definitely understood until Fourier crystallised Newton's rules in his *Theory of Heat*.

44. We have so far assumed a knowledge of some arbitrary but definite unit of electric charge which does not depend on the other fundamental units employed. We can however simplify the matter by regarding the physical law of action as defining an electric charge and then the constant γ can be chosen at will, provided that the electric charge is measured properly. The simplest plan is to take, after Gauss, $\gamma = 1$ so that the equation for the law of action is

$$F = \frac{q_1 q_2}{r^2}.$$

In this case the dimensions of an electric charge must clearly be

$$[q] = [l^{\frac{3}{2}} t^{-1} m^{\frac{1}{2}}]$$

and the unit charge is such that if condensed at unit distance in *vacuo* from a similar quantity it would exert unit force on it. This is Gauss's *absolute unit* of charge in which we shall henceforth assume all charges to be measured. We have then no further concern with the constant γ.

We have now sufficient information to enable us to formulate mathematically the general theory of electrostatic phenomena, and it was in fact on this basis that the earlier mathematical physicists of the French school (Poisson, Laplace and others) developed the theory. It will however be convenient for us to give first a short statement of the position of the more fundamental problem regarding the constitution of electricity itself before proceeding to our main problem.

45. The constitution of electricity. Previous to the researches of Faraday the generally accepted explanation of the phenomena of electrical action was based ultimately on Coulomb's experimental law for the interaction between electrical charges and involved the fundamental concept of the two electric fluids: these two fluids were assumed to be composed of very small particles of a non-gravitative subtile matter of a more refined and penetrating kind than ordinary liquids and gases: two particles of the same fluid repel each other with the law of action determined by Coulomb, whilst two particles of different fluids attract one another by the same law. It was then supposed that all bodies in their ordinary conditions contained equal amounts of both positive and negative electricity and that in rubbing two bodies together as described in § 36 a difference in the quantities of the two fluids in each of the bodies is produced, so that the one has an excess of positive fluid and is therefore positively charged, while the other has an excess of negative fluid and is negatively charged. This theory therefore effectively explains the phenomenon of electrification by friction and also the forces of attraction or repulsion between electrified bodies: it would also account for the phenomena of conduction and induction if it is assumed that either one or both of the fluids are freely moveable through good conducting media, being however more or less rigidly fixed to the elements in an insulating medium.

This two fluid theory may be said to have held the field until the time when Faraday began his researches on electricity. After educating himself by the study of the phenomena connected with his discoveries on electromagnetic induction, he applied his knowledge to electrostatic problems and finally came to the conclusion that the so-called charge of electricity on a conductor was not in reality anything of a material nature on the conductor or in the conductor, but consisted in a state of strain or polarisation, or a physical change of some kind in the particles of the dielectric medium surrounding the conductor and that it was this physical state in the dielectric which constituted electrification.

But while thus attempting to dispense with the necessity of the concept of the electric fluids, Faraday himself provided in his electrochemical researches the starting point for the next great development in electrical theory which culminated in J. J. Thomson's discovery and practical isolation and examination of the '*atom of negative electricity*,' the now famous '*electron*,' and the consequent formulation of the 'electron theory' of electrical (and optical) phenomena, which in many respects is very similar, although more explicit, than the old fluid theory. While thus crediting Thomson with the discovery of the electron it must not be forgotten that its existence had long previously been surmised by Helmholtz* and Maxwell† in a general way, and by Crookes in more explicit terms: and it had formed the basis for the fundamental theoretical researches of Larmor and Lorentz published long before Thomson's discovery‡.

The electron itself is an extremely minute electrically charged particle with an inertia mass of $9 \cdot 10^{-26}$ gms. and a charge of about $4{\cdot}69 \cdot 10^{-10}$ electrostatic units which is enormously large compared with its mass: in fact two grams of electrons placed at a distance of one metre apart would repel one another with a force equal to the weight of about 10^{21} tons. The really extraordinary thing about them is that however they are obtained, they are apparently always identical at least as regards their charges and masses, which is all that we are concerned with. It is of course beyond the powers of a physical science to say what is the ultimate cause of this exact identity of all the electrons.

No one has yet succeeded in isolating positive electrons, so that the term electron is temporarily applied only to the negatively charged particles

* Faraday Lecture, 1881.

† 'A dynamical theory of the electromagnetic field,' *Phil. Trans.* (1864).

‡ For full information the student may consult: J. J. Thomson, *Conduction of Electricity in Gases*; E. Rutherford, *Radioactivity*; Townsend, *Electricity in Gases*; which are the standard works on the physics of the electron and radioactivity disintegrations. The mathematical side is discussed with references by Larmor, *Aether and Matter*; Lorentz. *Theory of Electrons*; Richardson, *Electron Theory of Matter*. See also Campbell, *Modern Electrical Theory*; J. J. Thomson, *The Corpuscular Theory of Matter*.

discovered by Thomson, but experimental evidence is gradually tending to the view that the positively charged hydrogen atom is the ultimate element of the positive electricity which exists in all substances. The element of positive electricity if assumed to have the mass of a hydrogen atom also carries the same charge, viz. $3 \cdot 10^{-10}$ units; but its mass is 1700 times that of an electron.

46. There is now an enormous mass of experimental evidence, to which contributions are made, not only by the phenomena of electrostatics, but also by the phenomena of almost every branch of physics and chemistry tending to show that each chemical atom of matter contains as an essential part of its constitution a certain number of electrons grouped together in various more or less stable congeries; each atom also possesses in some as yet undetermined form the necessary positive electric charge to make it electrically neutral on the whole. In every solid body there is a continual process of atomic dissociation going on, the electronic configuration inside the atom being sufficiently unstable in many cases to be capable of breaking up on small provocation, with the consequent liberation of one or more electrons and occasionally of positive elements as well: the result of this is that mixed up with the atoms of chemical matter composing a body we have a greater or less percentage of negative electrons, and a few positive elements freely moveable in the interstices between the atoms. It is in fact to these free electrons and positive charges that the phenomena of electric conduction is due. An electrically charged body is one in which there is an excess or deficit of (negative) electrons. The action between the charges of the electrons and the charges in the atoms is precisely that specified by Coulomb's law provided the charges are at rest and at distances from one another large compared with ordinary molecular dimensions. The distinction between insulators and conductors as regards the phenomena of induction and conduction depends essentially on the fact that in the conductors there is a large number of the free dissociated electrons which can be pulled about from one part of the medium to another under the action of forces from other electrified bodies; whereas in insulatoıs there is such an extremely small number of these free electrons, that the phenomena depending on them can under most circumstances be neglected.

47. We shall in our future investigations discuss many facts which have led up to this conception of the essential electronic constituent of matter; but we may here just mention one important point in its favour to which we shall not have any need to refer to in our future work. In 1896 Becquerel discovered the so-called radioactive substances, which are continually and spontaneously emitting a complicated type of radiation, of which two of the main constituents have been proved to be composed, the one of rapidly moving electrons (β particles) and the other of more slowly moving positive

particles (α particles), the properties of which however suggest that they are attached to a molecule of helium. These phenomena were found by Becquerel to be associated with the element uranium, but Mme Curie working with her husband soon succeeded by a wonderful process of fractional distillation in isolating the chloride of a much more powerfully active element, then unknown to chemical science, and which they called *Radium*. Many other similar substances have subsequently been discovered, but radium is still the most active of them all.

The velocity with which the electrons are thrown off from radium is really astounding, in some cases it amounts to as much as $3 \cdot 10^7$ metres per second, i.e. one-tenth of the velocity of light. Moreover the phenomenon is still more remarkable on account of the great development of heat that accompanies the process of disintegration. This suggests, of course, that not only are the atoms emitting the electrons with this enormous velocity, but that what remains is also undergoing violent transformation; and it is now definitely established that this is actually the case. In fact, a detailed investigation of the phenomena associated with radioactivity has led to the result that a substance which is continually radiating electrons or positive particles or both spontaneously changes into another substance differing chemically from the former substance from which it arises; moreover the atomic weight of the new substance bears a definite relation to that of the old and the number of elements of charge lost in the transformation. This discovery probably provides the most direct evidence we have of a relation between the electrical constitution of an atom and its chemical composition, a relation which necessarily implies the essential existence inside any atom of matter of certain positive and negative charges.

48. Although the evidence thus deduced appears to be conclusive only as regards the radioactive substances there are a large number of facts which point to the conclusion that it is universally valid for all substances known to us. Of these facts the most direct and conclusive have recently been obtained by Profs. J. J. Thomson* and Rutherford† and their collaborators in an extensive examination of the electronic constitution of matter. The method employed consists in firing a stream of rapidly moving α- or β-particles into a piece of matter and examining the deflection and loss of energy of the individual particles caused by their collision with the atoms of matter. From the nature of the scattering of the heavier α-particles it is concluded that the positive charge in a simple atom, which is the active nucleus in producing the effect observed, is equal to the charge on an electron multiplied by the atomic number‡ of the substance; but that it is concentrated in an extremely

* *Camb. Phil. Soc. Proc.* xv. p. 465 (1910)

† *Phil. Mag.* xxi. p. 669 (1911).

‡ The ordinal number of the substance in the periodic table. In most cases this number is equal to half the atomic weight.

small volume, much smaller than the electron is presumed to occupy. The number of electrons in the neutral atom will therefore be equal to the atomic number of the substance.

The results obtained from the scattering of the smaller β-particles are not nearly so definite and are also not consistent with those just explained, but the discrepancy may be due to other causes*.

With the question as to whether there is anything else beyond the electric charges in the atom we are not at present concerned: the fact that it is impossible to obtain the positive particle unassociated with an atom of hydrogen or helium at the least, which are known for other reasons to be particularly complex systems, rather suggests that there is something else in an atom than mere electricity, but it is impossible to say what it is, chiefly because we are thus prevented from determining the true nature of the positive charge.

In the majority of our future discussions we shall have no special necessity to use this definite conception of *electricity* which now underlies the modern electrical theory, and we shall occasionally offer tentative illustrative explanations which are based on a less explicit conception. We shall not however refrain from resorting to the conception of an electron and, in fact, it will often be conducive to clearness if we elaborate various details in the usual expositions of the electron theory of the phenomena which arise.

49. The definition of the electric field of a system of point charges†. If an electrified body is brought into the space surrounding any system of charges it will in general produce a sensible disturbance in the electrification of the system by induction. But if the body is very small and its charge also very small the electrification of the other bodies will not sensibly be disturbed. The force acting on the body as a result of the action from the charges on the other bodies will then be proportional to its charge and will be reversed if the sign of the charge is reversed. That is if $\delta\mathbf{F}$ be the force and δq the charge, then when δq is an infinitesimal $\delta\mathbf{F}$ is proportional to δq or

$$\delta\mathbf{F} = \mathbf{E}\delta q,$$

where $\mathbf{E}$ is a vector function of the position of δq only, which is determinate when the system of charges is given. We may thus regard $\mathbf{E}$ as a property of the point.

The space in and around the given system of charges is called the *electric field* of those charges and the vector $\mathbf{E}$, which represents the force 'per unit charge' at the point in the field is called the *intensity of the electric force* at that point in the field.

In the neighbourhood of a single point charge q and at a distance r from it the electric force intensity is along the direction of r and of amount q/r^2.

* Cf. N. Bohr, *Phil. Mag.* xxx. p. 581 (1915).

† J. Lagrange, *Par. sav.* (*étr.*) 7, 1773 (Oeuvres, 6, p. 349).

If we refer to ordinary rectangular coordinates with the point charge q_1 at the point (x_1, y_1, z_1) then the components of the force intensity along the coordinate axes at the point (x, y, z) are

$$q_1\left(\frac{x-x_1}{r_1^3}, \quad \frac{y-y_1}{r_1^3}, \quad \frac{z-z_1}{r_1^3}\right),$$

where $$r_1^2 \equiv (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2.$$

These are simply

$$-\left(\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}\right)\frac{q_1}{r_1}.$$

In a similar manner it can be seen that if we have any system of point charges $q_1, q_2, \dots q_n$ at the points $(x_1, y_1, z_1;\ x_2, y_2, z_2; \dots;\ x_n, y_n, z_n)$ respectively, then the components of the total electric force at the point (x, y, z) of the field are

$$\begin{aligned}(\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z) &= \sum_{s=1}^{n}\left[\frac{q_s(x-x_s)}{r_s^3}, \quad \frac{q_s(y-y_s)}{r_s^3}, \quad \frac{q_s(z-z_s)}{r_s^3}\right] \\ &= -\left(\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}\right)\left(\sum_{s=1}^{n}\frac{q_s}{r_s}\right),\end{aligned}$$

where $$r_s^2 = (x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2.$$

The function $$\phi = \sum_{s=1}^{n}\frac{q_s}{r_s},$$

from which the components of the force intensity at any point of the field are obtained by simple differentiation along the axes, is called the *potential** of the electric field at the point (x, y, z). It has an important physical significance which we shall discuss later: for the present it is merely defined so that

$$(\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z) = -\left(\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}\right)\phi,$$

and thus

$$\begin{aligned}\mathbf{E}_x\frac{dx}{ds} + \mathbf{E}_y\frac{dy}{ds} + \mathbf{E}_z\frac{dz}{ds} &= -\frac{\partial\phi}{\partial x}\frac{dx}{ds} - \frac{\partial\phi}{\partial y}\frac{dy}{ds} - \frac{\partial\phi}{\partial z}\frac{dz}{ds} \\ &= -\frac{d\phi}{ds}.\end{aligned}$$

The component of the force intensity in any direction at a point is the space rate of fall or the negative gradient of the potential at that point and in that direction.

√**50.** If we choose rectangular axes with the origin O conveniently near the system of charges and if we write

$$r_0^2 \equiv x^2 + y^2 + z^2, \qquad r_{s0}^2 = x_s^2 + y_s^2 + z_s^2,$$

* This function was used first in the theory of attractions by Laplace. The name potential was given to it by Green and independently by Gauss, 'Allgemeine Lehrsätze über...anziehungs und Abstossungskräfte,' § 3. (*Collected Works*, 5, p. 200.)

and denote by θ_s the angle between the radii r_0 and r_{s0} from the origin then

$$r_s^2 = r_0^2 + r_{s0}^2 - 2r_0 r_{s0} \cos \theta_s,$$

so that
$$\phi = \Sigma \frac{q_s}{\sqrt{r_0^2 + r_{s0}^2 - 2r_0 r_{s0} \cos \theta_s}},$$

and so also
$$r_0 \phi = \Sigma \frac{q_s}{\sqrt{1 + \left(\frac{r_{s0}}{r_0}\right)^2 - 2\frac{r_{s0}}{r_0} \cos \theta_s}}.$$

The greatest interest attaches to the approximate values of ϕ at a considerable distance from the origin*. In this case $\left(\frac{r_{s0}}{r_0}\right)$ is small for each of the charges (assumed all to be at a finite distance) and we can expand each term of the above sum in an absolutely convergent series. We thus get in fact

$$r_0 \phi = \Sigma q_s \left[1 + \frac{r_{s0}}{r_0} \cos \theta_s + \frac{r_{s0}^2}{r_0^2}\left(\frac{3 \cos^2 \theta_s - 1}{2}\right) + \ldots\right].$$

Thus for points at a very large distance the first approximation to the potential is

$$\phi = \frac{\Sigma q_s}{r},$$

unless $\Sigma q_s = 0$ when it is

$$\phi = \frac{\Sigma q_s r_{s0} \cos \theta_s}{r_0^2}.$$

If we had chosen O at the centroid of the charges then $\Sigma q_s r_{s0} \cos \theta_s = 0$ and the second term in the expansion vanishes. The third term has as coefficient

$$\Sigma q_s r_{s0}^2 (3 \cos^2 \theta_s - 1),$$

and in this form it is easily identified with Gauss' term in the attraction of a gravitating system of masses. There is of course no centroid at a finite distance if $\Sigma q_s = 0$.

We have also

$$\frac{\partial \phi}{\partial r_0} = -\Sigma \frac{q_s (r_0 - r_{s0} \cos \theta_s)}{(r_0^2 + r_{s0}^2 - 2r_0 r_{s0} \cos \theta_s)^{\frac{3}{2}}},$$

so that
$$r_0^2 \frac{\partial \phi}{\partial r_0} = -\Sigma \frac{q_s \left(1 - \frac{r_{s0}}{r_0} \cos \theta_s\right)}{\left(1 + \left(\frac{r_{s0}}{r_0}\right)^2 - 2\frac{r_{s0}}{r_0} \cos \theta_s\right)^{\frac{3}{2}}},$$

and therefore also

$$\underset{r_0 \to \infty}{\mathrm{Lt}}\, r_0^2 \frac{\partial \phi}{\partial r_0} = -\Sigma q_s.$$

The action of the charges at a large distance is thus to a first approximation as if they were all collected at a point, which is their mean centre. These

* The expansion of the potential at a distant point is originally due to Poisson but was put into a convenient form by MacCullagh. *R. Irish Trans.* (1855).

remarks will help us in the elucidation of certain difficulties which crop up when we attempt to extend our definitions so as to apply to continuous distributions.

51. The definition of the electric field at points outside a continuous distribution of charge*. The discussion of the previous paragraph applies only to a system of discrete point charges or electrons and it is only in this sense that the analytical functions have any meaning. In actual practice however the distributions of charge with which we deal include such an enormous number of electrons that the expression of their field by functions of the type discussed above, even if it were possible, would be quite untractable. Fortunately however any such complete atomic analysis is useless in a physical theory, whose results can only be tested by observation and experiment on matter in bulk, for we are unable to take cognisance of the single molecule of matter, much less of the separate electrons inside it to which this analysis has regard. The development of the theory which is to be in line with experience must instead concern itself with an effective differential element of volume containing a crowd of molecules numerous enough to be expressible continuously, as regards their average relations as a volume density of matter.

Thus in any physical theory all that we are directly concerned with as regards the charge in any 'physically' small element of volume dv_1 is its total amount dq_1 and the ratio of these two magnitudes defines the density of the charge at the point, viz. ρ_1, where

$$dq_1 = \rho_1 dv_1.$$

The distinction here introduced between physically, as distinct from mathematically, small differential elements of volume is important and must be emphasised. In the speculations of pure mathematics there is no limit to the fineness of the subdivision of a region into volume elements, but in the physical theory there comes a limit when the element is so small that the number of elements of mass or charge in it is so small that the total mass included in the element depends appreciably upon its shape, so that the definition of density as the ratio of this total mass or charge to the volume ceases to have any meaning. The passage to the limit involved in a strict mathematical definition is thus not possible in a physical theory.

52. Now the element of charge $\rho_1 dv_1$ acts effectively at all points which are at a distance from it which is large compared with the linear dimensions of the element of volume dv_1 containing it, just like a charged particle so that

* The points here briefly dealt with are discussed at length by Leathem, *Volume and Surface Integrals used in Physics* (Camb. Univ. Press, 1st ed. 1905). Cf. also J. Boussinesq, *Journ. de math.* (3) 6, p. 89 (1880); H. Poincaré, *Amer. Journ. of Math.* 12, p. 284 (1890).

its field at such points is defined by the force vector $\delta\mathbf{E}$ and potential $\delta\phi$ which are determined by the relations

$$\delta\mathbf{E} = -\rho_1 dv_1 \operatorname{grad}\frac{1}{r_1}, \quad \delta\phi = \frac{\rho_1 dv_1}{r_1}.$$

In these expressions r_1 denotes the distance of the volume element dv_1 at the point (x_1, y_1, z_1) from the point (x, y, z) at which the functions are calculated. Thus for the whole system of charges grouped together in this way the field is defined by

$$\mathbf{E} = -\int \rho_1 dv_1 \operatorname{grad}\frac{1}{r_1},$$

whilst

$$\phi = \int \frac{\rho_1 dv_1}{r_1}*,$$

the integrals in each case being extended over the whole charge distribution.

The use of the definite integral expressions necessarily implies the possibility of endless subdivision in the strict mathematical sense of the electric charge and attributes to the density ρ at any point the value obtained by passing to a limit in the usual way. This inconsistency is however removed by the simple device of replacing the actual distribution of electric charge by a hypothetical perfectly continuous distribution with the same density at each point and referring the integral expressions to this distribution. Such a continuous distribution is effectively the same as the actual one, at least as regards its effect at all points which are not too near the distribution, the actual distribution of charge in any physically small volume element being then quite irrelevant.

53. So far the field-point at which the force and potential are calculated is restricted to be at a distance from the nearest charge element which is large compared with the linear dimensions of the physically small element of the charge distribution; it is however easy to see that the definitions remain valid up to a distance comparable with the dimensions of the physically small element. Let us consider the potential integral: the difference between the sum $\Sigma \frac{q}{r}$ for the elements of charge in any physically small element of volume of linear dimensions l and the integral $\int \frac{\rho_1 dv_1}{r_1}$ taken throughout the same element will be of the same order of magnitude as either quantity separately so long as r for all points of the element is of the same order of magnitude as l, but that the difference will diminish to a quantity smaller in the ratio $\frac{l}{r}$ when r becomes great compared with l. Thus for purposes of estimating

* J. Lagrange, *l.c.* p. 45.

the order of magnitude it is reasonable to represent the difference between these two expressions for the element under consideration as

$$\int \frac{\alpha l \rho}{r^2}\, dv,$$

where α is a finite number. Thus the difference between the representation by a sum or by an integral of the potential at any point due to the charge between the spheres of radii l and $a\,(> l)$ is

$$< 4\pi\alpha\rho' l \int_l^a dr$$
$$< 4\pi\alpha\rho' l\,(a - l),$$

where ρ' is the maximum value of ρ in the region : this difference is negligibly small if a is not too big (say 1 cm.) on account of the smallness (physical) of l. A similar proof also applies to the other integral.

It thus appears that the definitions of the potential and force intensity by means of integrals extended throughout the hypothetical continuous distribution of charge, which replaces the actual or discrete one, are completely effective and valid without sensible error not only for points well outside the charge, but also for points whose distance from the nearest portion of charge is small of the order of the physically small length l. This includes the case when the point is so close to the apparent outer surface of the charged body as to be sensibly just not in contact with it and also the case where the point is in a small but not imperceptibly small cavity of such a size that the piece excavated would have the properties of matter in bulk rather than the properties of a few molecules or electrons. Moreover the integrals involved in these definitions give rise to no mathematical difficulties. The subjects of integration are finite at all points of the region of integration and the integrals themselves are finite and differentiable with respect to the coordinates (x, y, z) of the external point by the method known as differentiation under the sign of integration. It thus follows that we still have on the modified definitions

$$\mathbf{E} = -\operatorname{grad}\phi,$$

so that the electric force intensity at an external point in the field is still equal to the negative gradient of the potential at that point.

54. The definition of the electric field at points inside the continuous charge distribution. The generalised specification of the electric field of a continuous charge distribution given in the previous paragraph is perfectly definite, but applies only to points external to the charge distribution. As however we shall want to extend our analysis also to points inside the continuous charge distributions we must see whether the definitions still hold for such points*.

* Cf. Gauss, 'Allg. Lehrsätze, etc.' § 6 (Footnote 7) (*Works*, 5, p. 202); O. Hölder, *Dissertation*, Tübingen, 1882, p. 6; J. Weingarten, *Acta math.* 10, p. 303 (1887); C. Neumann, *Leipz. Ber.* 42, p. 327 (1890).

Let us first assume, without preliminary justification, that the hypothetical continuous distribution with which the real distribution of charge was replaced, effectively represents this actual charge at any point of the field however near to the actual charge it may be. The force and potential at a point inside the distribution will then be defined by the same integral expressions if these have any meaning at all: we see that the integrands in both cases become infinite so that if the integrals are not convergent the expressions have no meaning whatever. We have however already indicated in the introduction that the two integrals are absolutely convergent in all cases if ρ is everywhere finite, so that there is no difficulty in the application of these expressions in this case.

We can regard the matter physically in the following manner. Imagine a small volume cut out of the charge distribution around the internal point at which it is desired to calculate the functions. The potential and force due to the remaining distribution have then definite values which may however be large. The question is now: do the values of the functions at this point depend appreciably on the size and shape of the cavity, if it is made very small? If they do, the integrals given are either divergent or semi-convergent and no meaning can be attached to the functions they represent. If on the other hand, as is the case in the present instance, they do not depend on the shape or size of the cavity, if it is only made small enough, the integrals, although of the type called improper, have distinct values and the definitions remain.

Thus if we can assume that the continuous charge distribution effectively replaces the real one at all points of the field the definitions of the force intensity and potential at an internal point are consistent and definite and moreover the removal of a physically small portion of this charge round the point does not appreciably affect the values of the functions at the point, so that in their definition it is immaterial whether this small portion of the charge is present or not. But any attempt to justify the use of this effective distribution in calculating the field at a point whose distance from the nearest element of charge is of a higher order of smallness than a physically small differential length (l, of our previous analysis) can only result in failure. For now the single electron contributes to the potential, for example, a term q/r which in spite of the smallness of q may become very great as r diminishes, so that the presence of a few such electrons might easily become so important as to make the potential quite different from the value obtained from the continuous distribution and expressed by

$$\int \frac{\rho_1 dv_1}{r_1},$$

to which, as we have just mentioned, the part of the distribution near the point contributes only a negligible amount. But there is from the physical

point of view no real motive for pursuing the enquiry further as we have in fact obtained a formulation of our field which is completely effective in a physical theory, and for the following reasons.

55. The real charge distribution may, as regards its action at any point inside the medium, be divided into two distinct portions by a physically small closed surface drawn round the point. The first part of the charge, viz. that outside the elementary surface, may be replaced by the continuous distribution as above which is, as regards its action at the point under consideration, effectively equivalent to it: to this we may also add, without appreciable modification, the continuation of this distribution throughout the interior of the small volume round the point. The second is the purely local distribution of charge elements inside the surface drawn. The contributions to the force and potential in the field at the internal point due to the former part of the charge are perfectly definite and are in fact expressed by the convergent integrals given above; but the local contribution from the elements of charge near the point is entirely unknown and may be continually changing. The only possible expressions for these functions are therefore the ones that omit altogether the contribution of these neighbouring elements.

If it were not possible thus to separate the physical functions into a molar and a molecular part a dependence would be involved between mechanical change and molecular structure, so that mechanical causes would alter the constitution of the medium and might even undermine its stability; whereas it is a postulate in ordinary mechanical theory that the physical properties of the medium are not affected by small forces*.

Thus for the purposes of a physical theory the force and potential at points inside the medium are properly defined as the corresponding quantities belonging to the field of the hypothetical distribution of charge which is thus concluded to be a completely effective representation of the real distribution. We have therefore both at external and internal points

$$\mathbf{E} = -\int \rho_1 dv_1 \,\text{grad}\, \frac{1}{r_1},$$

and

$$\phi = \int \frac{\rho_1 dv_1}{r_1}.$$

Moreover since the integrals in these expressions are both absolutely con-

* Cf. Larmor, *Aether and Matter* (particularly the footnote on p. 265), also *Phil. Trans.* 190 (1897). "The principle of D'Alembert, which is the basis of the dynamics of finite materi bodies necessarily involves this order of ideas. That part of the aggregate forcive on the moleculin the element of volume which is spent in accelerating the motion of that element *as a wh* is written off; and the regular part of the remainder must mechanically equilibrate. But wholly irregular parts of the molecular motions and forces are left to take care of themsel which they are known to do for the simple reason that the constitution of the material bod observed to remain permanent."

vergent it is legitimate in all cases to derive the former from the latter by the process of differentiation under the sign of integration so that we have

$$\mathbf{E} = -\operatorname{grad}\phi.$$

Thus the components of the force intensity at *any* point of the field in any direction is the space rate of fall of the potential in that direction.

56. On surface distributions and double-sheets. We must now pass to the consideration of certain important types of discontinuity in the volume charge distribution with which we shall have to deal in our future work. Such cases actually occur in nature and it seems necessary to consider what modifications are needed in the above definitions in order that they may apply to them.

The first example leads us to the idea of a surface density. If the volume density becomes very large in the neighbourhood of a surface f in the field we may separate the comparatively infinite values from the rest by drawing two surfaces parallel and very close to f one on each side. The layer between these surfaces is then of very small thickness Δn but the volume density ρ is so large that

$$\sigma = \int_0^{\Delta n} \rho\, dn$$

is finite when integrated across the common normal at any point of the surface. We then regard this part of the charge distribution as a surface distribution of density σ (i.e. amount per unit area) on the surface f. It is of course merely a big volume density concentrated in a shell of small thickness, but as we do not as a rule wish to be bothered about the constitution of the layer we treat it in this way.

The potential function associated with this part of the charge distribution is

$$\phi = \int_f df \int_0^{\Delta n} \frac{\rho\, dn}{r},$$

and since the shell is very thin this is practically

$$\phi = \int_f \frac{df}{r} \int_0^n \rho\, dn = \int_f \frac{\sigma\, df}{r}\text{*}$$

extended over the surface f, r denoting the distance of the element df from the point at which the potential is calculated.

The components of force are expressed in an analogous manner.

57. The second case of infinities in the volume density appears at first sight rather an artificial one, but as a matter of fact it actually exists in nature and the analysis associated with it is of immense importance for other branches of the work. Imagine a surface f' placed parallel and infinitely near to a surface f so that the small normal distance between them is Δn. Now

* G. Green, *Essay*, etc. Art. 4.

suppose the surface f' has a charge distribution of surface density σ' and the surface f one of density $-\sigma$. The potential of this distribution would be

$$\phi = \int_{f'} \frac{\sigma' df'}{r'} - \int_f \frac{\sigma\, df}{r};$$

now make $\sigma' df' = \sigma\, df$ so that there are equal and opposite charges on opposing elements of the surfaces and also put

$$\frac{1}{r'} = \frac{1}{r} + \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\Delta n,$$

so that

$$\phi = \int_f \sigma \Delta n \frac{\partial}{\partial n}\left(\frac{1}{r}\right) df^*.$$

We are therefore no longer concerned with the surface distributions separately, but must treat them together. They form what is called a *double sheet* distribution. The quantity which mathematically specifies the sheet is the product $\sigma\Delta n$, which is called the *moment* of the sheet and is denoted by τ. We must have a very large σ to get a finite τ for $\tau = \Delta n \,.\, \sigma$: the surface densities involved are therefore large compared with the usual ones which occur separately.

The potential of this double sheet is

$$\phi = \int_f \tau \frac{\partial}{\partial n}\left(\frac{1}{r}\right) df,$$

and the components of force analogously.

These represent the only types of distribution with which we have to deal in our theories. Other types may occur in nature, but they are of little importance, if they occur at all†.

58. If there are surface densities and double sheets in the field the ordinary considerations as to convergence and continuity of the integrals expressing the force and potentials still apply provided the point under consideration does not lie on any of the surface infinities. If the point is on an ordinary distribution of surface density the theorem stated in the introduction (§ 32) shows that the potential is quite definite, the integral expressing it being convergent, but that there is a certain indefiniteness in the expression of the force, which is given by a conditionally convergent integral. The significance of these results in the physical theory is however obvious from our former discussions and need not now be further elaborated.

There are certain discontinuities introduced as we approach the surface distributions thus specified but these are best attacked by the indirect method as discussed in the next chapter.

* Helmholtz, *Ann. Phys. Chemie*, 89 (1853); *Collected Works*, I. p. 491.

† Cf. however W. Voigt, *Lehrbuch der Kristallphysik*, ch. III. (Leipzig, 1910).

59. On the nature of the potential. It now seems necessary to enquire into the physical significance of the important analytical function called the potential; why should there be such a function?

Lagrange* and von Helmholtz† have given easy mathematical proofs of the existence of the potential function in electrostatic and allied theories, associating it with the potential energy function of any ordinary dynamical system. Their analysis is however based on the assumption that the action forces involved consist merely of attractions or repulsions in the direct lines of the particles and according to some function of the distance. We may however take up the standpoint that there is no reason at all why all mutual physical actions of this kind should be built up of direct attractions, instancing, for example, the action of a magnetic pole on a current filament discovered by Oersted, where the forces are certainly not radial attractions‡.

All these difficulties may be avoided and at the same time a far wider idea of the meaning of the potential obtained by connecting it with the doctrine of the conservation of energy. Let us examine a general problem from this point of view.

Suppose we have a number of physical systems M_1, M_2, ... acting upon one another across space and suppose also that we have a small element δm of a similar system.

We shall first suppose either that δm is so small that it does not disturb the finite systems when moved about in their neighbourhood, or that these finite systems are held rigid during the motion of δm, so that they are not thereby disturbed. When the element δm receives a small displacement δs the system does work on it of amount $T_s \delta s$, where T_s is the component of the force exerted on δm in the direction of its displacement. The work done in any finite displacement from the initial position 1 to a second position 2 is

$$\sum_1^2 T_s \delta s = \int_1^2 T_s ds.$$

But this work ought to be the same for all paths, if the general path is reversible, i.e. if the work gained in any displacement is equal to the work lost in the same displacement taken the reverse way. If the paths are all reversible and the work not equal for all of them we could take the element δm down one path and bring it back along another, so that the work lost in taking it down is less than that gained in bringing it back and thus on the whole there would be a gain of work. Where could this work come from? M_1, M_2, ... are all effectively rigid and so the work must have been created from nothing! This might be so but for the fact that we could repeat the

* *Par. sav.* (*étr.*) 7 (1773). (Oeuvres, 6, p. 349.)

† *Uber die Erhaltung der Kraft* (Berlin, 1847). Cf. Planck, *Das Prinzip der Erhaltung der Energie* (Leipzig, 1913).

‡ See chapter IX, where this particular case is discussed in detail.

process as often as we please and thus get an unlimited quantity of work, and all out of nothing. This is reasonably taken to be incredible as it involves the idea of perpetual motion: the essence of the matter is the unlimited extent.

Thus the argument from perpetual motion shows that for any natural law of action across space

$$\int_1^2 T_s ds$$

is independent of the path described in going from position 1 to position 2. The only other assumption involved is that of the reversibility of each path. In general this condition is satisfied.

The above argument is restricted to the case in which the systems M_1, M_2, ... were supposed to be uninfluenced by the motion of δm. We can however easily remove this restriction and consider for example the case where M_1, M_2, ... are systems of charged conductors, when the moving of a small charge δq about alters the distribution on the conductors by influence. However even in this case if we take δq round a closed path so that at the end the distribution is everywhere the same as the original one, the work done in the complete cycle must still be zero. There cannot be any loss of work for if it is reversible we could promptly turn it into a gain by reversing it.

Thus in the most general case

$$\int_1^2 T_s ds$$

is independent of the path from position 1 to 2. That is there must be a function Φ of the position of δm such that

$$\Phi_2 - \Phi_1 = \int_1^2 T_s ds$$

is true for any two positions of the points 1 and 2. If we can find this function Φ then

$$T_s = \frac{\partial \Phi}{\partial s}.$$

The function Φ represents for a definite position of the element δm the work done on it by the systems M_1, M_2, ... in bringing the element from a standard position assumed to correspond to the value $\Phi = 0$ and this work may be utilised by external agents for any ulterior purpose that may be desired. This is interpreted in the usual way by saying that the element δm possesses a store of potential energy in virtue of its position relative to the systems M_1, M_2, ..., the amount in the typical position being $-\Phi$ more than that in the standard position for which $\Phi = 0$.

60. If we now regard the physical systems as the system or systems of charged bodies acting across space in the manner specified, or in fact in any

manner, and if δm is a small charged body carrying the quantity δq of electricity then we know that

$$T_s = -\delta m \frac{\partial \phi}{\partial s},$$

and thus

$$\Phi = -\phi \delta m + \text{const.}$$

and the existence of the potential energy function Φ implies the existence of the analytical potential function ϕ and *vice versa*.

This proof of the existence of the potential function of our analytical theory rests on a physical basis. It assumes nothing about the method of transmission of the force, but merely that the effects are reversible and that perpetual motion does not exist. If there were friction effects or if the charge δq were moved about rapidly electric currents and perhaps also electromagnetic waves would be excited which would result in heat production and the essential condition of reversibility would then be lost.

In the whole of this discussion we have neglected altogether the store of energy possessed by every system in the form of heat. Why should not work come out of this store of heat? Such a question is easily answered by an appeal to Carnot's principle. The Carnot Axiom states that if we have systems in thermal equilibrium then it is impossible for work to be done at the expense of the heat they contain. The test of thermal equilibrium is equality of temperature. Thus to make the above argument correct we must put in the criterion of equal temperatures. If two bodies were at different temperatures they could be used as a heat engine from which work could be obtained.

61. The energy in the electrostatic field. We have found that in order to establish an electric field by bringing the charges which define it into position (by friction, conduction, etc.) a certain amount of work has to be done but that when once the field is established its maintenance requires the expenditure of no work, provided of course there is no leakage to be counteracted. Thus there is a certain amount of work associated with each electric field, and this amount must be independent of the method of establishing the field in order that the energy principle may be verified: it measures the amount of energy in the electrostatic field relative to the same group of masses in their uncharged state, which would be transformed into other forms of energy if the charges were removed or cancelled.

A theory of the present type regards the electrical conditions in any field as characteristic of the electric charges in the field, the distribution of these charges being the most essential thing required for the specification of the system. In such a theory we therefore require a definition of the electric energy which makes it depend on the charges and potentials. This is readily obtained in the following way.

The work required to bring up a small charge δq to any place where the potential is ϕ is, by the generalised definition of the potential function, equal to

$$\phi \delta q.$$

This statement is valid and consistent whatever the complexity of the field.

Thus the work required to increase the density of the volume charge at any point of the field by $\delta\rho$ and the density of any surface charge by $\delta\sigma$ is

$$\delta W_1 = \int \phi \delta\rho \, dv + \int_f \phi \delta\sigma \, df,$$

the first integral being extended to all points of space where there is a volume charge ρ and the second over all surfaces f on which there is a surface charge σ*.

This is the fundamental differential equation of the subject, representing as it does, in a differential form, the characteristic equation of energy for the system. If we can by any process succeed in integrating this equation we shall be in a position to know the complete mechanical circumstances of the system.

But in bringing up these charges the potential at each point of the field is increased by $\delta\phi$ so that the potential energy of the charges already existing in the field is increased by the amount δW_2 where

$$\delta W_2 = \int \rho \delta\phi \, dv + \int_f \sigma \delta\phi \, df.$$

But if the mechanical process of establishing the charges in the field is a reversible one so that all operations involved in it can be reversed—and this is essential to the existence of a potential function—the work which is done in charging the system must all be stored up as potential energy of a purely electrical nature in the field so that

$$\delta W_1 = \delta W_2,$$

and thus each of these is equal to the half of their sum or

$$\begin{aligned} \delta W_1 = \delta W_2 &= \tfrac{1}{2}(\delta W_1 + \delta W_2) \\ &= \tfrac{1}{2}\delta \left\{ \int (\rho\delta\phi + \phi\delta\rho)\, dv + \int_f (\phi\delta\sigma + \sigma\delta\phi)\, df \right\} \\ &= \tfrac{1}{2}\delta \left\{ \int \phi\rho \, dv + \int_f \phi\sigma \, df \right\}. \end{aligned}$$

We have therefore in such a case

$$W_1 = W_2 = \tfrac{1}{2}\int \rho\phi \, dv + \tfrac{1}{2}\int \sigma\phi \, df \dagger.$$

* Double sheets are excluded as they are of relatively small importance in the present aspects of the theory. It is however quite easy to generalise the discussion to include them.

† Kelvin, *Glasgow Phil. Soc. Proc.* III. (1853); Helmholtz, *l.c.* p. 55; Clausius, *Die Potential Funktion* (Leipzig, 1859), §§ 63 and 64.

Thus if we multiply each element of charge by half the potential at its position and add up over the whole distribution we get the general value for the potential energy of the electrical system referred to the state in which $\rho = \sigma = 0$ everywhere as the zero state.

62. The result here deduced in a physical manner can also be obtained analytically as follows. We know that

$$\phi = \int \frac{\rho\, dv}{r} + \int_f \frac{\sigma\, df}{r}.$$

Thus if we use dashed letters to denote these integrals in ϕ we see that

$$\delta W_1 = \iint \frac{\rho' \delta\rho\, dv\, dv'}{r'} + \int_f \int \frac{\sigma' \delta\rho\, dv\, df'}{r'} + \iint_f \frac{\rho'\, d\sigma\, df\, dv'}{r'} + \int_f \int \frac{\sigma'\, d\sigma\, df\, df'}{r'},$$

where r' denotes the distance between the typical elements of the double integrations over the volume of the field and the surfaces of discontinuity in it.

It follows immediately from this form of the expression, by integrating with respect to the undashed elements first—a process that is fully justified in view of the absolute convergence of the integrals concerned—that

$$\delta W_1 = \int \rho' \delta\phi' dv' + \int_f \sigma' \delta\phi' df',$$

where

$$\delta\phi = \int \frac{\delta\rho\, dv}{r} + \int_f \frac{\delta\sigma\, df}{r}$$

is the increment of the potential at the typical field point consequent on the addition of the charges $\delta\rho$ and $\delta\sigma$. We thus see that

$$\delta W_1 = \delta W_2.$$

and the result deduced above now follows immediately by the same argument.

63. We have thus succeeded in establishing the existence of a mechanical potential energy function associated with our electrically charged system of a type similar to that with which we are familiar in ordinary mechanics and the usual properties of this function now follow as a matter of course. A system of charges on a rigid system of masses will adjust themselves in such a way as to make the electrical potential energy function W_1 stationary subject only to the constancy of the total charge and the conditions implied by the natural restraints of the system. In fact the mutual forces exerted between the various charge elements tend to produce motion of these elements and if the energy of such motions can be obtained at the expense of the internal store of potential energy they are almost certain to take place. Thus equilibrium is possible only in those configurations in which the potential energy has a stationary value as regards small displacements from the configuration, because it is only then that the small initial displacement from the configuration results in no appreciable change in the store of potential

energy from which the kinetic energy of any further motions that take place must be derived.

Moreover it appears that if any configuration of the system is a configuration of stable equilibrium as regards the charge distribution throughout it, the stationary value of the potential energy must be an absolute minimum value, for it is only in such a case that the system will not, if slightly disturbed, depart widely from the configuration by the action of its own internal forces. Thus it is only when the natural restraints* of the system prevent any further running down of the electrical potential energy that equilibrium among the charges is permanently possible.

64. If we apply this condition for the equilibrium of a system of charges some or all of the elements in which are capable of free movement within certain limited spaces we see that the potential function ϕ must be constant throughout any space in which the charges are freely movable. This follows at once because the condition for equilibrium is that the variation of the potential energy consequent on a slight rearrangement of the charges, which is

$$\delta W_1 = \int \phi \delta\rho \, dv + \int_f \phi \delta\sigma \, df,$$

must vanish subject to the condition that the total charge in each partial space must be constant, or that

$$\int \delta\rho \, dv + \int_f \delta\sigma \, df = 0,$$

where the volume integral is taken throughout the partial space and the surface integral over the boundary of that space and any surfaces of discontinuity inside it. This leads to the result that ϕ has a constant value throughout the partial space.

65. The importance of this last result lies in its application to the field of a number of charged conductors. The elements of a charge on a conductor are, effectively speaking, freely movable throughout the material of the conductor but not beyond its outer surface. Thus in order that the charge distribution may be one of stable equilibrium it must be such that the potential ϕ is constant throughout the volume of each conductor. We shall see later that this means that the charge on the conductor exists entirely on its outer surface; this of course also results from the fact that the greatest dispersion then exists in the group of elements constituting the charge and the least value of the potential energy is attained.

In the sequel these fundamental results here deduced as consequences of the energy principle, will be discussed from a rather different standpoint and

* The insulation of a conductor acts as a 'restraint' to prevent the charge on it getting across to another conductor.

certain additional details will be obtained respecting them. For the present it is sufficient once more to emphasise the fact that so long as we confine ourselves to static systems everything is summed up in the doctrine of energy.

66. Gauss' Reciprocal Theorem. There is an important reciprocal theorem due to Gauss* which is worth quoting at this stage: we shall first give it in terms of discrete charge elements and then indicate its extension to the more general case.

Let q be an element of charge in any distribution and ϕ the potential of that distribution at any point and in a second system let these be q' and ϕ'; we have then

$$\Sigma q\phi' = \Sigma\phi' q,$$

where in $\Sigma q\phi'$ every element of the first distribution is multiplied by the potential of the second distribution at the position of the element, and similarly in $\Sigma q'\phi$.

The usual proof of this theorem in the present case is that each sum is equal to the double sum

$$\Sigma\frac{qq'}{r},$$

wherein the summation extends to each element of charge of the one system with each element of the other system and r is the distance between them.

A more fundamental interpretation of the relation is obtained by noticing that

$$\Sigma q\phi'$$

is the work required to bring up the first charged system supposed rigidly fixed in its final relative configuration into its position relative to the second, and that therefore this must be equal to the work required to bring up the second charge system into its position relative to the first which is expressed by the second sum $\Sigma q'\phi$; these are in fact merely two different ways of establishing the combined field.

In this sense we see that it must also be true in the more general case with effectively continuous charge distributions, or in other words if the two systems of charges are specified by their charge densities ρ and ρ' at the typical point of space and if the potentials of their respective fields are ϕ and ϕ' then

$$\int\rho\phi' dv = \int\rho'\phi\, dv,$$

the integrals in each case being taken throughout the entire charge distribution.

If the second system of charges is the first increased by very small increments $\delta\rho$ then ϕ' differs from ϕ only by a differential amount $\delta\phi$ and thus we may put

$$\rho' = \rho + \delta\rho, \qquad \phi' = \phi + \delta\phi,$$

* 'Allgemeine Lehrsätze über...Anziehungs- und Abstossungskräfte,' Art. 19 (*Collected Works*, v. p. 200).

and on substituting these values in the above relation we see at once that

$$\int \rho \delta\phi \, dv = \int \phi \delta\rho \, dv,$$

a relation deduced above from the energy principle.

67. The mechanical forces on the matter in the field. The analytical theory up to the present stage has been concerned merely with the electromotive forces of the field, the question of the ponderomotive forces has so far not arisen: we have only explained how charges are separated and not how electrified bodies attract one another.

It is however at once evident that such forces must exist. The *electromotive* force acting on a charge connected with a material body and in equilibrium must be counterbalanced by an equal and opposite force resulting from the action of the material body on the same charge: and the reaction to this latter force will be an equal and opposite force exerted from the charge on the material medium with which it is rigidly connected. Thus any material body carrying a charge and in equilibrium will be acted upon by a force equal to and in the same direction as the resultant electromotive force on the system of charges contained in it. Thus if ρ denote the density of the charge distribution at the point (x, y, z) in the body where the intensity of the electric force is $\mathbf{E}$, then there will be a ponderomotive force on the body determined by

$$\mathbf{F}_1 = \int \rho \mathbf{E} \, dv,$$

the integral being extended throughout the volume of the body at no point of which is ρ infinite.

This is the general form; but it is convenient to have the specialisation of it applicable when there are surface distributions associated with the body. This is easily obtained by considering such a surface distribution as the limiting case of a volume density concentrated in a thin layer. On this view the force on the small element df of the surface is practically

$$\int \mathbf{E} \rho \, dv,$$

this integral being taken throughout the small volume of the sheet standing on df. Now $\mathbf{E}$ varies continuously throughout the sheet from a value $\mathbf{E}_1$ on one side to a value $\mathbf{E}_2$ on the other and thus the average value throughout the sheet is $\frac{1}{2}(\mathbf{E}_1 + \mathbf{E}_2)$* and with this the force on df is

$$\mathbf{F}_2 \, df = \tfrac{1}{2}(\mathbf{E}_1 + \mathbf{E}_2) \int \rho \, dv = \tfrac{1}{2}(\mathbf{E}_1 + \mathbf{E}_2) \, \sigma df,$$

* This statement and the whole proof depending on it is perhaps not as be desired. The usual argument divides the force close up to the surface into a local and a general part; the local part is due to the small portion of the surface charge near the point and the general

and thus for the whole surface charge

$$\mathbf{F}_2 = \tfrac{1}{2}\int (\mathbf{E}_1 + \mathbf{E}_2)\,\sigma df,$$

and a proper combination of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ will give the correct form of the force on any electrified body or at least the linear components of the resultant force. The angular components can be obtained in an analogous manner or they may be calculated in a general way from the results deduced below where the question of these forces is regarded from another and more general point of view.

68. The mere existence of mechanical forces on the matter in the field is involved in the idea of the energy of the charged system. When two charged bodies are moved relative to each other the total electrical energy in the field is altered and if the charges are kept constant the loss (or gain) of energy is due to some other system linked with the electrical one. It reappears in fact as a gain (or loss) in the mechanical energy of the charged bodies, which determines the mechanical forcive between them. Thus to obtain the forces we need only give the bodies small virtual displacements and include the virtual work in these displacements in the general expression of the work done on the system during a general virtual change in its configuration.

If the positions of the material bodies of the system are determined by the generalised coordinates $\theta_1, \theta_2, \ldots$ in the usual Lagrangian sense and if the internal force components corresponding to the coordinates are $\Theta_1, \Theta_2, \ldots$ respectively, then the work done by external agency (which is applying forces $-\Theta_1, -\Theta_2, \ldots$ in the various coordinates) during a displacement is

$$-\Theta_1\delta\theta_1 - \Theta_2\delta\theta_2 - \ldots.$$

This is the generalised Lagrangian definition of a force component in statics: it is defined so that the force multiplied into the small change in the coordinate is the work done, provided none of the other coordinates vary.

The work done in increasing the charge distribution of the system is

$$\int \phi\,\delta\rho\, dv + \int \phi\,\delta\sigma\, df,$$

if the bodies are fixed; if however in addition the matter receives a small virtual displacement as above we must add the work done against the forces acting in these displacements. The general form for the work done on the system during the most general virtual change in its configuration (a complete

part to the remainder. For two near points equidistant from the surface but on opposite sides the local parts are equal in magnitude but opposite in direction whilst the general parts are the same at both points. The general part of the force which alone is mechanically effective is then equal to the mean of the total forces at the two points.

definition of a configuration involving a knowledge of the charge distribution and the positions of the material bodies) is thus

$$\delta W = \int \phi \delta \rho \, dv + \int_f \phi \delta \sigma \, df - \Theta_1 \delta \theta_1 - \Theta_2 \delta \theta_2 - \dots;$$

and again the usual argument based on the assumption of reversibility and the negation of perpetual motion requires that δW should be a complete differential of some function W which ultimately measures relative to some standard configuration the potential energy which the electrified system possesses in virtue of its charge. In other words W is the electrical potential energy of the system.

We know however from the discussions of the previous section that the electrical potential energy of the system can be expressed in the form

$$W = \tfrac{1}{2} \int \rho \phi \, dv + \tfrac{1}{2} \int_f \sigma \phi \, df,$$

so that as soon as these integrals can be effected W is known and the complete mechanical relations of the system are theoretically determinate.

69. If the charge distribution on the system of masses is maintained constant during the slight displacement of the system the first part of the total expression for δW does not occur and the increase in the electrical potential is simply given by

$$\delta W_c = - \Theta_1 \delta \theta_1 - \Theta_2 \delta \theta_2 \dots,$$

where W_c denotes exactly the same quantity as W above but the suffix implies that the charge distribution throughout each body is maintained constant during a displacement of that body. In this case the work of the internal forces, viz.

$$\Theta_1 \delta \theta_1 + \Theta_2 \delta \theta_2 + \dots$$

which can be used to drive a machine outside the system is derived solely from the store of internal energy which the system of masses possesses in virtue of the charges rigidly attached to them.

We have also in this case

$$\frac{\partial W_c}{\partial \theta_s} = - \Theta_s,$$

so that the force in any one of the material coordinates under the specified conditions is determinate.

70. In some important cases the total charge and its distribution are altered in such a way as to maintain the potential distribution throughout the various masses constant. In this case the work done on the system during a small virtual displacement will involve the complete expression given above but this is not now in a convenient form as it requires a knowledge of the distribution of $\delta \rho$ and $\delta \sigma$ necessary to secure the maintenance of the potential

distribution. To obtain a more suitable form for such cases we have only to rewrite it in the form

$$\delta\left(\int \rho\phi\, dv + \int_f \sigma\phi\, df - W\right) = \int \rho\delta\phi\, dv + \int_f \sigma\delta\phi\, df + \Theta_1\delta\theta_1 + \ldots,$$

and then notice that since in the present instance

$$\tfrac{1}{2}\int \rho\phi\, dv + \tfrac{1}{2}\int_f \sigma\phi\, df = W,$$

the term on the left is still δW, so that

$$\delta W = \int \rho\delta\phi\, dv + \int_f \sigma\delta\phi\, df + \Theta_1\delta\theta_1 + \ldots.$$

71. We now see immediately that when the potential distribution throughout the various masses is maintained constant throughout the displacement the change of the internal potential energy is

$$\delta W_p = \Theta_1\delta\theta_1 + \Theta_2\delta\theta_2 + \ldots$$

where we have used W_p to denote the value of W in which the potential distribution in the various masses is maintained constant.

Again, as above, $$\Theta_1\delta\theta_1 + \Theta_2\delta\theta_2 + \ldots$$

represents the energy expended by the system in the mechanical work of shifting the masses and $$\delta W_p$$

is the increase in the internal potential energy of the system of charged masses. These quantities are not now of opposite sign so that some outside source must provide both parts. The only available source of energy is, generally speaking, that which is used to maintain the potential distribution and thus the amount of energy derived from it is double the amount of the mechanical work gained from the system; the other half of the total supply goes to increase the internal potential energy of the charge distribution.

We have also in this case

$$\frac{\partial W_p}{\partial \theta_s} = \Theta_s, \qquad s = 1, 2, \ldots\ldots$$

so that

$$\frac{\partial W_p}{\partial \theta_s} = -\frac{\partial W_o}{\partial \theta_s}.$$

Thus a variation of the configuration of the system which increases the internal energy when the charge distribution is maintained constant decreases this energy when the potential distribution is maintained constant.

CHAPTER II

THE CHARACTERISTIC PROPERTIES OF THE ELECTRIC FIELD

72. Some particular types of electric fields. Having now obtained consistent definitions of the analytical functions determining the field of any distribution of electric charges we may proceed to examine the properties of these functions which are characteristic of the fields to which they appertain. Before however entering on the general examination it seems desirable to consider the form which the definitions assume in the case of certain simple specified distributions, with the view principally to obtaining some insight into the analytical nature of the functions involved. Most distributions of charge may be regarded as more or less approximately composed of a certain number of simple distributions of standard type, so that if we know the nature of the fields associated with these simple distributions we shall be in a position to obtain an approximate estimate at least of the nature of any more complex distribution.

It is of course not necessary in each case to determine all the integrals discussed in the last chapter. When the integral for the potential is known in any case the components of force in any direction may be most simply derived as the component of the vector

$$\mathbf{E} = -\nabla\phi$$

or as the negative gradient of the potential in that direction, and it is not necessary to evaluate separately the integrals for the force components. We shall therefore merely discuss the integral for the potential function in the separate cases.

73. *The potential of a linear distribution of charge.* The charge is continuously distributed along a line of continuous curvature. The charge on the element of length ds is dq and

$$v = \frac{dq}{ds}$$

is the line density; the conception of which depends on the existence of the differential coefficient expressing it. Physically the charge is so concentrated around a line that for all practical purposes it is convenient to regard it as actually on the line. The potential function is

$$\phi = \int \frac{dq}{r} = \int \frac{v\,ds}{r}.$$

The simplest case is the straight line with constant linear density v. Choose coordinate axes as usual and suppose the line lies in the axis of x between $x = a$ and $x = -b$; the point P at which we wish to calculate the potential being at a distance r along the z-axis; then

$$\phi = \left[\int_0^a + \int_0^b\right] \frac{v\,dx}{\sqrt{r^2 + x^2}}$$

$$= v\left|\log x + \sqrt{r^2 + x^2}\right|_0^a + v\left|\log x + \sqrt{x^2 + r^2}\right|_0^b$$

$$= v\left(\log\frac{a + \sqrt{a^2 + r^2}}{r} + \log\frac{b + \sqrt{b^2 + r^2}}{r}\right).$$

The equi-potentials i.e. the surfaces over which ϕ is constant are in this case confocal spheroids. For we can write

$$\phi = v\left(\log\frac{a + \sqrt{a^2 + r^2}}{r} - \log\frac{\sqrt{b^2 + r^2} - b}{r}\right)$$

$$= v\log\frac{\sqrt{a^2 + r^2} + a}{\sqrt{b^2 + r^2} - b} = v\log\frac{\sqrt{b^2 + r^2} + b}{\sqrt{a^2 + r^2} - a},$$

so that if we use $r_1 = \sqrt{a^2 + r^2}$, $r_2 = \sqrt{b^2 + r^2}$, $c = a + b$. then

$$a + r_1 = l^{\frac{\phi}{v}}(r_2 - b),$$

$$b + r_2 = l^{\frac{\phi}{v}}(r_1 - a),$$

or

$$r_1 + r_2 = c \,.\, \frac{l^{\frac{\phi}{v}} + 1}{l^{\frac{\phi}{v}} - 1} = c\coth\frac{\phi}{2v}.$$

The interesting result for our purposes is however the behaviour of ϕ as the point P approaches the line; we make r very small and thus find that

$$\underset{r \to 0}{\mathrm{Lt}}\ \frac{\phi}{\log r} = -2v,$$

and also since generally

$$\frac{\partial\phi}{\partial r} = \frac{r}{(a + \sqrt{a^2 + r^2})\sqrt{a^2 + r^2}} + \frac{r}{(\sqrt{b^2 + r^2})(\sqrt{b^2 + r^2} + b)} - \frac{2}{r},$$

we see that

$$\underset{r \to \infty}{\mathrm{Lt}}\ r\frac{\partial\phi}{\partial r} = -2v$$

as well. Thus the potential becomes infinite as P approaches the line, but the degree of the infinity is definitely calculable.

If the line is of infinite length we have similarly

$$\phi = C - 2v\log r,$$

$$-\frac{\partial\phi}{\partial r} = \frac{2v}{r},$$

where C is a very large constant of the order $\log a$.

Results exactly similar to these can be proved to hold whatever the form of the line on which the distribution is given. As the point P approaches the curve any infinities and discontinuities in ϕ or $\frac{\partial \phi}{\partial r}$ can only arise from the small portion of the curve in the neighbourhood of the foot of the normal from P, which piece may, under the assumption of continuous curvature, be considered as practically straight and of very great length compared with the distance of P from it.

74. *The potential of a constant surface density σ over a sphere of radius a.* In this case the field is obviously symmetrical about the centre of the sphere. To calculate the potential at the point P distant r from the centre C of the sphere we resolve the sphere into small rings with their planes perpendicular to CP. The elements of charge on any one of these rings are at the same distance from P and thus the contribution to the potential due to the ring of radius $a \sin\theta$ is

$$\delta\phi = \frac{2\pi a \sin\theta \, . \, a\delta\theta\sigma}{\sqrt{a^2 + r^2 - 2ar\cos\theta}}.$$

Now write $$z^2 = a^2 + r^2 - 2ar\cos\theta,$$

so that $$z\,dz = ar\sin\theta\, d\theta,$$

and then this element of the potential is

$$\delta\phi = \frac{2\pi a\sigma}{r}\frac{z\,dr}{z} = \frac{2\pi a\sigma}{r}\,dz.$$

Thus $$\phi = \frac{2\pi a\sigma}{r}\int_{z_1}^{z_2} dz = \frac{2\pi a\sigma}{r}\Big| z \Big|_{z_1}^{z_2}.$$

We must now distinguish the cases when P is inside or outside the sphere.

(i) P inside: the limits for z are $z_1 = a - r$ and $z_2 = a + r$ and thus

$$\phi_i = 4\pi a\sigma.$$

(ii) P outside: the limits for z are $z_1 = r - a$ and $z_2 = a + r$ and thus

$$\phi_0 = \frac{4\pi a^2\sigma}{r},$$

or using $Q = 4\pi a^2\sigma$ as the total charge on the sphere

$$\phi_i = \frac{Q}{a},$$

$$\phi_0 = \frac{Q}{r}.$$

The external potential thus appears as if the single charge Q were collected at the centre.

The internal and external potentials have the same value at the surface of the sphere and are otherwise continuous functions in their respective regions.

The force intensity in the field is now derived as the negative gradient of the potential. The form of this latter function suggests that it is convenient to employ a spherical polar coordinate system of reference for the field, with the pole at the centre of the sphere. If this is done it is seen that the resultant force intensity is purely radial and of amount

$$E = \frac{Q}{r^2} = -\frac{\partial \phi}{\partial r}$$

outside the sphere and zero inside it.

The normal differential coefficients or the force intensity at the surface are

$$E_i = -\left(\frac{\partial \phi_i}{\partial r}\right)_{r=a} = 0,$$

$$E_0 = -\left(\frac{\partial \phi_0}{\partial r}\right)_{r=a} = +\frac{Q}{a^2},$$

so that at the surface there is a sudden jump in the value of the normal differential rate of variation of the potential

$$E_i - E_0 = 4\pi\sigma,$$

which is independent of the size of the sphere.

75. *The potential and force of a flat circular disc uniformly charged at a point on its axis.* We again make the calculation by dividing the disc into concentric rings: r is the radius of the elementary ring and we calculate the potential at a point distant z along the axis.

The contribution to the potential from the elementary ring is

$$\delta\phi = \frac{2\pi r\,\delta r\,\sigma}{\sqrt{z^2 + r^2}}.$$

Thus in all

$$\phi = 2\pi\sigma \int_0^a \frac{r\,dr}{\sqrt{z^2 + r^2}} = 2\pi\sigma\,(\sqrt{a^2 + z^2} - z).$$

The force is directed along the axis and is of amount

$$E = -\frac{\partial \phi}{\partial z} = 2\pi\sigma\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right)$$

quite near the disc

$$\phi = 2\pi\sigma\,(a - z) \quad \text{and} \quad E = 2\pi\sigma\left(1 - \frac{z}{a}\right).$$

Thus on the disc itself the force is $2\pi\sigma$ away from the disc and so jumps by $4\pi\sigma$ as we pass through in a definite direction. The potential is continuous from one side to the other.

For a very large plate the values are approximately

$$\phi = C - 2\pi\sigma z,$$

$$-\frac{\partial \phi}{\partial z} = 2\pi\sigma,$$

C being a large constant of the order of the radius of the plate.

The results for a point not on the axis of the disc are very complex and will be derived by another method in the next section.

76. *The uniform volume distribution of charge through a sphere.* The volume density ρ of the distribution of charge is constant throughout the sphere $r = a$ so that the conditions in the field are again obviously symmetrical round the sphere. We calculate the potential at a point P distant r from the centre C of the sphere. To do this we divide the sphere into small concentric spherical shells, which will be uniformly charged; if the radii of a shell are t and $t + dt$, then the total charge inside it is $4\pi\rho t^2 \delta t$ and this is distributed over the surface like a surface charge of density $\rho\,\delta t$. We can thus apply the results obtained for a uniform spherical shell.

We must distinguish the cases where P is inside or outside the sphere.

(i) If P is outside the sphere it is outside all the elementary shells, and the corresponding contribution to the potential from each shell is

$$d\phi_0 = \frac{4\pi\rho t^2\,dt}{r},$$

so that

$$\phi_0 = \int_0^a \frac{4\pi\rho t^2\,dt}{r} = \frac{4\pi a^3 \rho}{3} \cdot \frac{1}{r},$$

and

$$-\frac{\partial\phi_0}{\partial r} = \frac{4\pi a^3\rho}{3} \cdot \frac{1}{r^2},$$

or if we use $Q = \frac{4\pi a^3 \rho}{3}$ for the total charge on the sphere

$$\phi_0 = \frac{Q}{r}, \quad -\frac{\partial\phi_0}{\partial r} = \frac{Q}{r^2}.$$

(ii) If P is inside the sphere it will be an external point for the shells of radii $t < r$ and an internal point for shells of radii $t > r$, and we shall therefore have the potential at P consisting of two parts

$$\begin{aligned}\phi_i &= \int_0^r \frac{4\pi\rho t^2\,dt}{r} + \int_r^a 4\pi t\rho\,dt \\ &= \tfrac{4}{3}\pi\rho r^2 + 4\pi\rho\left(\frac{a^2}{2} - \frac{r^2}{2}\right) \\ &= \tfrac{4}{3}\pi\rho\left(\frac{3a^2}{2} - \frac{r^2}{2}\right)^*\end{aligned}$$

and

$$-\frac{\partial\phi_i}{\partial r} = \tfrac{4}{3}\pi\rho r.$$

Thus at the surface both force and potential are continuous.

Also notice that if we use Cartesian coordinates with the origin at C

$$r^2 = x^2 + y^2 + z^2,$$

* Laplace, *Théorie du mouvement et de la figure des planètes*, Paris, 1784, p. 89 (*Œuvr.* x. p. 349).

and thus $$\phi_i = \tfrac{4}{3}\pi\rho\left(\frac{3a^2}{2} - \frac{x^2+y^2+z^2}{2}\right),$$

so that $$\frac{\partial^2\phi_i}{\partial x^2} + \frac{\partial^2\phi_i}{\partial y^2} + \frac{\partial^2\phi_i}{\partial z^2} = -4\pi\rho,$$

a result which will be proved to be more general than the particular conditions here implied would indicate.

77. *The potential of a homogeneous charge distribution throughout the interior of the volume of the ellipsoid bounded by*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1^*.$$

We first calculate the potential for an internal point P. Consider the attraction of an elementary cone EE' with its vertex at P and of solid angle $d\omega$. The potentials of the two portions of charge at P are

$$\frac{\rho}{2}PE^2 d\omega \text{ and } \frac{\rho}{2}PE'^2 d\omega,$$

or, say, $$\frac{\rho}{2}\cdot(r_1^2,\ r_2^2)\,d\omega.$$

Thus $$\phi = \frac{\rho}{2}\int (r_1^2 + r_2^2)\,d\omega.$$

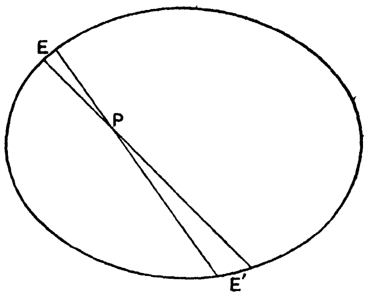


Fig. 10

Now if (x, y, z) are the coordinates of P relative to the principal axes of the ellipsoid and (l, m, n) the direction cosines of PE, then r_1 and r_2 are the roots of the quadratic equation

$$\frac{(x+rl)^2}{a^2} + \frac{(y+rm)^2}{b^2} + \frac{(z+rn)^2}{c^2} = 1.$$

Thus

$$\tfrac{1}{2}(r_1^2 + r_2^2) = \frac{\dfrac{l^2}{a^2}\left(\dfrac{2x^2}{a^2} + U\right) + \dfrac{m^2}{b^2}\left(\dfrac{2y^2}{b^2} + U\right) + \dfrac{n^2}{c^2}\left(\dfrac{2z^2}{c^2} + U\right) + V}{\left(\dfrac{l^2}{a^2} + \dfrac{m^2}{b^2} + \dfrac{n^2}{c^2}\right)^2},$$

where $$U = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2},$$

$$V = 4\left(\frac{mnyz}{b^2c^2} + \frac{nlzx}{c^2a^2} + \frac{lmxy}{a^2b^2}\right).$$

Now obviously $$\int \frac{V\,d\omega}{\left(\dfrac{l^2}{a^2} + \dfrac{m^2}{b^2} + \dfrac{n^2}{c^2}\right)^2} = 0,$$

and thus we have

$$\phi = U\Phi + \frac{x^2}{a}\frac{\partial\Phi}{\partial a} + \frac{y^2}{b}\frac{\partial\Phi}{\partial b} + \frac{z^2}{c}\frac{\partial\Phi}{\partial c},$$

* J. Somoff, *St Petersburg Bull.* XIX. (1873), p. 215. See also Gauss, *Werke*, v. p. 296.

where
$$\Phi = \int \frac{d\omega}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}}.$$

To evaluate this last integral we use ordinary spherical polar coordinates with the x-axis as polar axis and P as origin. We have then

$$l = \cos\theta, \qquad m = \sin\theta\cos\phi, \qquad n = \sin\theta\sin\phi, \qquad d\omega = \sin\theta\, d\theta\, d\phi,$$

so that

$$\Phi = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \frac{\sin\theta\, d\theta\, d\phi}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta\cos^2\phi}{b^2} + \frac{\sin^2\theta\sin^2\phi}{c^2}}$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{2\pi} \frac{d\phi}{\sin^2\phi\left(\frac{\sin^2\theta}{c^2} + \frac{\cos^2\theta}{a^2}\right) + \cos^2\phi\left(\frac{\sin^2\theta}{b^2} + \frac{\cos^2\theta}{a^2}\right)}$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta}{\sqrt{\left(\frac{\sin^2\theta}{c^2} + \frac{\cos^2\theta}{a^2}\right)\left(\frac{\sin^2\theta}{b^2} + \frac{\cos^2\theta}{a^2}\right)}}.$$

Now put $t = a^2\tan^2\theta$ and then the integral reduces to

$$\Phi = \pi abc \int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}}.$$

The potential at the point P is therefore

$$\phi = \pi\rho abc \int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}}\left(1 - \frac{x^2}{a^2+t} - \frac{y^2}{b^2+t} - \frac{z^2}{c^2+t}\right),$$

or, say,
$$\phi = \phi_0 - Ax^2 - By^2 - Cz^2,$$

where
$$A = \pi\rho abc \int_0^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}}.$$

The components of force are

$$(E_x,\ E_y,\ E_z) = -2\,(Ax,\ By,\ Cz).$$

The internal equipotential surfaces are ellipsoidal, the squares of the axes being in the inverse ratio of $A : B : C$.

78. The potential at an external point is now best obtained by a theorem due to Maclaurin which may be stated in the following form:

Any two confocal homogeneous solid ellipsoids of equal total charge produce equal attraction throughout all space external to both*.

To prove this we notice that of two uniformly charged confocal ellipsoids (a, b, c and a', b', c') a line element PQ of the inner one parallel to the a-axis

* Maclaurin proved special cases of this theorem. Cf. Grube, *Zeitschr. Math. Phys.* XIV. (1869), p. 261. Laplace proved the general theorem, *l.c.* p. 70.

will attract a point R' on the outer with a force in the direction of its length equal to $\rho\, dy\, dz\left(\frac{1}{R'Q}-\frac{1}{R'P}\right)$, if $dy\,dz$ is its cross sectional area and ρ the density of charge in it. But if P', Q', R are the points corresponding to (P, Q, R') the line charge $P'Q'$ will pull the corresponding point R in the same direction with a force

$$\rho' dy' dz' \left(\frac{1}{RQ'}-\frac{1}{RP'}\right),$$

but these are in the ratio $\rho\, dy\, dz : \rho' dy' dz'$, i.e. $\rho bc : \rho' b' c'$. We thus conclude that the attraction of the inner ellipsoid parallel to the x-axis at the point R

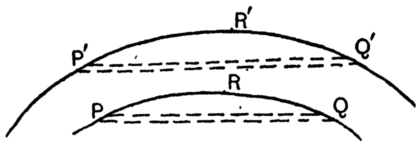


Fig. 11

outside it (x, y, z) is the same as that of the outer confocal in the same direction at the internal point $xyz \left(=\frac{ax}{a'}, \frac{by}{b'}, \frac{cz}{c'}\right)$ increased in the ratio $\rho bc : \rho' b' c'$; but this latter is

$$E_x' = +\pi\rho' a' b' c' \frac{x}{a'}\int_0^\infty \frac{dt}{(a^2+t)\sqrt{(a'^2+t)(b'^2+t)(c'^2+t)}},$$

and so the former is

$$E_x = +\pi\rho abcx\int_0^\infty \frac{dt}{(a'^2+t)\sqrt{(a'^2+t)(b'^2+t)(c'^2+t)}},$$

or if $a'^2 = a^2+\lambda$, $b'^2 = b^2+\lambda$, $c'^2 = c^2+\lambda$ this is

$$= +\pi\rho abcx\int_\lambda^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

with similar results for E_y, E_z; Maclaurin's theorem results immediately.

Thus if we take two such charged confocals and a point external to both of them, the separate attractions at that point are in the same direction and proportional to the total charges. They must therefore have their external potentials in the ratio of their charges. The potential at the external point is therefore*

$$\phi = \pi\rho abc\int_\lambda^\infty \left(1-\frac{x^2}{a^2+t}-\frac{y^2}{b^2+t}-\frac{z^2}{c^2+t}\right)\frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

where of course λ is the positive root of the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1;$$

the point (x, y, z) being on this confocal.

* Chasles, *Jour. de math.* v. (1840), p. 465.

This is the general result. One or two particular forms of it as applied to special cases are however of particular interest for our future work

79. *The homoeoidal ellipsoidal shell with a uniform volume charge throughout its thickness.* The potential of a uniform charge distribution in the shell between two ellipsoids is now easily obtained. We shall assume that the ellipsoids are similar and similarly situated, their axes being (a, b, c) and $(\sqrt{1+\delta})\,(a, b, c)$. The potential of this shell at an internal point is obviously

$$\phi_i = \pi\rho abc \int_0^\infty \left(1 - \frac{x^2}{a^2+t} - \dots\right) \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}}$$

$$- \pi\rho\,(1+\delta)^{\frac{3}{2}}\, abc \int_0^\infty \left(1 - \frac{x^2}{a^2(1+\delta)+t} - \dots\right) \frac{dt}{\sqrt{(a^2\,\overline{1+\delta}+t)(\qquad)(\qquad)}};$$

write $t\,(1+\delta)$ for t in the second integral and it reduces to the first except for one term so that

$$\phi_i = \pi\rho abc\delta \int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

and the potential is constant throughout the interior of the shell.

A similar argument shows that the potential at an external point (x, y, z) on the λ confocal is

$$\phi = \pi\rho abc \int_{\lambda'}^{\lambda} \left(1 - \frac{x^2}{a^2+t} - \dots\right) \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}}$$

$$+ \pi\rho abc\delta \int_{\lambda'}^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

where λ' is the positive root of

$$\frac{x^2}{a^2+t} + \frac{y^2}{b^2+t} + \frac{z^2}{c^2+t} = 1+\delta.$$

The most important application of these results is to the limiting case when the shell is very thin but ρ so very big that we have practically a surface charge distribution on the one ellipsoid. The surface density at any point will be ultimately proportional to the normal distance between the two surfaces. To obtain this distance notice that the tangent plane to the first ellipsoid normal to the direction (l, m, n) is at a distance p from the origin given by

$$p^2 = a^2l^2 + b^2m^2 + c^2n^2,$$

the parallel tangent plane to the other surface being at a distance $(p + dp)$ where

$$(p+dp)^2 = (a^2l^2 + b^2m^2 + c^2n^2)(1+\delta)$$
$$= p^2(1+\delta),$$

so that ultimately $dp = \dfrac{p\delta}{2}$ is the thickness of the shell.

The total charge is

$$Q = \tfrac{4}{3}\pi\rho abc\,(\overline{1+\delta}^{\frac{3}{2}} - 1)$$
$$= 4\pi\rho abc\,\frac{\delta}{2}.$$

The surface density is

$$\sigma = \rho\, dp = \frac{Qp}{4\pi abc},$$

and varies as p.

The potential function of such a distribution is :

(i) Internal

$$\phi_i = \frac{Q}{2}\int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}}.$$

(ii) Outside

$$\phi_0 = \frac{Q}{2}\int_\lambda^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}}.$$

80. We write the last equation but one in the form

$$\phi_i = \frac{Q}{C},$$

so that
$$\frac{1}{C} = \tfrac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}}.$$

The values of this function in special cases will be frequently required.

(i) Oblate spheroid : $b = c$ the larger axes and a is the smaller : if we use $\epsilon^2 = \dfrac{b^2 - a^2}{a^2}$ we easily verify that

$$\frac{1}{C} = \tfrac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)^2}} = \frac{1}{a\epsilon}\tan^{-1}\epsilon,$$

or if we use
$$\zeta^2 = \frac{b^2 - a^2}{b^2}, \qquad a\epsilon = b\zeta$$

and then
$$\frac{1}{C} = \frac{1}{b\zeta}\tan^{-1}\zeta\,*.$$

(ii) For the circular plate $a = 0$ and

$$\frac{1}{C} = \frac{2b}{\pi}.$$

(iii) For the sphere $a = b = c$ and

$$C = a.$$

(iv) For the *prolate spheroid*: c is the axis of rotation and $a = b\ (< c)$; put

$$\eta^2 = \frac{c^2 - a^2}{a^2}, \qquad \zeta^2 = \frac{c^2 - a^2}{c^2},$$

* Cf. Riemann-Weber, *Partielle Differentialgleichungen der mathematischen Physik.* I. p. 236. R. Gans, *Zeitschr. für Math. u. Phys.* XLIX. (1903), p. 298, LIII. (1906), p. 434.

then we get
$$\frac{1}{C} = \frac{1}{a\eta}\log \eta + \sqrt{1+\eta^2}$$
$$= \frac{1}{2c\zeta}\log\frac{1+\zeta}{1-\zeta}.$$

If a is small compared with c we have a very long thin ellipsoid

$$\zeta = 1 - \frac{a^2}{2c^2} \text{ approximately,}$$

$$\frac{1}{C} = \frac{1}{c}\log\frac{c}{a}.$$

It is important to notice that in the case of the circular plate the value of the surface density of the charge is

$$\underset{a\to 0}{\mathrm{Lt}} \frac{Q}{4\pi ab^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{1}{2}}} = \frac{Q}{4\pi b\sqrt{b^2-r^2}},$$

where
$$r^2 = y^2 + z^2.$$

The density becomes infinite at the edge of the plate.

81. Green's analysis of the electric field. We can now proceed to a discussion of the more important properties characteristic of the general electrostatic field in which the force and potential are defined by the integrals given in the last chapter. The most direct method of approach is that provided by Green's analysis, an account of which has already been given above. In fact we have seen that any function ϕ defined analytically in the whole of space by

$$\phi = \int\frac{\rho\, dv}{r} + \int_f \frac{\sigma\, df}{r} + \int_f \tau\frac{\partial}{\partial n}\left(\frac{1}{r}\right) df,$$

where the first integral is taken over the whole of space and the second and third over those surfaces on which σ and τ exist is subject to the usual conditions of continuity and regularity, that

$$\nabla^2\phi = -4\pi\rho$$

at every point of the field and

$$\left(\frac{\partial\phi}{\partial n}\right)_+ - \left(\frac{\partial\phi}{\partial n}\right)_- = -4\pi\sigma,$$

$$\phi_+ - \phi_- = 4\pi\tau$$

at any point of the surface or surfaces f: and conversely any regular function ϕ satisfying these three differential conditions is uniquely determined in the integral form.

But we have seen that such a function ϕ is the potential of the ideal charge distribution specified by a volume density ρ throughout the whole

of space and a surface density σ and double sheet τ on the surfaces f. We therefore conclude that the potential function of any electric field defined in the manner previously specified must be subject to the following conditions.

(i) It must be such that

$$\nabla^2\phi + 4\pi\rho = 0$$

at all points of the field where there is a finite volume density ρ: and where $\rho = 0$

$$\nabla^2\phi = 0.$$

The first of these equations contains Poisson's extension* of the second, which is Laplace's equation†. It expresses the general characteristic property of the potential function in its differential form and provides us with a test that any stated function is a potential function.

(ii) At any point on the surface distributions of charge

$$\left(\frac{\partial\phi}{\partial n}\right)_+ - \left(\frac{\partial\phi}{\partial n}\right)_- + 4\pi\sigma = 0 \ddagger,$$

and
$$\phi_+ - \phi_- - 4\pi\tau = 0 \S,$$

where σ is the density of the surface distribution and τ the moment of the double sheet at the point of the surfaces.

These two conditions are in reality merely the particular forms which the general property expressed by Poisson's equation assumes when applied to the respective limiting forms of distribution. Notice that in crossing a simple surface distribution σ the normal force only is discontinuous, the potential and tangential forces being continuous, whereas in crossing a double sheet the potential and tangential forces are discontinuous, but the normal force is continuous.

(iii) ϕ must be an otherwise continuous function regular everywhere in the field; and also if the charge distribution is a finite one it satisfies the conditions that the limiting values of

$$R\phi \text{ and } R^2\frac{\partial\phi}{\partial R} \|,$$

remain finite when R, the distance of the field point from the finite origin of coordinates, increases indefinitely.

* *Nouveau Bulletin des Sciences par la Société Philomathique de Paris*, III. (1813), p. 388. Gauss gave the first correct proof, *Allgemeine Lehrsätze*, ..., §§ 9, 10.

† *Par. Hist.* 1782 [85], pp. 135, 252 (*Œuvr.* x., pp. 302, 278). Cf. also *Mécanique Céleste*, t. II.

‡ Poisson, *Par. Mém.* 1811 [XII.], p. 30. Cauchy, *Bull. Soc. Phil.* 1815, p. 53. Green, *Essay etc.*, § 4.

§ Helmholtz, *Ges. Abh.* I. p. 489.

|| These conditions are quoted by Lejeune-Dirichlet, *Jour. f. Math.* XXXII. (1846), p. 80 (*Werke*, II. p. 40).

We might sum this last condition up by saying that the function ϕ is regular everywhere and at infinity*.

Moreover the analysis shows that there is *only one* solution of these conditions and that is *the one* we have got.

82. If therefore we can by any means obtain a solution of these conditions for given values of ρ, σ and τ, then we have a complete specification of the field of the given charges. The inverse problem in electrostatics is the determination of such solutions. It is easily seen that the solution required assumes the form

$$\phi = \int \frac{\rho dv}{r} + \Phi,$$

where Φ is an appropriate solution of the differential equation of Laplace,

$$\nabla^2 \Phi = 0$$

which must be so chosen that ϕ satisfies the specified boundary conditions at the surface infinities in the charge distribution.

Now it appears that the first part of the complete solution thus obtained is a perfectly continuous function of position with continuous first derivatives. The complementary function Φ has therefore to take full account of the discontinuities in ϕ and it is in fact the potential of the surface distributions which give rise to these discontinuities.

The problem thus resolves itself into a determination of the function Φ which satisfies the equation

$$\nabla^2 \Phi = 0,$$

at all points of the field and

$$\frac{\partial \Phi}{\partial n_+} - \frac{\partial \Phi}{\partial n_-} + 4\pi\sigma = 0,$$

$$\Phi_+ - \Phi_- - 4\pi\tau = 0,$$

at the surface distributions.

The nature of the solution required will of course depend essentially on the type of surface or surfaces on which the charge infinities exist, and it is in fact only in the cases where these surfaces are of the simplest geometrical form that a solution can be obtained at all†. It is moreover clear that any desired solution will be obtained in its simplest form in terms of those coordinates in which the equation to the surface or surfaces to which it is to be appropriate is in its simplest form. Thus in dealing with any particular type

* I am reminded that this last expression is not a usual one: it appears however a useful and concise method of expressing a definite property of such functions and saves detailed repetition of the conditions in every case where they occur.

† Cf. Heine, *Handbuch der Kugelfunktionen*, 2nd ed., Berlin, 1878. Byerly, *Fourier's Series and Spherical, Cylindrical and Ellipsoidal Harmonics*, Boston (U.S.A.), 1893. Whittaker, *Modern Analysis*, 2nd ed., Cambridge, 1915.

of surface it is desirable to begin by transferring the fundamental differential equation to coordinates suitable for that surface, and then to tabulate the different types of solution. We can then choose the particular solution appropriate to the problem in hand by bringing in the surface conditions.

83. In its general cartesian form the equation

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

is obviously satisfied by the function

$$\Phi = f(lx + my + nz;\ l,\ m,\ n),$$

where f is any arbitrary* function of its argument and (l, m, n) are any constants subject to the condition that

$$l^2 + m^2 + n^2 = 0.$$

We might for instance take

$$l = \cos\omega, \quad m = \sin\omega, \quad n = i = \sqrt{-1},$$

so that the function

$$\Phi = F(x\cos\omega + y\sin\omega + iz,\ \omega)$$

is a solution of the equation. To generalise this solution we can multiply it by any arbitrary function of ω and integrate between any two constant limits†; thus

$$\Phi = \int_0^\pi F(x\cos\omega + y\sin\omega + iz,\ \omega)\,d\omega$$

is a solution which is due to Whittaker ‡.

If the solution thus obtained is a homogeneous polynomial in (x, y, z) it is called a *solid harmonic* of order equal to its degree in these variables. The most general solid harmonic of order n is of the form

$$\Phi_n = \int_0^\pi (x\cos\omega + y\sin\omega + iz)^n\,\psi(\omega)\,d\omega,$$

and particular cases are obtained when $n = 1$ and 2 of the form

$$\Phi_1 = \lambda_1 x + \lambda_2 y + \lambda_3 z$$
$$\Phi_2 = \lambda_1(x^2 - y^2) + \lambda_2(y^2 - z^2) + \lambda_3(z^2 - x^2) + 2\mu_1 yz + 2\mu_2 zx + 2\mu_3 xy.$$

84. When any solution of the equation has been obtained other solutions can be derived from it in various ways. For instance the function obtained by differentiating the given solution any number of times with regard to the

* When we speak of an arbitrary function it is understood that it may be restricted so as to render the processes of differentiation and integration intelligible operations.

† The limits need not be constant; they can for instance be any two roots of an equation of the type

$$x\cos\theta + y\sin\theta + iz = g(\theta).$$

‡ *Math. Ann.* LVII. p. 333 (1902).

coordinates (x, y, z) is also a solution. By adding together arbitrary constant multiples of the solutions thus obtained we may derive a very general type of solution. Again if any solution involves one or more arbitrary parameters we may obtain others from it by differentiating with regard to the parameters, or by integrating with regard to them after having multiplied the expression by any arbitrary function of the parameters. For instance we know that

$$\frac{1}{r} = \frac{1}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}}$$

is a solution of the equation; we may thus infer that the function

$$\Phi = \iiint \frac{\rho(x_1, y_1, z_1)}{r} \, dx, \, dy, \, dz,$$

is also a solution, whatever the arbitrary function ρ may be.

In addition to these linear transformations there are certain other special transformations which enable us to pass from one solution of Laplace's equation to another. The principal transformation of this kind is that of inversion with respect to the arbitrary origin of coordinates. It is in fact easy to verify that if $\Phi(x, y, z)$ is a solution of Laplace's equation then the function

$$\frac{1}{r} \Phi\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$$

is also a solution if $\qquad r^2 = x^2 + y^2 + z^2.$

In particular if $\Phi_n(x, y, z)$ is a solid harmonic of order n, then

$$\Phi = \frac{1}{r^{2n+1}} \Phi_n(x, y, z)$$

is a solution of the potential equation. Thus

$$\frac{\lambda_1 x + \lambda_2 y + \lambda_3 z}{r^3}, \quad \frac{\lambda(x^2 - y^2) + \mu xy}{r^5}$$

are solutions*.

85. In dealing with spherical distributions it is usually much more convenient to use spherical polar coordinates. The transformation to this case is easily effected by the substitution

$$x = r\cos\theta, \quad y = r\sin\theta\sin\phi, \quad z = r\sin\theta\cos\phi,$$

or it may be more readily obtained by noticing that

$$\nabla^2\Phi = \operatorname{div}\nabla\Phi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right],$$

so that the equation becomes

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\cot\theta}{r}\frac{\partial\Phi}{\partial\theta} = 0.$$

* Cf. Thomson and Tait, *Treatise on Natural Philosophy*, Vol. II., where several applications of this transformation to definite problems are made.

The general type solution corresponding to that given above for cartesian coordinates is

$$\Phi = \int_0^{\pi} f(r\cos\overline{\theta - \omega}\sin\phi + ir\cos\phi,\ \omega)\, d\omega,$$

particular cases giving $\quad \frac{\lambda}{r},\ \frac{\mu\cos\theta}{r^2}$

as solutions.

Elementary solutions are obtained of the type

$$\Phi = U(r).\quad V(\theta).\quad W(\phi).$$

where
$$\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} - n(n+1)U = 0,$$

$$\frac{d^2W}{d\phi^2} + m^2W = 0,$$

and
$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}V = 0.$$

The first of these equations is satisfied by

$$U = A_n r^n + B_n r^{-n-1},$$

and the second by $\quad W = \cos m(\phi - \phi_m),$

whilst the solutions of the third are the associated Legendre functions $P_n{}^m(\cos\theta)$ and $Q_n{}^m(\cos\theta)$. In the usual notation for the Gamma and hypergeometric functions*,

$$P_n{}^m(\cos\theta) = \frac{\cot^m\frac{\theta}{2}}{\Gamma(1-m)} F\left(-n,\ n+1;\ 1-m;\ \sin^2\frac{\theta}{2}\right),$$

$$Q_n{}^m(\cos\theta) = \frac{\pi}{2}\left[\cot(n+m)\pi P_n{}^m(\cos\theta) - \frac{\tan^m\frac{\theta}{2}}{\sin(n+m)\pi\Gamma(1-m)} F\left(-n,\ n+1;\ 1-m;\ \cos^2\frac{\theta}{2}\right)\right]$$

When m is a positive integer

$$P_n{}^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}(P_n(x)),$$

$$Q_n{}^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}(Q_n(x)),$$

and when n is a positive integer

$$P_n(x) = \frac{2n!}{(n!)^2 2^n}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \ldots\right].$$

* E. W. Hobson, *Phil. Trans.* A, Vol. CLXXXVII. (1896), pp. 443–531.

The general normal solution of the equation is thus of the form

$$(A_n r^n + B_n r^{-n-1})\{A_n^m P_n^m(\cos\theta) + B_n^m Q_n^m(\cos\theta)\}\cos m(\phi - \phi_0),$$

the part depending on r^n being regular at the origin if $n > 0$ when the other part is regular at infinity. Any other solution of the equation can always be reduced to this form by the application of the theorem of Fourier and the corresponding theorems for the expansion of arbitrary functions in terms of the Legendre functions.

86. In cylindrical coordinates the equation assumes the form

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = 0.$$

The general solution is

$$\Phi = \int_0^\pi f(r\cos\theta - \omega + iz, \omega)\, d\omega,$$

and normal solutions can be obtained in the form

$$\Phi = U(r).\quad V(\theta).\quad W(z),$$

when

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} + \left(p^2 - \frac{m^2}{r^2}\right)U = 0,$$

$$\frac{d^2V}{d\theta^2} + m^2V = 0,$$

$$\frac{d^2W}{dz^2} - p^2W = 0.$$

The second and third equations are satisfied by

$$V = \cos m(\theta - \theta_0),$$

$$W = Ae^{pr} + Be^{-pr},$$

respectively whilst the solutions of the first are the Bessel functions $J_m(pr)$ and $K_m(pr)$ of the first and second kind. When m is integral

$$J_m(x) = x^m\left(-\frac{1}{x}\frac{d}{dx}\right)^m J_0(x),$$

$$K_m(x) = x^m\left(-\frac{1}{x}\frac{d}{dx}\right)^m K_0(x),$$

where for small values of x*

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \dots$$

and

$$K_0(x) = -\frac{2}{\pi}\left(\log\frac{x}{2} + \gamma\right)J_0(ix) - \frac{x^2}{2^2} + \left(1 + \frac{1}{2}\right)\frac{x^4}{2^2 . 4^2} - \dots,$$

* Whittaker, *Modern Analysis*, 2nd ed. (1915), ch. XVII.

and $\gamma = \cdot 577\ldots$ is Euler's constant. For large values we have

$$J_0(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left(x - \frac{\pi}{4}\right) + \ldots,$$

$$K_0(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\left[1 - \frac{1}{8x} + \ldots\right].$$

The elementary solutions of the potential equation in cylindrical coordinates are thus of the type

$$\Phi = e^{\pm pz}(aJ_m(pr) + bK_m(pr)) \cos m(\theta - \theta_0),$$

the part involving J_m being regular near the axis and the other part at a large distance from it.

87. Let us now consider one or two specific problems illustrating the usefulness of some of the analytical solutions of the potential equation just obtained. We firstly find the potential in the field of a simple surface distribution of charge on concentric spherical surfaces $r = a$ and $r = b$. $(a < b)$ when the surface density at any point on the inner sphere can be expanded in the form

$$\sigma = \Sigma a_n P_n(\cos\theta),$$

and that on the outer sphere in the form

$$\sigma = \Sigma b_n P_n(\cos\theta).$$

The type of functions that occur in these expressions suggest the types of solutions to try for the potential function; but it must be remembered that the complete function representing the potential must be regular at each point of the field. The most general type of function regular inside the inner sphere and consistent with the values for σ is

$$\Phi_1 = \Sigma a_n' r^n P_n(\cos\theta);$$

between the spheres we may take

$$\Phi_2 = \Sigma(a_n'' r^n + b_n'' r^{-n-1}) P_n(\cos\theta),$$

whilst outside the bigger sphere we can only have

$$\Phi_3 = \Sigma b_n''' r^{-n-1} P_n(\cos\theta).$$

As there is no double sheet distribution the potential is continuous throughout the whole of the field so that these functions must agree at the surfaces separating their regions of validity. Thus we must have

$$a_n' a^n = a_n'' a^n + b_n'' a^{-n-1},$$

$$b_n''' b^{-n-1} = a_n'' b^n + b_n'' b^{-n-1}.$$

In addition to this the difference of the normal gradients in the potential at either side of the two spherical surfaces must be equal respectively to the value of $4\pi\sigma$ there. This requires that

$$n(a_n' - a_n'') + \frac{(n+1) b_n''}{a^{2n+1}} = \frac{4\pi a_n}{a^{n-1}},$$

$$na_n'' - \frac{n+1}{b^{2n+1}}(b_n'' - b_n''') = \frac{4\pi b_n}{b^{n-1}}.$$

There are thus four equations from which the arbitrary constants a_n', a_n'', b_n'', b_n''' can be determined for each value of n. The potential function in each part of the field is thus completely determined, so that our problem is solved.

If both surface densities are uniform and equal respectively to $Q_a/4\pi a^2$ and $Q_b/4\pi b^2$ on the two spheres we find that

$$\Phi_1 = \frac{Q_a}{a} + \frac{Q_b}{b},$$

$$\Phi_2 = \frac{Q_b}{b} + \frac{Q_a}{r},$$

$$\Phi_3 = \frac{Q_a + Q_b}{r}.$$

88. We next examine the fields of certain distributions of charge on a plane ($z = 0$) which are symmetrical about a point in that plane. If this latter point is taken as origin of coordinates the field will be symmetrical about the axis of z, so that it will now be convenient to use cylindrical polar coordinates. The field will moreover be symmetrical on either side of the plane $z = 0$ so that the condition that

$$\frac{\partial \phi}{\partial n_+} - \frac{\partial \phi}{\partial n_-} = -4\pi\sigma$$

may be interpreted as implying that on the positive side of $z = 0$

$$\frac{\partial \phi}{\partial z} = -2\pi\sigma.$$

Now σ is a function of r only and by a well known theorem in Bessel's functions* it may be written in the form

$$\sigma = \int_0^\infty J_0(kr)\, kdk \int_0^\infty \sigma(x)\, J_0(kx)\, xdx.$$

We now see that the solution of the potential equation given by

$$\phi = -2\pi \int_0^\infty e^{-kr} J_0(kr)\, dk \int_0^\infty \sigma(x)\, J_0(kx)\, xdx,$$

which is regular at all points on the positive side of the plane $z = 0$, satisfies the condition that

$$\frac{\partial \phi}{\partial z} = -2\pi\sigma,$$

on this plane. It is therefore the proper potential function of the specified distribution. Two particular cases of this general result are worth noticing.

(i) If σ is constant within the circle $r = a$ and zero at all other points we have

$$\phi = -2\pi\sigma \int_0^\infty e^{-kr} J_0(kr)\, dk \int_0^a J_0(kx)\, xdx.$$

* Due substantially to C. Neumann. Cf. Heine, *Handbuch etc.* I. p. 442. Nielsen, *Handbuch der Theorie der Cylinderfunktionen* (Leipzig, 1904), p. 360. Lamb, *Hydrodynamics* (4th ed. 1916), ch. V.

Now*
$$\int_0^a J_0(kx)\,x\,dx = \frac{a}{k} J_1(ka),$$

so that
$$\phi = -2\pi a\sigma \int_0^\infty e^{-kr} J_0(kr)\, J_1(ka)\,\frac{dk}{k}.$$

This is the general value of the potential in the field surrounding the uniformly charged circular plate. It is easy to verify that it reduces to the value
$$2\pi\sigma\left(\sqrt{a^2+z^2}-z\right),$$
on the axis ($r = 0$) in agreement with the result of the last section.

(ii) If $\sigma = \dfrac{Q}{a\sqrt{a^2-r^2}}$ for values of $r < a$ and vanishes for all other values

then
$$\phi = \frac{2\pi Q}{a}\int_0^\infty e^{-kr} J_0(kr)\,dk \int_0^a \frac{J_0(kx)\,x\,dx}{\sqrt{a^2-x^2}}.$$

Now it is easily verified† by using the substitution $x = a\sin\theta$ and the series for $J_0(kx)$ that
$$\int_0^a \frac{J_0(kx)\,x\,dx}{\sqrt{x^2-a^2}} = \frac{\sin ka}{k},$$

so that
$$\phi = \frac{2\pi\phi}{a}\int_0^\infty e^{-kr} J_0(kr)\sin ka\,\frac{dk}{k}.$$

This is the formula for the potential of a circular plate over which ϕ is constant; it is obtained in a different form in the previous section as the limiting case of an ellipsoidal distribution.

89. The circumstances in the physical problems with which we usually deal are, as a rule, simpler than the previous general discussion would imply. The bodies in the field carrying the charges are usually composed of conducting materials (metals) which admit of the free passage of any elements of charge which may exist in them under the action of any external force. Thus in such a system equilibrium of the charge distribution is possible only when the force intensity is zero at any point in the interior of the charged bodies; but if the force intensity is everywhere zero, the potential, of which it is the negative gradient, must be constant so that
$$\nabla^2\phi = 0$$
there; there can therefore be no volume distribution of charge inside the conductor. There is however a distribution on the surface of each conductor and its density at any point is determined simply by the normal gradient of the potential in the external field just outside the point
$$\sigma = -\frac{1}{4\pi}\frac{\partial\phi}{\partial n}\ ‡.$$

* By series.

† Cf. Rayleigh, *Scientific Papers*, t. III. p. 98. Hobson, *Proc. L.M.S.* t. xxv. p. 71 (1893).

‡ Coulomb *Par. Hist.* 1788 (1791), p. 676.

The total charge on the surface is simply

$$\int_f \sigma df = -\frac{1}{4\pi}\int \frac{\partial \phi}{\partial n} df.$$

Moreover if, as is usually assumed to be the case, there is no double sheet distribution the potential outside must be continuous with the constant interior value at the surface of each conductor. In other words the external potential reduces to a constant value on the surface of the conductor; this condition provides a usually sufficient criterion for its determination in all practical cases.

Thus for instance the potential function

$$\Phi_0 = \frac{Q}{r}$$

reduces to the constant value $\frac{Q}{a}$ at the surface of the sphere $r = a$; it is therefore the potential in the field of an isolated conducting sphere whose potential is

$$\frac{Q}{a}.$$

The density of the charge at any point of the sphere is

$$-\frac{1}{4\pi}\left(\frac{\partial \phi}{\partial r}\right)_{r=a} = \frac{Q}{4\pi a^2},$$

and is uniformly distributed over its surface. Of course we might have guessed this to be the case, on account of the symmetry, and then we could have used the result of the integration carried out in the previous section.

90. We can similarly use any of the results obtained by direct integration which fulfil the necessary conditions implied in the physical circumstances of the problem. As a final example let us briefly consider the case of a charged conducting ellipsoid whose boundary surface is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

In this case the distribution of charge on the outer surface must be such that the potential function throughout its interior is constant and outside the ellipsoid it must satisfy the usual conditions and agree with the constant value on the surface. But if we refer to the results obtained in the first section we see at once that the charge distribution which satisfies these conditions is identical with the uniform distribution throughout a very thin homoeoidal shell at the surface of the ellipsoid in question provided only that the total charge is right. Thus if Q is the total charge on the ellipsoid the constant internal potential will be

$$\phi_i = \frac{Q}{2}\int_0^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

whilst the potential at the point (x, y, z) outside is

$$\phi_0 = \frac{Q}{2}\int_\lambda^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

λ being the positive root of the equation

$$\frac{x^2}{a^2+t} + \frac{y^2}{b^2+t} + \frac{z^2}{c^2+t} = 1.$$

The density of charge at any point on the ellipsoid is given by

$$\sigma = \frac{Qp}{4\pi abc},$$

p being the central perpendicular on the tangent plane at the point. From this we see that the density is greatest where p is largest. If the ellipsoid is very long in one direction the charge will be practically all concentrated at the ends: in the case of a very flat ellipsoid with one very short axis the charge will be concentrated round the edge. These remarks illustrate the general theorem that the charge always tends to disperse itself as much as possible so as to secure the minimum condition for the potential energy in the final field.

91. Gauss' Theorem: normal induction. So far our basis is purely analytical and as a consequence the physical significance of the results is not very clear. In order to obtain a better insight into this other side of the matter we shall proceed from a different standpoint and along more elementary lines.

The chief characteristic property of the electric field contained in Poisson's equation is expressed in an integral form by a theorem usually ascribed to Gauss* but which was probably first stated by Faraday in a physical manner.

If any closed surface f is taken in the electric field and if $\mathbf{E}_n$ denote the component of the electric force intensity at any point of the surface in the direction of the outward normal dn, then

$$\int_f \mathbf{E}_n\, df = 4\pi Q,$$

where the integration extends over the whole of the surface and Q is the total charge enclosed by it.

This theorem is a mere mathematical verification for a single point charge q: for at any point of the closed surface distant r from q

$$\mathbf{E}_n = \frac{q}{r^2}\cos(\widehat{nr}),$$

* *Allg. Lehrsätze.* The theorem was also given by Kelvin in 1842. Cf. *Papers on Electricity and Magnetism.* The present demonstration is due to Stokes.

where $(\widehat{nr})$ denotes the angle between the positive directions of n and r; but if $d\omega$ is the element of solid angle subtended by the surface element df at the point charge q then

$$df \cos(\widehat{nr}) = r^2 d\omega,$$

and thus

$$\mathbf{E}_n df = q\, d\omega,$$

so that

$$\int_f \mathbf{E}_n df = q \int d\omega$$

$$= 4\pi q \text{ if } q \text{ is inside } f$$

$$= 0 \quad \text{if } q \text{ is outside.}$$

If there are any number of charges $q_1, q_2, \dots q_n$ present in the field, then

$$\mathbf{E}_n = \mathbf{E}_{1_n} + \mathbf{E}_{2_n} + \dots$$

is the sum of the normal components of force due to the separate point charges and thus we get by simple addition

$$\int_f \mathbf{E}_n df = 4\pi \text{ (total charge inside } f).$$

Moreover although we have proved this theorem for a system of point charges it remains valid when the charges are merged into continuous volume or surface distributions as the following analysis, which exhibits the theorem as an immediate consequence of Poisson's equation, proves. If we consider the integral

$$\int_v \operatorname{div} \mathbf{E}\, dv$$

taken throughout the space v inside the surface f its value is

$$4\pi \int \rho\, dv;$$

but by Green's lemma it also consists of

$$\int_f \mathbf{E}_n df,$$

together with the surface integrals arising at the surface distributions of charge. These are simply

$$\int_{f'} (\mathbf{E}_{n_+} - \mathbf{E}_{n_-})\, df',$$

f' referring to the surfaces on which the charge infinities are distributed. But

$$\mathbf{E}_{n_+} - \mathbf{E}_{n_-} = -4\pi\sigma,$$

so that this latter integral is

$$-4\pi \int_{f'} \sigma\, df,$$

and we have

$$4\pi \left[\int_v \rho\, dv + \int_{f'} \sigma\, df'\right] = \int_f \mathbf{E}_n df,$$

which is precisely Gauss' theorem. If we use

$$\mathbf{E}_n = -\frac{\partial \phi}{\partial n},$$

then the equation may be written in the form

$$-\int_f \frac{\partial \phi}{\partial n}\, df = 4\pi Q,$$

which exhibits it as the integral form of the characteristic property of the potential function of an electrostatic field.

The integral $\int \mathbf{E}_n df$ is defined as the *total normal induction* through the surface f.

92. Many of the results of the previous section may be obtained very simply by an application of Gauss' theorem of normal induction.

(*a*) Uniformly charged infinite circular cylinder.

By symmetry the field is radial and symmetrical round the axis.

Choose as the arbitrary surface in Gauss' theorem the portion of a concentric cylinder (radius r) cut off by two parallel planes unit distance apart and parallel to the axes. If the force at a distance r is E then the total normal induction over this surface is

$$2\pi r \,.\, E,$$

the flat ends contributing nothing.

If $r < a$, the radius of the cylinder

$$2\pi r E = 0, \quad E = 0;$$

if $r > a$, then

$$2\pi r F = 4\pi Q, \qquad E = \frac{2Q}{r},$$

Q being the charge per unit length.

(*b*) Uniformly charged infinite plate.

By symmetry again we see that the force is everywhere normal to the plate.

Now apply Gauss' theorem to a cylinder of unit sectional area perpendicular to the plane and bisected by it and we easily deduce that

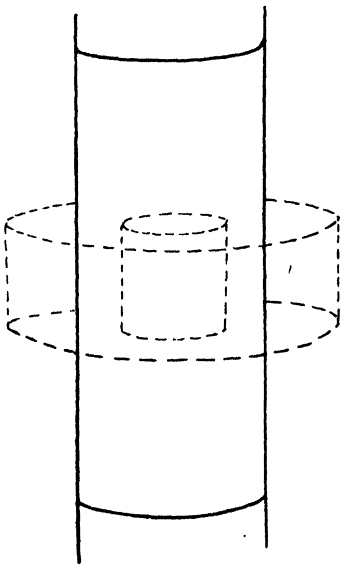


Fig. 12

$$2E = 4\pi\sigma,$$

(*c*) The uniformly charged conducting sphere.

The method is the same: apply Gauss' theorem to concentric spherical surfaces.

(*d*) We can determine in the same way the field of force for two concentric spherical conductors (radii a_1 and a_2) carrying charges Q_1 and Q_2 and prove that if $Q_1 = -Q_2$ the field is confined to the space between the conductors.

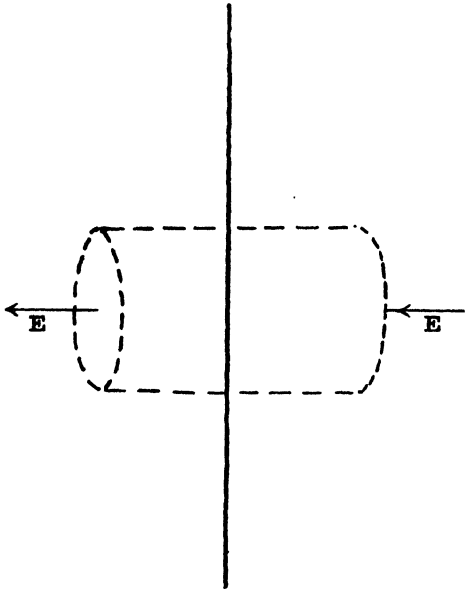


Fig. 13

The force inside the inner sphere is zero: in the space between the spheres it is radial and symmetrical and at a distance r from the centre it amounts to

$$\frac{Q_1}{r^2}.$$

Outside both spheres the force is also symmetrical and radial but is now of intensity

$$\frac{Q_1 + Q_2}{r^2},$$

at a distance r from the centre.

93. We now make use of the general theorem to analyse the properties of electrostatic fields in such a way as will lead most directly to a consideration of Faraday's speculations on the origin of all electrical actions; but before going on to this it is interesting to notice that when we consider

as in Gauss' theorem a part only of a closed surface instead of the whole of the surface and take the surface integral of normal induction for it then the contribution of each element q is

$$q\omega,$$

where ω is the solid angle the boundary of the surface subtends at q.

The solid angle of a right circular cone of angle θ is $2\pi(1-\cos\theta)$; thus the surface integral of the normal flux through a circle due to a system of point charges ranged on its axis is

$$2\pi\Sigma q\,(1-\cos\theta).$$

This result will be of use to us later on.

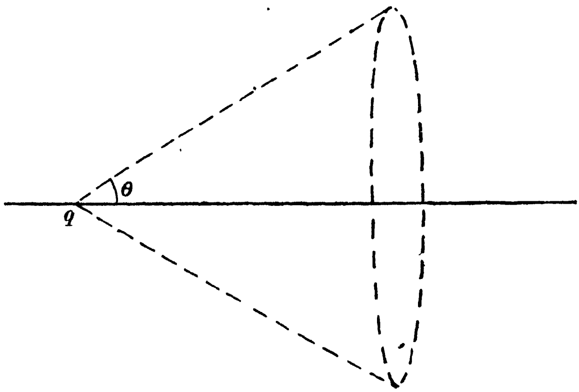


Fig. 14

94. Lines of Force and Equipotential Surfaces*. The electric field due to any static system of charges is completely specified if we know the magnitude and direction of the electric force intensity at any point of the field. The direction of the force at any point is understood to be the direction in which a small point charge δq would be displaced if put there. If we follow this direction from point to point we obtain curves which are called lines of force.

A line of force in an electric field is a curve such that along it the force is always tangential. It follows that the positive direction of the line is always that of decreasing potential. Hence a line of force cannot return into itself, but must have a beginning and an end. We shall prove presently that it can only begin on a positively charged surface and end on a negatively charged one.

Now suppose an electric field to be given and all the lines of force drawn in it. At any position P in the field where there is no electricity we place

* Faraday, *Roy. Soc. Trans.* 141 (1831), p. 2.

a small surface element df, perpendicular to the lines of force at that place. Then all the lines of force which pass through the edges of this surface element will form the sides of a tube called a *tube of force.*

The inside of a tube of force is also filled with lines of force, i.e. through each point in its interior we can draw one line of force but only one. If there were several they would cut at this point and this is impossible because at each point the line of force gives the direction of the resultant force intensity of the field and this is uniquely determinate unless the force is zero, an exceptional case which is reserved for future consideration. Thus as a general rule lines of force cannot cut one another and none of them can pass through the sides of a tube of force.

At another point P' of our tube of force let us draw another surface element df' also perpendicular to the direction of the lines of force there. Now apply Gauss' theorem to the portion of the tube PP'. In summing up the induction over the surface we can neglect the curved sides of the tube since the electric force intensity is at each point tangential and thus has no normal component. Let now $\mathbf{E}$, $\mathbf{E}'$ be the force intensities at P and P' respectively: their directions are along tangents to a mean axis in the tube and may therefore be considered normal to the end surfaces df, df' both of which are very small. A simple application of Gauss' theorem thus gives

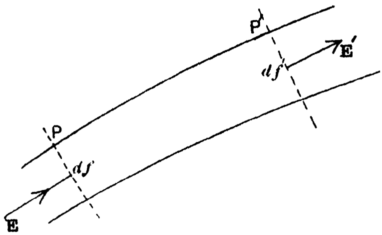


Fig. 15

$$\mathbf{E}df = \mathbf{E}'df',$$

provided there is no electricity in the part of the tube between P and P'.

The intensity of force at every point of the tube is inversely proportional to the cross section of the tube at the point. We have thus a convenient method of graphically representing the intensity of a field of electric force. We fill up the space of the field by drawing tubes of force, choosing their cross sections so that the constant value of $\mathbf{E}df$ along each is the same for all. The density of the tubes, or merely their thickness would then give a graphical measure of the strength of the field. The tubes are thickest in the positions of small intensity.

95. A still simpler method of representing the field is obtained by considering the potential function*. The force intensity is a vector with three components at each point of the field and so it is much easier if we notice that this vector is always the gradient of the scalar quantity, which

* Maclaurin, *Treatise on Fluxions* (Edinb. 1742), § 640. Clairaut, *Figure de la terre* (Par. 1743).

we have called the potential, so that instead of having three things to consider we have only one. The field of force is completely specified by this one function ϕ. We could therefore map out the field by plotting the function ϕ, the simplest method being to draw the surfaces over which ϕ is constant: such surfaces are called *equi-potential surfaces, level surfaces* or simply *equi-potentials.*

Since the potential is as a general rule a one-valued function of position we see that two equi-potentials cannot in general cut one another. Now let us choose two adjacent equi-potentials whose potential difference is $\Delta\phi$ and let Δs be a small line drawn from a point on the surface of higher potential to a point on the other surface. We know then that $\frac{\Delta\phi}{\Delta s}$ is in the limit the force intensity in the direction of Δs; but $\frac{\Delta\phi}{\Delta s}$ is a maximum when Δs is a minimum. Since we may regard the two adjacent surfaces as parallel, at least when we confine ourselves to small opposing regions on them, Δs will be a minimum when it is the normal distance Δn between the surfaces. Thus the resultant force intensity is

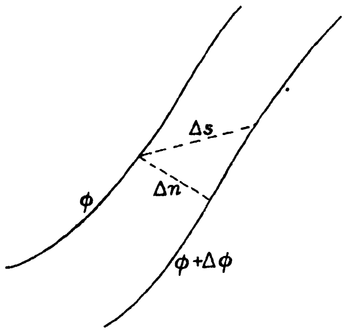


Fig. 16

$$\mathbf{E} = \frac{\Delta\phi}{\Delta n},$$

and is normal to the equi-potential surfaces.

The lines of force are therefore everywhere normal to the equi-potentials. Conversely, of course, we may conclude that if we can find a surface everywhere normal to the lines of force, it must be a level surface.

Thus if we draw all the level surfaces in the field so that the potential difference for any two succeeding ones is the same, the density of the surfaces so drawn is everywhere proportional to the force intensity in the field. We thus obtain another very real representation of the field. If we draw the tubes of force as well we notice that the cross sections of the tubes are proportional to the distances between the level surfaces. This would provide a good test as to whether the field had been correctly mapped. The tubes are thickest where the surfaces are widest apart.

96. As an illustration of the usefulness of the conceptions here introduced we may discuss in terms of them some very important properties of the fields in question*.

* Gauss, *Allg. Lehrsätze etc.* (1840). Stokes, *Camb. and Dublin Math. Journal,* IV. (1849). See also papers by Lord Kelvin in the same journal (1842–3) and a memoir by Chasles, *Connaissances des Temps* (1845).

The potential cannot be a maximum or minimum at a point of the field where there is no charge. For supposing that at any point P there is a true maximum value of ϕ, then at all other points in the immediate neighbourhood of P the value of ϕ is less than its value at P. Hence P will be surrounded by a series of closed equi-potential surfaces, each outside the one before it, and at all points of any one of these surfaces the electrical force will be directed outwards so that the total normal induction through the surface will be positive and cannot be zero: it follows then that there must be a positive charge inside the surface, and since we may take the surface as near to P as we please, there is a positive charge at P.

In the same way we may prove that if ϕ is a minimum at P there must be a negative charge at P.

This enables us also to complete the proof of the statement made above that lines of force must start from a place where there is positive electricity and end at a place where there is negative electricity, for it can only begin at a position of maximum potential and end at a position of minimum potential.

It is of course possible for a line of force to begin on a positive charge and go to infinity, the potential decreasing all the way, in which case the line of force has, strictly speaking, no second end at all. So also a line may come from infinity and end on a negative charge.

97. To obtain a still closer insight into the significance of the various properties of electric field we may examine one or two cases where it is possible to determine by simple methods the full details of the electric field in terms of the lines of force and equi-potential surfaces. The examples in each case are typical of the general problem where the field of a number of point charges on an axis is under investigation.

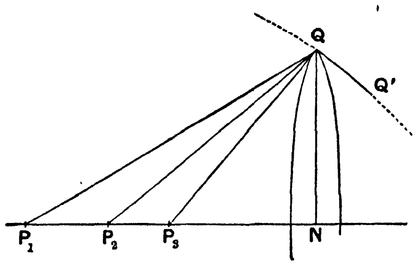


Fig. 17

The equation to the lines of force are easily determined: for the curves in each plane through the axis for which

$$\Sigma 2\pi q\,(1 - \cos\theta) = \text{const.}$$

lie on a tubular surface with the given line of charges as axis: moreover from the theorem quoted at the end of § 93 the normal induction across

any section of this tube is constant: it is therefore a tube of force and its curve of section by an axial plane gives the equations of the lines of force in that plane: they are therefore the curves

$$\Sigma q \cos \theta = \text{const.}$$

The equi-potential surfaces are of course those on which

$$\Sigma \frac{q}{r}$$

is constant.

When a line of force passes through one of the point charges, the radius vector at that point becomes a tangent and θ is then the angle that tangent makes with the positive direction of the axis. Let a line of force pass through the hth and kth particle, then we have

$$q_1 + q_2 + \dots + q_h \cos \theta_h - q_{h+1} - \dots$$
$$= q_1 + q_2 + \dots + q_h + \dots q_k \cos \theta_k - q_{k+1} - \dots .$$

Thus $$q_h \sin^2 \tfrac{1}{2}\theta_h + q_{h+1} + \dots + q_{k-1} + q_k \cos^2 \tfrac{1}{2}\theta_k = 0.$$

If all the charges have the same sign the only line of force which can pass from one particle to another is the straight line along the axis.

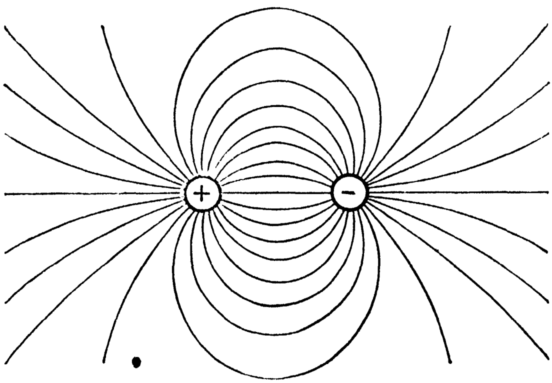


Fig. 18

Now let a line of force pass from the particle q_s to a point at an infinite distance in a direction which ultimately makes an angle a with the axis. We have then in the same way

$$q_1 + q_2 + \dots + q_{s-1} + q_s \cos \theta_s - q_{s+1} - \dots = (\sum_{s=1} q) \cos a.$$

If $\Sigma q = 0$ no line of force can in general pass to an infinite distance.

98. Fig. 18 represents the lines of force due to two equal and opposite charges. In this case all the lines of force start from the positive charge and end on the negative charge. The force in the field is strongest in the

neighbourhood of the charges and especially between them; it decreases rapidly as the point gets farther and farther from the charges.

When the two point charges are very close together they are collectively described as an *electric doublet.*

99. Figs. 19 and 20 represent respectively the lines of force and equi-potential surfaces in a plane section of the field due to two equal positive

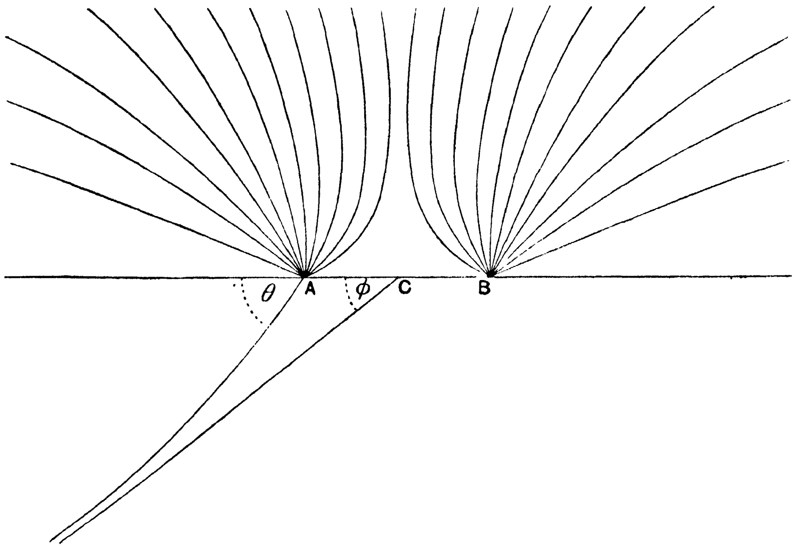


Fig. 19

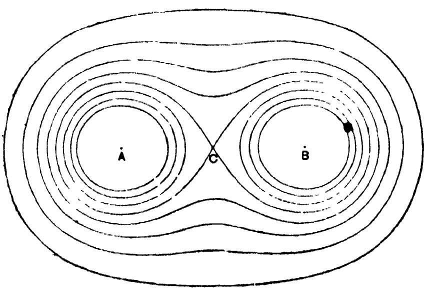


Fig. 20

charges: the lines of force being represented by continuous lines and the sections of the equi-potentials by dotted lines. The properties of the lines of force and equi-potential surfaces are again clearly indicated, except perhaps in the neighbourhood of the point C. The behaviour at this point is reserved for future consideration.

100. Figs. 21 and 22 represent the similar section of the field due to a positive charge of 4 units at A and a negative charge equal to -1 at B. In this case all the lines of force which fall on B start from A, but

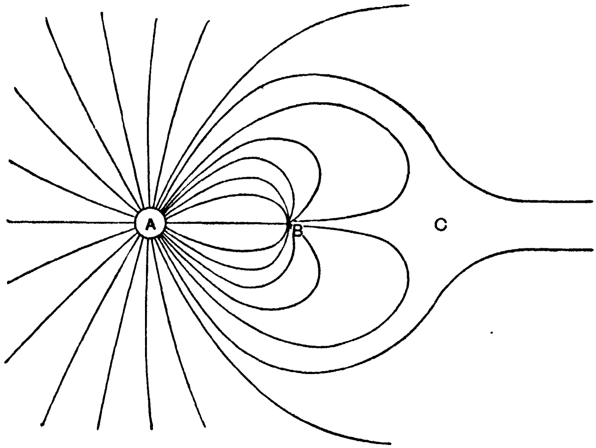


Fig. 21

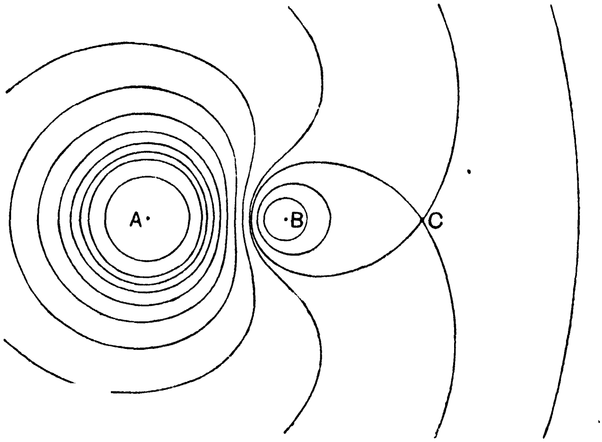


Fig. 22

since the charge at A is numerically greater than that at B, lines of force will start from A which do not fall on B but travel off to an infinite distance*.

* These and several other diagrams of lines of force of point charge systems are given by Maxwell, *Treatise*, I.

101. We have seen that there may be in an electrostatic field certain points where the resultant force intensity vanishes: and since the existence of such points invalidates the general applicability of certain simple types of argument expounded above it seems necessary to enter into a closer investigation of the field in the neighbourhood of such points, called *points of equilibrium,* when they do not coincide with any part of the charge distribution.

We shall refer the discussion to convenient rectangular axes chosen at the point under consideration. Now it can be proved that the potential function ϕ at any point external to the charge distribution is a holomorphic function of the position of the point so that in the neighbourhood of the point under special consideration it may be expanded in the form

$$\phi = \phi_0 + lx + my + nz + \tfrac{1}{2}(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + \dots.$$

As the point is one of equilibrium we know that

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial z} = 0,$$

i.e. $$l = m = n = 0.$$

Thus $$\phi = \phi_0 + \tfrac{1}{2}(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + \dots.$$

Also from the condition that $\nabla^2\phi = 0$ we conclude that

$$a + b + c = 0;$$

the asymptotic cone of a surface of equal potential has therefore three mutually perpendicular generators.

If we turn the axes round and make

$$\phi = \phi_0 + \tfrac{1}{2}(\alpha x^2 + \beta y^2 + \gamma z^2) + \dots,$$

then $$\alpha + \beta + \gamma = 0,$$

and the surfaces of equi-potential in the neighbourhood of such a point are similar hyperboloids all with the same asymptotic cone

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 0.$$

If there is an axis of symmetry (say the axis x) in the field, as is the case with the three examples examined above then $\beta = \gamma$ and thus

$$\alpha + 2\beta = 0.$$

The sections of the level surface in the plane (x, y) are then hyperbolas

$$x^2 - \tfrac{1}{2}y^2 = \frac{\phi - \phi_0}{\alpha},$$

the asymptotes being

$$x \pm \frac{1}{\sqrt{2}}y = 0.$$

The lines of force in the same plane are the orthogonal trajectories of this system of hyperbolas and they appear as the curves

$$y^2 x = \text{const.}$$

The accompanying diagram illustrates the general course of the lines of force and equi-potentials in the neighbourhood of such a point.

There thus appears to be no real difficulty such as that anticipated in the previous discussion except perhaps in connection with the single line of force or equi-potential which actually goes through the point of equilibrium; but even in this case we can regard these particular curves or surfaces as the limiting cases of others exhibited in the figure so that they result as the limiting form of two systems of curves or surfaces moving up to coincidence in part of their length or surface, so that they do not in reality cut across one another. The general argument given above therefore remains valid if care be taken to consider this point when it arises. As a general rule it is

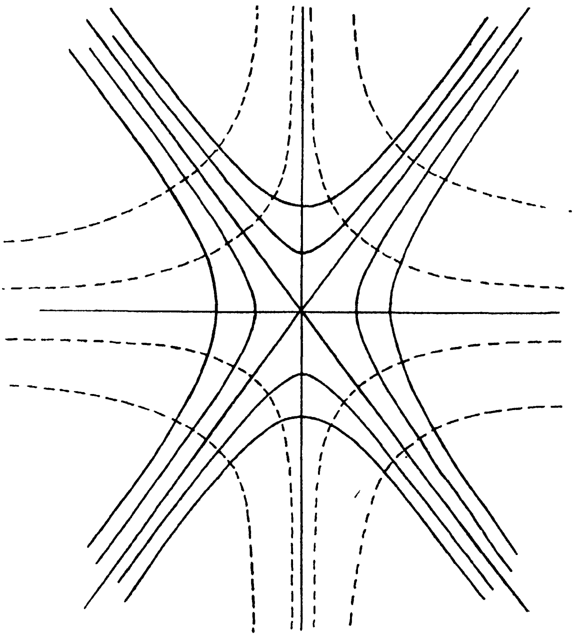


Fig. 23

however safer, in discussing any particular example, to determine the point of equilibrium and the line of force which goes through it and then to discuss the portions of the field inside and outside this line separately: all difficulty is then avoided.

102. The field of a system of conductors. A very important class of electrostatic problem is concerned with the field of a system of charged metallic conductors, as we may gather from the experiments described at the beginning of the previous chapter. The field surrounding such a system possesses

certain characteristic properties which we can very easily interpret in terms of lines of force and equi-potential surfaces. Metallic conductors as we know them are chiefly characterised by the presence in them of an exceedingly large number of electrons (about 10^{23} per c.c.) which are freely moveable in the space between the atoms. Thus as long as any electric force acts on a free electron inside a conductor equilibrium is not possible.

Thus if such a body be brought into an electric field the electric force will act in opposite directions on the positive electricity (in the atoms of the body) and the negative electrons at each point of the metal and separate them so that they no longer cancel one another's effects. A number of the negative electrons in any volume element A will be driven by the electric force into a neighbouring element B; the consequence of this is that the element A becomes positively charged whilst the element B is negatively charged, the total quantity of electricity remaining the same.

Between the two neighbouring elements A and B a new field will arise due to their charge. This new field is superposed on the old one, but is in the opposite direction to it, as lines of force always go from a positive to a negative charge. In this way the field in the interior of the metal will tend to annul itself, and moreover this process will go on until the field in the whole of the body is zero.

We may now enquire as to the whereabouts of the charges which arise from the separation in each element and which give rise to the field which when superposed on the original field makes the resultant intensity zero throughout the whole of the interior of the conductor. Now since the force intensity at every point of the interior of the conductor is zero, the total normal induction, in Gauss' sense, through any closed surface entirely in the metal must also be zero and therefore this surface can contain no electricity. There is therefore no charge in the interior of the conductor. The charge is concentrated on its surface. Thus any conductor introduced into a field of intensity $\mathbf{E}_0$ will have *induced* on its surface a charge which gives rise to a field of intensity $\mathbf{E}_1$ which must be such that at every point in the conductor

$$\mathbf{E}_0 + \mathbf{E}_1 = 0.$$

Moreover at a point on the surface of the conductor the total electric force $\mathbf{E}_0 + \mathbf{E}_1$ can have no component tangential to the surface, as otherwise an unending separation of charge would take place in the surface itself. The electric force intensity of the total field just outside the conductor is therefore entirely normal to the surface.

Even if we introduce a charge to the metallic body from any source this charge must distribute itself over the surface of the body so that the above conditions are still satisfied.

103. In the interior of the conductor the force intensity of the total field is everywhere zero. Therefore the potential of the field must be constant throughout the interior: this is the *potential of the conductor.* The surface of the conductor is therefore an equi-potential surface in the field. This is another reason why the lines of force just outside a conductor are normal to its surface and it also shows the great importance of the potential function introduced on general lines as above.

Since there are no lines of force inside a conductor, the tubes of force of the external field must end on the conductors. We can now show quite simply they must begin on positive electricity and end on negative and also that the quantities of electricity on the two ends of the tube are equal and opposite. Consider the portion of a small tube of force from a conductor up to the cross section df of it and apply Gauss' theorem to the tube closed by a slight extension into the metallic conductor. We see at once that

$$\mathbf{E}df = 4\pi dq,$$

where dq is the charge on the portion of the conductor included in the tube. Similarly from the other part of the tube

$$-\mathbf{E}df = 4\pi\delta q'.$$

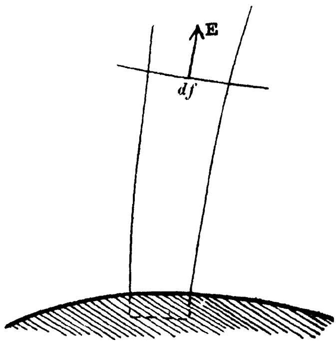


Fig. 24

Thus
$$\delta q = -\delta q' = \frac{\mathbf{E}df}{4\pi}.$$

We may thus regard an electrified system as consisting always of positive and negative electricity in equal amounts, each element of electricity being associated with an equal and opposite element and connected with it by a tube of force running through the dielectric medium.

Moreover since along a tube of force

$$\mathbf{E}df = \text{const.},$$

we see that a line of force is always a line of ascending or of descending potential throughout its length; the sign of $\mathbf{E}$ cannot change along any one line. From this we conclude that the potential cannot be an absolute maximum or minimum in free space. The greatest potential in the field must occur on a conductor (among the conductors we must include the earth, or infinity, as we say, if necessary). Moreover this conductor must have its charge all of the positive kind because if there were negative electricity at any point of its surface lines of ascending potential would pass from the conductor and this is impossible.

Similarly the least potential must occur on a conductor on which the electricity is wholly negative.

104. We also see that if the potential in a field is constant over any surface not enclosing any charge, it must be the same constant value throughout the interior of that surface, because if it were not we could draw lines of force from one point of the surface to another and we should then have an ascending and descending potential (the initial and final values at the ends are the same) in the same line. There can therefore be no charge on the interior surface of a hollow in a charged conductor unless of course there were a charge placed somewhere inside the hollow.

As may be gathered from the experiments described in the previous chapter this last exceptional case is of great practical importance and therefore deserves special examination.

We can easily show that if any number of charged bodies exist in a hollow in a closed conductor the charge on the inner surface of the conductor (i.e. round the hollow) will be equal in magnitude but opposite in sign to the total charge on the system of bodies inside. This can be seen in various ways: we can draw a closed surface entirely inside the material of the conductor and surrounding the hollow; the normal force at every point of this surface being a point in the interior of the metal must be zero so that the total normal induction through the surface is zero; the total charge inside the surface must therefore also be zero. The total charge is, however, the sum of the charge induced on the inner surface of the conductor and the charges on the bodies in the hollow: these must therefore be equal and opposite.

But in the experiments mentioned the total charge of the metallic vessel into which the charged bodies were inserted was zero; so that the charge on the outer surface of the vessel must be equal and opposite to that on the inner surface: it will therefore be equal to the total charge on the bodies inserted in the interior. This is the result deduced experimentally.

105. Referring back again to the result just established, that the quantity of electricity at the end of the tube of force multiplied by 4π is numerically equal to the induction along the tube, we may conclude that the density of the charge at any point of a conductor in any field is given by

$$\sigma = \frac{\mathbf{E}_n}{4\pi} = -\frac{1}{4\pi}\frac{\partial \phi}{\partial n},$$

where $\mathbf{E}_n = -\frac{\partial \phi}{\partial n}$ determines the normal component of the electric force at a point in the field just outside the conductor near the point where the density is examined. This is of course a particular case of the more general result established above that

$$\left(\frac{\partial \phi}{\partial n}\right)_+ - \left(\frac{\partial \phi}{\partial n}\right)_- + 4\pi\sigma = 0,$$

because $\left(\frac{\partial \phi}{\partial n}\right)_-$ is zero the potential inside the conductor being constant.

The total charge on the conductor is

$$-\frac{1}{4\pi}\int_f \frac{\partial\phi}{\partial n}\,df$$

taken over its surface.

106. Electrostatic problems in two dimensions. In many cases which occur in actual practice the configuration of the field both as regards the distribution of masses on it and the distribution of charge is uniform in a certain direction over a considerable range: this is for instance the case for a very long conducting cylinder carrying any charge and placed near to and parallel to other cylindrical bodies, conductors or dielectrics, charged or uncharged. If the cross dimensions of the field are small compared with the length over which it is uniform (i.e. compared with the lengths of the cylindrical bodies involved) the circumstances are so simple that they are worthy of separate consideration. From the results obtained we can of course infer the general nature of the result to be expected from a more complex case.

In all these cases the variation of the field is so slight in the direction of the axes of the field that it is entirely negligible as compared with its variation in the cross directions: thus if we choose the z-axis of the rectangular coordinate system parallel to the direction of the field axis and use ϕ as the potential in the field the general equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=-4\pi\rho,$$

will reduce practically to*

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-4\pi\rho.$$

In this case the density ρ of the charge distribution is of course independent practically speaking, of the z-coordinate. The general integral of the fundamental equation is known to be of type

$$\phi=\int\rho\frac{dv}{r},$$

with the usual notation: if we write

$$dv=dx\,dy\,dz=df\,dz$$

and then integrate along the cylindrical axis of the field we get

$$\phi=\int\rho\,df\int_{-\infty}^{+\infty}\frac{dz}{r},$$

or writing (x, y, z) for the coordinates of the volume element in the integral and (x_ρ, y_ρ, z_ρ) for the coordinates of the point in the field at which the function is calculated we have

$$\begin{aligned} r^2 &= (x-x_\rho)^2+(y-y_\rho)^2+(z-z_\rho)^2 \\ &= r_1^2+(z-z_\rho)^2. \end{aligned}$$

* Laplace, *Mécanique céleste* (Paris, 1799), II. 13.

Thus
$$\frac{1}{r} = \frac{\partial}{\partial z}\log(z - z_\rho + r),$$

$$\phi = \int \rho\, df \left| \log z - z_\rho + r \right|_{-\infty}^{+\infty},$$

and this is easily verified to be of the form*

$$= C - \int \rho \log r_1 df,$$

where C is a very large constant of the order $\log l_0$, l_0 being the length of the cylinder.

This is the general form of the logarithmic potential applicable in problems of the present type: in it ρdf is to be interpreted as the amount of charge per unit length of cylinder standing on the element of surface $df = dxdy$ in the (x, y) plane which may be conveniently chosen across the central section of the length along which the field is uniform.

The components of force are derived in the usual way: that along the direction of the arcs of the field is practically zero and the others are

$$\mathbf{E}_x = -\frac{\partial\phi}{\partial x}, \quad \mathbf{E}_y = -\frac{\partial\phi}{\partial y}.$$

The function ϕ is of course constant throughout the interior of all conducting masses and the density of charge on any one of these reckoned as the amount per unit length of cylinders is

$$\sigma = -\frac{1}{4\pi}\frac{\partial\phi}{\partial n},$$

the total charge being

$$-\frac{1}{4\pi}\int\frac{\partial\phi}{\partial n}ds,$$

taken round the cross section curve of the cylindrical conductor.

107. In the cases which usually occur there is no volume distribution of charge so that the potential function satisfies an equation of the type

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0,$$

and this fact enables us to introduce all the beautiful results of Cauchy's theory of analytic functions. In fact we know from the general theory of these functions that the real part of any differentiable analytic function of the type
$$f(z) \equiv f(x + iy) = 0,$$
must satisfy the fundamental equation: in fact if we write

$$w = \phi + i\psi = f(x + iy),$$

* C. Neumann [*Jour. f. Math.* LXIX. (1861), p. 335] calls this the *logarithmic* potential. Cf also his *Untersuchungen über das logarithmische und Newtonsche Potential* (Leipzig, 1877).

we see that
$$\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=i\left(\frac{\partial\phi}{\partial y}+i\frac{\partial\psi}{\partial y}\right),$$
so that
$$\frac{\partial\phi}{\partial x}=-\frac{\partial\psi}{\partial y},\quad \frac{\partial\psi}{\partial x}=\frac{\partial\phi}{\partial y},$$
and thus
$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0.$$
Thus either the functions ϕ or ψ will serve as a solution of the fundamental potential equation. If we take ϕ then the curves
$$\phi=\text{const.}$$
are the sections by the (x, y) plane of the cylindrical equi-potential surfaces n the field : in this case also the curves
$$\psi=\text{const.},$$
which are the orthogonal trajectories of the former set, represent the lines of force in the plane section of the field.

108. The simplest example of this type is provided by taking
$$w=z^n,$$
or introducing polar coordinates in the (x, y) plane so that
$$x=r\cos\theta,\qquad y=r\sin\theta,$$
$$z^n=Ar^n(\cos\theta+i\sin\theta)^n$$
$$=Ar^n\cos n\theta+r^n i\sin n\theta,$$
and thus
$$\phi=Ar^n\cos n\theta,$$
are the equi-potential curves
$$\psi=Ar^n\sin n\theta,$$
the lines of force, or *vice versa* : particular cases are worth considering.

(i) $n=1$ gives
$$\phi=Ax,\qquad \psi=Ay,$$
and the field is uniform.

If we take any two of the surfaces $\phi=$ const. and place on them charges to make their potentials assume the appropriate values then the field between will be completely specified as to its potential by the function
$$\phi=Ax,$$
which satisfies the fundamental equation and all conditions as to regularity besides taking the proper values on the surfaces.

Of course one of the two surfaces may be taken a very long distance off when the field is that of the single conductor by itself.

The present example determines the field between two uniformly charged planes.

(ii) $n = 2$ gives

$$\phi = Ar^2 \cos 2\theta$$
$$= A(x^2 - y^2),$$
$$\psi = 2Axy,$$

so that the equi-potentials are rectangular hyperbolic cylinders, including as a special case two planes intersecting at right angles.

This transformation gives the field in the immediate neighbourhood of two conducting planes meeting at right angles in any field of force, or freely charged. It also gives the field between two coaxal rectangular hyperbolic cylinders.

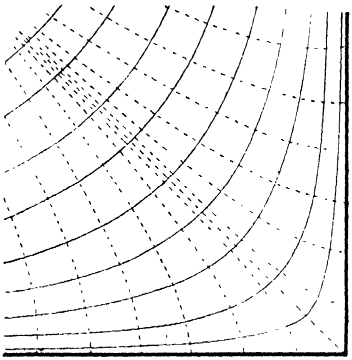


Fig. 25

(iii) $n = \frac{1}{2}$ gives

$$x + iy = (\phi + i\psi)^2,$$
$$x = \phi^2 - \psi^2, \quad y = 2\phi\psi,$$

and thus also

$$y^2 = 4\phi^2 (x + \phi^2).$$

Thus the equi-potential curves are confocal parabolas, including as a special case ($\phi = 0$) a semi-infinite plane bounded by the line of foci.

This transformation gives the field in the immediate neighbourhood of a conducting sharp straight edge in any field.

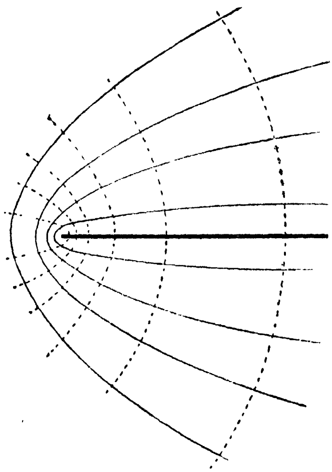


Fig. 26

(iv) $w = \log z$ gives

$$\phi + i\psi = A \log r (\cos\theta + i \sin\theta)$$
$$= A \log re^{i\theta}$$
$$= A \log r + iA\theta,$$

so that the equi-potentials are the cylinders

$$r = \text{const.},$$

the lines of force are

$$\theta = \text{const.},$$

they radiate out in straight lines from the axes.

This is the field due to a uniform line charge or to a uniformly charged right circular cylinder: the charge per unit length on the cylinder is equal to

$$-\frac{1}{4\pi}\int\frac{\partial\phi}{\partial n}ds = -\frac{1}{4\pi}\int\left(\frac{\partial\phi}{\partial x}\frac{dy}{ds}-\frac{\partial\phi}{\partial y}\frac{dx}{ds}\right)ds$$
$$=\frac{1}{4\pi}\int\left(\frac{\partial\psi}{\partial y}\frac{dy}{ds}+\frac{\partial\psi}{\partial x}\frac{dx}{ds}\right)ds$$
$$=\frac{[\psi]}{4\pi},$$

ψ denoting the total change in ψ on going round the boundary curve of the section : in the present case

$$[\psi]=[A\theta]=2\pi A,$$

so that
$$Q=\frac{A}{2},$$

and thus
$$\phi=2Q\log r,$$

apart from a constant.

109. The relation
$$w=A\log\frac{z-a}{z+a}$$

determines a field equivalent to the superposition of the fields given by

$$w=A\log z-a \quad \text{and} \quad w=-A\log z+a;$$

the transformation is accordingly that appropriate to two equal and opposite line charges along the parallel lines

$$z=a \quad \text{and} \quad z=-a.$$

We may generalise this because if we introduce two sets of polar coordinates with these lines as axes we get

$$w=A\log\frac{z-a}{z+a}$$
$$=A\log\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}},$$
$$\phi+i\psi=A\log\frac{r_1}{r_2}+iA(\theta_1-\theta_2).$$

The equi-potentials are the surfaces

$$\frac{r_1}{r_2}=\text{const.},$$

that is they are cylinders standing on two members of a coaxial system of circles with the points on the lines

$$z=a, \qquad z=-a,$$

as limiting points.

The solution is therefore that appropriate to any two of such cylinders equally charged and influencing one another. The lines of force are the orthogonal circles in any plane : the charge per unit length on the cylinders is as before $=\frac{A}{2}$, so that

$$\phi=2Q\log r_1/r_2.$$

110. If $w = \cosh^{-1}\frac{z}{c}$, then

$$x = c \cosh\phi \cos\psi,$$
$$y = c \sinh\phi \sin\psi.$$

The curves $\phi =$ const. are the ellipses

$$\frac{x^2}{c^2\cosh^2\phi} + \frac{y^2}{c^2\sinh^2\phi} = 1,$$

and the curves $\psi =$ const. are the hyperbolas

$$\frac{x^2}{c^2\cos^2\psi} - \frac{y^2}{c^2\sin^2\psi} = 1.$$

These conics have the common foci $(\pm c, 0)$.

The field in which $\phi =$ const.

are the equi-potential surfaces is therefore that appropriate to the case of either a single elliptic cylinder freely charged or of two such cylinders confocal with one another and carrying equal charges: a particular cylinder ($\phi = 0$) is the flat plate between the line of foci.

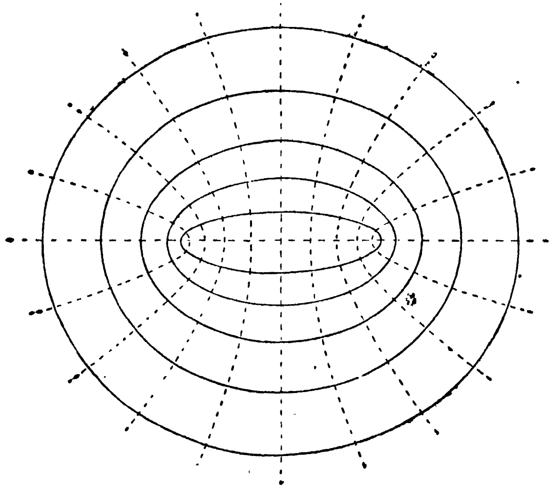


Fig. 27

We might however also take the surfaces $\psi =$ const. as the equi-potential surfaces: the field is then that appropriate to a freely charged hyperbolic cylinder including as a particular case the infinitely extended plate with a slit in it between the focal lines.

111. Let w be defined as a function of z by the implicit relation

$$z = w + e^w,$$

so that $$x = \phi + e^\phi \cos\psi, \quad y = \psi + e^\phi \sin\psi.$$

The line
$$\psi = 0$$
coincides with the axis of x: the line
$$\psi = \pi$$
is the line $y = \pi$ between the points $x = -\infty$ and -1: the line
$$\psi = -\pi$$
is the line
$$y = -\pi,$$
also between the same values of x.

Thus if we take the surfaces
$$\psi = \text{const.}$$
as the equi-potential surfaces in the field we shall have the field of two similarly charged semi-infinite plates placed symmetrically parallel to one another with a difference of potential equal to 2π.

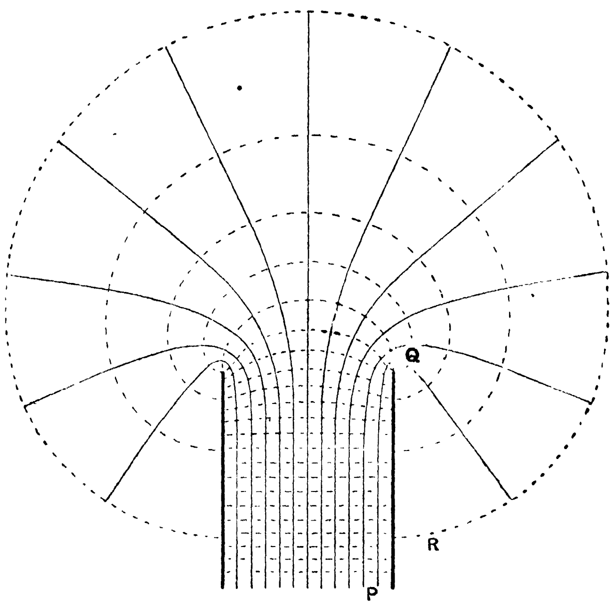


Fig. 28

This solution provides us with the approximate nature of the field in the neighbourhood of the edges of the parallel plate condenser to be discussed at a later stage provided the distance apart of the plates is small compared with the curvature of the edges.

The methods here illustrated depend of course on a knowledge of suitable forms of relations between z and w and they exhibit no systematic method for obtaining the relation when the particular form of the surfaces is given.

There is no method of doing this for the most general cases, but when the curves of intersection of the cylinders on a cross section plane are linear polygons a general method suggested by Schwarz* will always, theoretically at least, effect the transformation. A consideration of this method would however take us beyond the scope of this work; it will be quite familiar to those students acquainted with the methods of conformal representation.

112. On the mathematical theory of electrification by induction. We have in the previous chapter briefly explained the phenomenon of electrification by induction: it consists essentially in the fact that if any good conducting body is introduced into the electric field surrounding any system of charged bodies it will in general be found to be electrified, certain parts of its surface however exhibiting positive electrification and the other parts negative electrification. This phenomenon arises from the action of the electric forces in the field in pulling the free electrons in the metal about, causing them to concentrate in certain parts of the metal, which will thus be negatively electrified, whilst the parts from which they have been driven will be positively electrified. Such a process will go on until the force due to the original field in the interior of the metal is balanced by the force in the new *induced* field which arises from the distribution of charges thus brought about, which of course, as explained above, can only exist on the surface of the metal. The question then naturally arises: can we determine the distribution of charge thus induced on any conductor if the original inducing distribution is specified and also the mechanical force of attraction of the conductor which results from it?

Now whatever distribution is induced on the conductor the potential ϕ of the new total field must be a regular function satisfying the following conditions:

(i) $$\nabla^2\phi = 0$$

at all points of space where there is none of the inducing charge with a finite volume density ρ, at other points it satisfies the equation

$$\nabla^2\phi + 4\pi\rho = 0.$$

We shall temporarily assume that the inducing charge can be completely specified by the distribution of volume density ρ.

(ii) ϕ must be constant on the surface and throughout the interior of the conductor and it must be the same as the potential of the inducing field at a great distance from the conductor where the effect of the induced charge is negligibly small.

(iii) There is also the further condition that the total charge on the conductor is unaltered by the induced charge and is (generally) zero.

* *Jour. f. Math.* LXX. (1869), p. 105. Cf. also E. B. Christoffel, *Ann. di. mat.* (2), I. (1867); IV. (1870).

The solution of a problem of this kind thus turns on a determination of a regular solution of the differential equation

$$\nabla^2\phi + 4\pi\rho = 0$$

which shall satisfy the remaining conditions of the problem. It is obvious that it is only in certain cases where the inducing field is expressible in analytical form and the shape of the conductor is of a geometrically simple type that the solution can be effected.

113. As an example of the method we may consider the comparatively simple case of a spherical conductor in a part of the field far removed from all the inducing charges (which are therefore at infinity, as we say), where the electric force intensity is practically constant in both magnitude and direction over a region large enough to contain the sphere. We shall refer the field to rectangular axes with the origin at the centre of the sphere and the x-axis parallel to the direction of the lines of force in its neighbourhood. Thus near the sphere the potential of the given field can be written in the very approximate form

$$\phi = \phi_0 - Ex,$$

ϕ_0 being the potential at the origin and

$$E = -\left(\frac{\partial\phi}{\partial x}\right)_0$$

the force there.

This is the inducing potential. When the sphere is introduced certain charges will be induced on its surface and their field will be superposed on the field just specified. Let ϕ_1 be the potential of these induced charges so that the potential of the total field is

$$\phi \equiv \phi_1 + \phi_0 - Ex.$$

This must satisfy the following conditions:

(i) $\nabla^2\phi = 0$ everywhere in the field at a finite distance from the origin.

(ii) $\phi = \phi_0 - Ex$ at a comparatively great distance from the sphere where the effect of the local disturbance produced by the introduction of the sphere is negligible. This of course assumes that the form $\phi_0 - Ex$ represents the original field at such distances: if it does not the proper value must be inserted; in any case the argument is not appreciably modified.

(iii) ϕ must be constant over the surface and throughout the interior of the sphere; and finally

(iv) the total charge on the sphere is zero so that

$$\int \frac{\partial\phi}{\partial n}\, df = 0,$$

the integral being taken over the surface of the sphere.

114. Let us try a solution of these conditions with

$$\phi_1 = \frac{Ax}{r^3} + \frac{B}{r}$$

outside the sphere and $$\phi_1 = Ex$$

inside. These forms are specially chosen to satisfy the first two conditions: they satisfy the third if

$$\frac{A}{a^3} - E = 0,$$

i.e. if $$A = a^3E.$$

We have finally to put down the condition that the total charge on the sphere is zero; it is of course all on the surface and its density is determined by

$$\sigma = -\frac{1}{4\pi}\frac{\partial\phi}{\partial n} = -\frac{1}{4\pi}\frac{\partial\phi}{\partial r}$$

$$= \frac{1}{4\pi}\left(\frac{B}{a^2} + \frac{3Ex}{a}\right),$$

where we have used the condition that $A = a^3E$. Thus the condition for no charge requires that $B = 0$. We thus satisfy all the conditions in the field with the potential function

$$\phi = \phi_0 - Ex\left(1 - \frac{a^3}{r^3}\right)$$

outside the sphere and $$\phi = \phi_0$$

inside, these two values agreeing as they must do at the surface. We have therefore completely determined the circumstances for this special case of the general problem. The density of the charge induced on the sphere is given as above by

$$\sigma = \frac{3Ex}{4\pi a}$$

being positive on one side and negative on the other.

Before proceeding to a discussion of the mechanical relations of this field we shall illustrate by a diagram the type of disturbance here obtained. This is done in the figure below where the lines of force in the part of the field near the sphere are drawn.

This diagram is easily constructed if it is noticed that the field induced outside the sphere which is superposed on the original uniform field, viz. the field of the potential

$$\phi_1 = -a^3E\frac{x}{r^3},$$

is precisely the field of an electric doublet at the centre of the sphere with its axis along the direction of the field and of amount $-Ea^3$.

The method followed in this simple case is typical of the more general one to be followed in any case of the present type. The given field involves

a harmonic of the first order and so we try for the additional field harmonics of the same type: outside the sphere we must use $\frac{x}{r^3}$ instead of x because the latter is regular at infinity, whereas inside the sphere the latter type is the correct one to try since the other one is not regular at the origin.

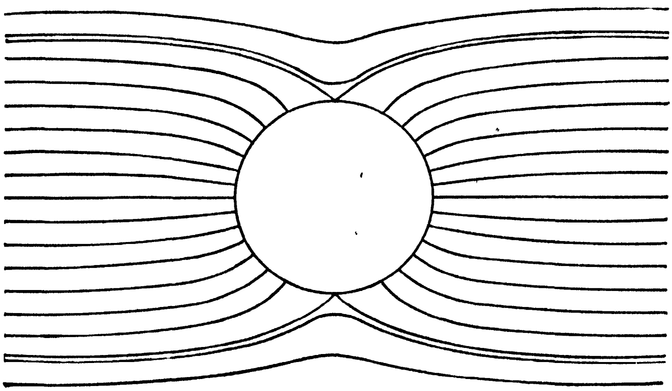


Fig. 29

115. Now let us examine the mechanical relations of the field thus determined. The total work required from external systems to establish the sphere in its position in the field is represented by the integral

$$W = \int \sigma\phi\, df$$

taken over its surface; it is of course presumed that the inducing system is either rigid or at least too far away to be affected by the introduction of the sphere. Inserting the values for ϕ and σ we get

$$W = -\int \frac{3Ex}{4\pi a}\phi_0\, df$$

$$= -\frac{3E\phi_0}{4\pi a}\int x\, df = 0.$$

Thus no energy at all is required from external systems. The energy for the mechanical work actually performed will thus all come from the store of internal energy in the sphere, part of which is set free by the separation of the charges on the conductor. The amount of energy thus set free is

$$W_1 = -\tfrac{1}{2}\int \phi_1\sigma\, df$$

$$= -\frac{a^3\mathbf{E}^2}{2},$$

and thus the amount gained in mechanical work (raising weights, etc.) during the introduction of the conductor is

$$\frac{a^3\mathbf{E}^2}{2}.$$

We may conclude that the force tending to move the sphere is

$$\frac{a^3}{2}\operatorname{grad}\mathbf{E}^2,$$

so that it will tend to move into regions of stronger force.

116. We can verify this result directly but it requires a closer investigation taking into account the next approximation of the original field. Owing however to the intrinsic interest of the problem it seems worth while indicating the analysis. The inducing field must now be taken in the more general form*

$$\phi = \phi_0 + x\left(\frac{\partial\phi}{\partial x}\right)_0 + y\left(\frac{\partial\phi}{\partial y}\right)_0 + z\left(\frac{\partial\phi}{\partial z}\right)_0$$
$$+ \tfrac{1}{2}\left\{x^2\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \ldots + \ldots + 2xy\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_0 + \ldots\right\}\dagger.$$

The field after the introduction of the sphere must satisfy the usual fundamental conditions besides reducing to the constant value (ϕ_0) at the surface of the sphere and agreeing with the above value at a great distance. Bearing in mind the remark made above we see that we may take for the new field outside the sphere

$$\phi = \phi_0 + \left(1 - \frac{a^3}{r^3}\right)\left\{x\left(\frac{\partial\phi}{\partial x}\right)_0 + \ldots\right\}$$
$$+ \tfrac{1}{2}\left(1 - \frac{a^5}{r^5}\right)\left\{x^2\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \ldots + 2xy\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_0 + \ldots\right\}.$$

the inside potential being still ϕ_0. The density of the charge on the sphere will be similarly given by

$$4\pi a\sigma = 3\left\{x\left(\frac{\partial\phi}{\partial x}\right)_0 + \ldots\right\} + 5\left\{x^2\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \ldots\right\}.$$

The force on the sphere will be identical with the force exerted by the external system on the system of charges induced on it and its x-component will be

$$\mathbf{F}_x = \int\sigma\left\{\left(\frac{\partial\phi}{\partial x}\right)_0 + x\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + y\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_0 + z\left(\frac{\partial^2\phi}{\partial x\partial z}\right)_0\right\}df,$$

which to the second order of approximation reduces to

$$\mathbf{F}_x = a^3\left\{\left(\frac{\partial\phi}{\partial x}\right)_0\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \left(\frac{\partial\phi}{\partial y}\right)_0\left(\frac{\partial^2\phi}{\partial y\partial x}\right)_0 + \left(\frac{\partial\phi}{\partial z}\right)_0\left(\frac{\partial^2\phi}{\partial x\partial z}\right)_0\right\},$$

* It is not now necessary to assume that the x-axis is parallel to the direction of the field at the point.

† The suffix 0 denotes the values of the functions at the origin.

provided that $$\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \left(\frac{\partial^2\phi}{\partial y^2}\right)_0 + \left(\frac{\partial^2\phi}{\partial z^2}\right)_0 = 0,$$

which is necessarily satisfied. Thus

$$\mathbf{F}_x = \frac{a^3}{2}\frac{\partial}{\partial x}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\}_0,$$

or generally $$\mathbf{F} = \frac{a^3}{2}\operatorname{grad}\mathbf{E}^2,$$

as above.

The theory thus completely accounts in a general way for the whole series of phenomena associated with the electrification of a conductor by induction.

117. As a second example of these principles we may consider the slightly more general case of a conducting ellipsoid (axes a, b, c) introduced into a field which is practically uniform in the neighbourhood of the ellipsoid. Again choosing axes at the centre of the ellipsoid, with however the principal directions now along the principal axes of the ellipsoid, the field in the neighbourhood of the origin may be expanded in the approximate form

$$\phi = \phi_0 - E_x x - E_y y - E_z z.$$

We shall first consider the case where the inducing field is parallel to the axis of x so that the potential is

$$\phi = \phi_0 - E_x x.$$

When the ellipsoid is introduced charges will be induced on it and the potential function ϕ of the new field will be a regular function satisfying the fundamental equation

$$\nabla^2\phi = 0$$

at all points of space, reducing to a constant on the surface of the ellipsoid and agreeing with the original field at a great distance. The appropriate type of solution could be obtained directly but we have indirectly obtained it in the first section where the fields of certain ellipsoidal distributions were directly examined. Remembering the results there given we are induced to try solutions of the type

$$\phi = \phi_0 - E_x x - L_x x\int_\lambda^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}}$$

outside the ellipsoid and $$\phi = \phi_0$$

inside: in the first of these expressions λ is the positive root of the equation

$$\frac{x^2}{a^2+t} + \frac{y^2}{b^2+t} + \frac{z^2}{c^2+t} = 1.$$

This form of ϕ satisfies all the conditions except constancy of ϕ on the surface of the ellipsoid, and it satisfies this if

$$E_x + L_x \int_0^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}} = 0.$$

We use as above

$$A = \tfrac{1}{2} a^3 bc \int_0^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

so that

$$L_x = \frac{a^3 bc E_x}{2A}.$$

We are thus enabled to satisfy all conditions with the outside potential in the form

$$\phi = \phi_0 - E_x x + \frac{a^3 bc E_x x}{2A} \int_\lambda^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

and the inside one in the form $\phi = \phi_0$.

118. The density at the point (x, y, z) on the surface is given by

$$-\frac{1}{4\pi}\frac{\partial \phi}{\partial n} = -\frac{1}{4\pi}\left(\frac{px}{a^2}\frac{\partial \phi}{\partial x} + \frac{py}{b^2}\frac{\partial \phi}{\partial y} + \frac{pz}{c^2}\frac{\partial \phi}{\partial z}\right),$$

where p is the central perpendicular on the tangent plane at the point,

$$\left(\frac{px}{a^2}, \frac{py}{b^2}, \frac{pz}{c^2}\right)$$

being then the direction cosines of the normal to the surface; but remembering the rule for the differentiation of a definite integral function with respect to a parameter occurring in one of its limits we see that this is

$$-\frac{a^3 bc E_x x}{4\pi \cdot 2A}\left(\frac{2p}{a^3 bc}\right)$$

$$= -\frac{pE_x x}{4\pi A},$$

and

$$\int \sigma df$$

taken over the surface of the ellipsoid vanishes, as it should do, there being no charge on it.

In the more general case we may easily see that the potential in the field outside the ellipsoid is

$$\phi_0 - E_x x - E_y y - E_z z$$
$$+ \frac{abc}{2}\int_\lambda^\infty \left\{\frac{a^2 E_x x}{A(a^2+t)} + \frac{b^2 E_y y}{B(b^2+t)} + \frac{c^2 E_z z}{C(c^2+t)}\right\} \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

whilst the density on the surface is

$$\sigma = -\frac{p}{4\pi}\left(\frac{xE_x}{A} + \frac{yE_y}{B} + \frac{zE_z}{C}\right).$$

119. The energy function of the mechanical forces on the ellipsoid can be calculated as before either directly or as the equivalent of the energy required to establish the separation of charges on the ellipsoid: it turns out to be

$$W = \int \frac{p}{8\pi}\left(\frac{x^2E_x^2}{A} + \frac{y^2E_y^2}{B} + \frac{z^2E_z^2}{C}\right) df,$$

the integral being taken over the surface of the conductor, and this is equal to

$$W = \frac{abc}{6}\left(\frac{E_x^2}{A} + \frac{E_y^2}{B} + \frac{E_z^2}{C}\right).$$

The mechanical forces tending to drag the ellipsoid into the field are obtained as the gradients of this function. In addition to this there are forces tending to rotate the ellipsoid. To obtain some idea of these we can specialise our field slightly and put

$$E_x = E\cos\theta, \quad E_y = E\sin\theta, \quad E_z = 0;$$

we have then $$W = \frac{abcE^2}{6}\left(\frac{\cos^2\theta}{A} + \frac{\sin^2\theta}{B}\right),$$

and the couple tending to increase the angle θ is

$$\frac{\partial W}{\partial\theta} = -\frac{abcE^2}{3}\sin\theta\cos\theta\left(\frac{1}{A} - \frac{1}{B}\right).$$

Now if $a > b$, then $A < B$, so that the couple is negative, in other words the tendency is to set the long axis of the ellipsoid along the field.

There are other general methods of attacking problems of the type just examined; but as they depend essentially on some rather important analytical results deduced from Green's theorem we shall find it more convenient to discuss these first.

120. Green's equivalent stratum—general theory of images*. According to the general formula of Green applied to space external to a given surface f, the value of ϕ satisfying the usual continuity conditions at any point P outside the surface is given by

$$4\pi\phi_P = -\int \nabla^2\phi\frac{dv}{r} - \int_{f'}\left(\frac{\partial\phi_1}{\partial n} - \frac{\partial\phi_2}{\partial n}\right)\frac{df'}{r} + \int_{f'}(\phi_1 - \phi_2)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)df'$$
$$+ \int_f\left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]df,$$

where the first integral extends to all space outside the surface f, the second and third to all surfaces of discontinuity outside f and the fourth to f itself.

* Cf. Kelvin. *Cambridge and Dublin Math. Jour.* 1848, 1849, 1850. *Reprint,* § 55 et seq. also § 208 et seq. See also Thomson and Tait, *Treatise on Natural Philosophy,* II. secs. 499–518.

Now suppose that the surface f is so charged that at each point of it there is a surface density σ and a double sheet distribution τ, where

$$\sigma = -\frac{1}{4\pi}\frac{\partial \phi}{\partial n}, \qquad \tau = +\frac{\phi}{4\pi};$$

the potential of this distribution is

$$-\frac{1}{4\pi}\int\left[\frac{\sigma}{r} - \tau\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]df$$
$$= -\frac{1}{4\pi}\int\left[\frac{1}{r}\frac{\partial \phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]df.$$

We now interpret ϕ_P, in the above formula, in the language of electric theory as a potential function of an electric field arising from a charge distribution obtained from it in the usual manner. Now superpose on this electrical field the electrical field of the imaginary distribution on the surface f defined above. The total potential is now

$$4\pi\phi_P' = -\int \nabla^2\phi\frac{dv}{r} - \int_{f'}\left(\frac{\partial \phi_1}{\partial n} - \frac{\partial \phi_2}{\partial n}\right)\frac{df'}{r} + \int_{f'}(\phi_1 - \phi_2)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)df'.$$

That is ϕ_P' is due solely to the charge distribution external to the surface f. The imaginary distribution therefore completely cancels the effect of the charge distribution inside f at all points external to that surface. This distribution with its sign changed thus completely represents the external system at all external points.

This is Green's equivalent stratum. Any electrical system can be replaced by a distribution similar to that above, on any surface completely surrounding it, as long as its action at external points is under review.

121. If in the above example the surface f had been an equi-potential of the whole original field the double sheet distribution would not be required, since as ϕ is constant over the surface the potential due to this part of the distribution at external points is

$$\phi\int_f \frac{\partial}{\partial n}\left(\frac{1}{r}\right)df = 0.$$

The distribution on the surface f would then be the same as if the surface were conducting, because the distribution

$$\sigma = -\frac{1}{4\pi}\frac{\partial \phi}{\partial n}$$

on it is in agreement with the fact that it is an equi-potential surface.

We know for example that the potential of the point charge q at O is

$$\frac{q}{r},$$

and the equi-potential surfaces are concentric spheres: if we take the one whose radius is $\left(\frac{q}{\phi_0}\right)$ its potential is ϕ_0, and thus the external potential of this

sphere charged to potential ϕ_0 is the same as that of the point charge q at its centre and the charge distribution on it is uniform : this agrees with what we have found above.

Another example is provided by the case of three point charges $a_1\phi_0$ at O_1, $a_2\phi_0$ at O_2 and $-\frac{a_1a_2}{c}\phi_0$ at O, where however O divides the line O_1O_2* in the ratio of the squares of a_1 and a_2 and $c^2 = O_1O_2^2 = a_1^2 + a_2^2$. In this case the equi-potential surface in the field on which $\phi = \phi_0$ is the outer portion of two intersecting orthogonal spheres of radii a_1, a_2, centres at O_1 and O_2 respectively, and O is the mid-point of their common chord. Thus

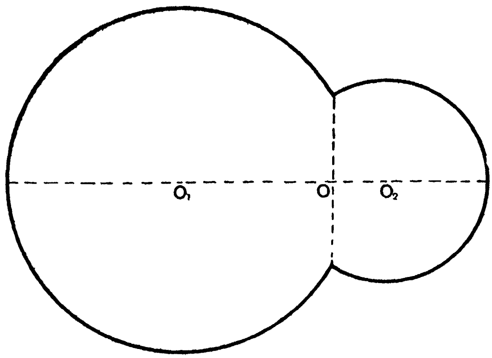


Fig. 30

these three point charges effectively represent the external electric field of the free equilibrium distribution of charge on the conductor which makes its potential ϕ_0; the amount of the charge is equal to the sum of the point charges inside and is therefore

$$\left(a_1 + a_2 - \frac{a_1a_2}{c}\right)\phi_0.$$

The analysis for this case, which is quite straightforward, will be obvious when we have treated the more general problem now to be discussed.

122. In a case like this last it is often desirable to know the actual distribution of the charge on the conductor and the relative amounts on the different parts of the surface. This of course is easily obtained as soon as the potential function is specified completely, as is for instance always the case when the image system can be reduced to a series of point charges finite in number. We can however find the total charges on different portions of the surface by an application of Gauss' theorem to non-closed

* Maxwell, *Treatise*, I. § 166. This author also solves the general case of spheres cutting at any angle π/n.

surfaces. For example, in the case of the conductor formed by the outer parts of two orthogonal spheres the total induction through the sphere of radius a is

$$\Sigma q\,(1 + \cos\theta),$$

where θ is the angle of the cone subtended by the image q at the circle of intersection of the spheres: in this case this is

$$N = 2\pi\left\{a_1\phi_0\left(1 + \frac{a_1}{c}\right) - \frac{a_1 a_2}{c}\phi_0 + a_2\phi_0\left(1 - \frac{a_2}{c}\right)\right\},$$

and thus the charge on this sphere is

$$\frac{N}{4\pi} = \frac{\phi_0}{2}\left\{a_1 + a_2 - \frac{a_1 a_2}{c} + \frac{a_1^2 - a_2^2}{c}\right\}.$$

The case where a_2 is small compared with a_1 enables us to approximate to the effect of a small knob on an otherwise perfectly spherical surface.

If a_1 is very large the solution is of the type appropriate to a plane conductor with a hemispherical boss on it.

123. Now consider the problem in another form. Suppose we have any two electrical systems A and B for which combined we can determine the equi-potential surfaces by calculation. Suppose that $\phi = \phi_0$ is one of these surfaces which divides the system A from the system B, the system A being inside the surface and B outside it.

Now consider the surface $\phi = \phi_0$ as a conductor charged to potential ϕ_0 and under the influence of the electrical system B outside it: what is the potential function and distribution of charge. The function has to take the value $\phi = \phi_0$ on the given conductor and $\phi = 0$ at infinity and has also to include the discontinuities of the system B in the region between. Now the original potential function ϕ of the combined systems is a function which outside $\phi = \phi_0$ satisfies all these conditions and must therefore be the required solution of the problem. The surface density on the conductor ϕ_0 is

$$\sigma = -\frac{1}{4\pi}\frac{\partial\phi}{\partial n}$$

at any point.

Thus the distribution on the conductor just obtained has at all external points the same potential as the old system A and consequently also the same force intensity. Also the total quantity of electricity on the conductor is the same in amount as the total of the system A.

For internal points the charge on the surface and the B system produce a constant potential ϕ_0 and therefore the force is zero: thus the charge and the system B produce the same force inside but not the same potential unless in the special case $\phi_0 = 0$.

When ϕ_0 is not zero we may however regard the charge on the conductor $\phi = \phi_0$ as consisting of two charges superposed, viz. (i) a charge which would

be exactly equal and opposite to B in its internal effect and (2) a charge which would produce a constant potential ϕ_0 on the conductor in a field by itself.

In a generalised sense the distribution in the system A may be regarded as the image of the distribution B in the conductor $\phi = \phi_0$ when it is at a potential ϕ_0.

Notice that all electrical images are virtual; the system A must be completely enclosed by the surface $\phi = \phi_0$.

124. If the system A consists of the single point charge q' at A' and the system B of the point charge q ($> q'$) at A, then the equi-potentials of the combined system are determined by

$$\phi = \frac{q}{r} + \frac{q'}{r'},$$

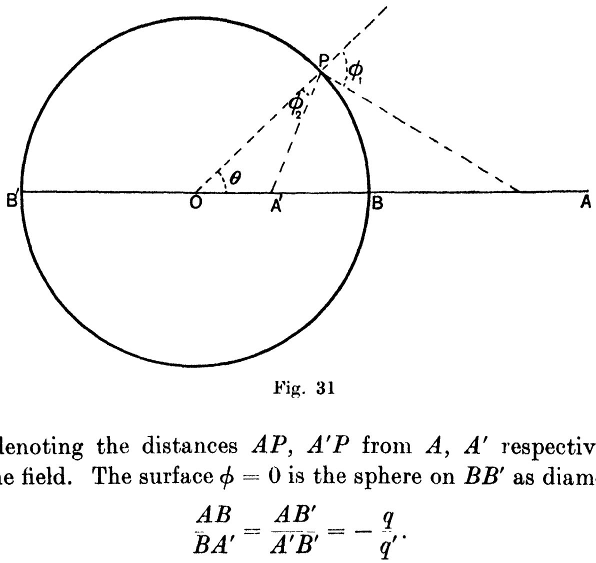


Fig. 31

r and r' denoting the distances AP, $A'P$ from A, A' respectively to any point in the field. The surface $\phi = 0$ is the sphere on BB' as diameter, where

$$\frac{AB}{BA'} = \frac{AB'}{A'B'} = -\frac{q}{q'}.$$

If the centre of this sphere is O and $OA = f$, then $OA' = \frac{a^2}{f}$, where $2a = BB'$, and therefore

$$q' = -\frac{aq}{f}.$$

We thus see that if a sphere of radius a is kept at zero potential under the influence of the point charge q at A, the charge on the sphere acts at all external points just like the charge $-\frac{aq}{f}$ at the inverse point A' would do.

Under these circumstances the normal component of force outside the sphere must be the same as that due to the two charges and we can therefore calculate the density of the induced charge on the sphere; for at any point P on it

$$\sigma = -\frac{1}{4\pi}\text{ (normal force inwards)}$$

$$= -\frac{1}{4\pi}\left\{\frac{a}{AP^2}\cos\phi_1 + \frac{qa}{f\,.\,A'P^2}\cos\phi_2\right\},$$

ϕ_1, ϕ_2 being the angles marked in the figure. Thus

$$\sigma = -\frac{q}{4\pi}\left[\frac{1}{AP^3}(f\cos\theta - a) + \frac{a}{f\,.\,A'P^3}\left(a - \frac{a^2\cos\theta}{f}\right)\right]$$

$$= -\frac{q}{4\pi AP^3}\left[f\cos\theta - a + \frac{a}{f}\cdot\frac{f^3}{a^3}\left(a - \frac{a^2\cos\theta}{f}\right)\right]$$

$$= -\frac{q}{4\pi a}\frac{f^2 - a^2}{AP^3}.$$

The density varies inversely as the cube of the distance from the external point. We know that the total charge on the sphere is $-\frac{aq}{f}$ and so we conclude that a distribution of density $\frac{\lambda}{r^3}$ on any sphere of radius a, where r is the distance from a fixed external point distant f from the centre, has a total mass or quantity

$$\frac{4\pi a^2\lambda}{f(f^2 - a^2)},$$

and that it acts on all external points as if it were condensed at the inverse point; but it acts on all internal points as if $\frac{4\pi a\lambda}{f^2 - a^2}$ were condensed at the external point.

125. In considering the more general case, with the same notation, in which the system A consists of $-\frac{aq}{f}$ at A' and $a\phi_0$ at O, the general potential function with the same system B is

$$\phi = \frac{q}{r} - \frac{aq}{fr'} + \frac{a\phi_0}{R},$$

R denoting the distance from O to any point in the field. Now the surface

$$\phi = \phi_0$$

is the sphere $R = a$: and this shows that the image of the charge on the sphere still under the influence of the point charge q at A, but now at a potential ϕ_0, consists of the point charges $-\frac{aq}{f}$ at A' and $a\phi_0$ at O.

The total charge on the sphere in this case is the sum of the images and is therefore

$$Q = a\phi_0 - \frac{aq}{f},$$

and thus if Q had been given instead of ϕ_0, we could determine ϕ_0 from this relation as

$$\frac{Q}{a} + \frac{q}{f}.$$

The density on the sphere is determined as before and is given by

$$4\pi\sigma = \frac{\phi_0}{a} - \frac{q}{a} \cdot \frac{f^2 - a^2}{AP^3}.$$

In the particular case of the first example, where $q = -q'$, the sphere degenerates into the infinite plane which bisects perpendicularly the line AA'.

126. The mechanical forces tending to move the conductor can also be simply determined from the image system, for it is equal and opposite to the force of reaction on the external inducing system and this depends only on the field in its immediate neighbourhood; so that it is independent of whether it arises from the actual distribution on the conductor or the internal image distribution which effectively represents it at all external points.

Thus, for example, there is a force tending to pull the charged sphere of the above example towards the point charge q of amount

$$-\frac{q\left(Q + \dfrac{aq}{f}\right)}{f^2} + \frac{q \cdot \dfrac{aq}{f}}{\left(f - \dfrac{a^2}{f}\right)^2}$$

$$= -q\left[\frac{Q}{f^2} + \frac{aq}{f^2} - \frac{aqf}{(f^2 - a^2)^2}\right];$$

if q, Q are of the same sign it is positive when the charge is near the sphere but negative when it is at a great distance away.

127. Returning now to the case of the conductor formed by the larger portions of two orthogonal spheres we see at once the reason for the particular choice of the point charges: the point charge $-\dfrac{a_1 a_2}{c}\phi_0$ is the image of the charge at either centre in the other sphere. Let us now consider the case of the conductor uninsulated and under the influence of a point charge q at an external point P. Now we know by geometry that if P_1 is the inverse point of P in the first sphere O, and P_2 is the inverse point of P in the second sphere and if O_1P_2 and O_2P_1 intersect in P_3, then P_3 is the inverse of P_2 in the first sphere and of P_1 in the second sphere and also that

$$O_1P \cdot O_2P_1 = O_2P \cdot O_1P_2.$$

Thus if $O_1P = f_1$ and $O_2P = f_2$ and we put charges at P_1, P_2 and P_3 respectively of amounts

$$-\frac{a_1 q}{f_1}, \quad -\frac{a_2 q}{f_2}, \quad \frac{a_1 a_2 q}{\sqrt{a_2^2 f_1^2 + a_1^2 f_2^2 - a_1^2 a_2^2}},$$

the potential over the surface of the conductor is constantly zero. The denominator in this last fraction is the common value of $O_1P \,.\, O_2P_1$ and $O_2P \,.\, O_1P_2$.

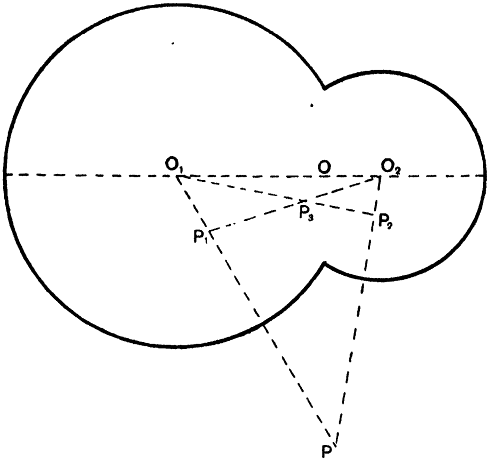


Fig. 32

If we want the more general case when the conductor is insulated and charged we must insert the appropriate images at O_1, O and O_2 to give the total charge right.

The force on the conductor can also be obtained as the resultant of the forces on the images inside it.

128. Finally let us briefly consider the simple case of the charges induced on two infinite plane conductors at right angles to one another and under the influence of a point charge q in the angle between them. The system of images is obvious, they will be $-q$, $+q$, $-q$ respectively at the corners of a rectangle symmetrical round the angle and with one corner at the position of q. Thus if r_1, r_2, r_3, r_4 denote the respective distances from the four corner charges the external potential of the charge distribution induced on the planes is

$$-\frac{q}{r_2} + \frac{q}{r_3} - \frac{q}{r_4},$$

whereas the internal potential is

$$-\frac{q}{r_1}.$$

The law of distribution of charge on the conductor can now readily be obtained as above. The total charge on the horizontal conductor turns out to be $-\frac{2q}{\pi}\tan^{-1}\frac{a}{b}$, where a and b are respectively the distances of q from the vertical and horizontal planes.

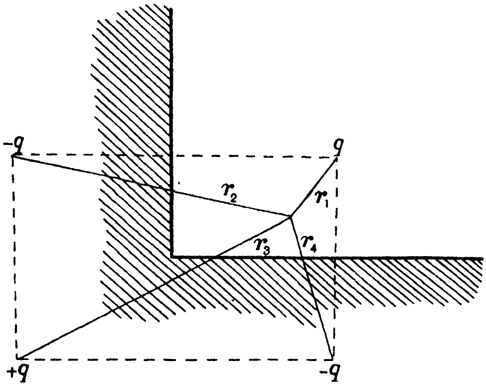


Fig 33

CHAPTER III

THE ELECTRICAL AND MECHANICAL RELATIONS OF A SYSTEM OF CONDUCTORS

129. On the relations between the potentials and charges of a system of conductors. The discussion of the two former chapters indicates as the main theoretical problem in electrostatic theory, the determination of the distribution of charge on a given system of conductors under given conditions and the deduction finally of the mechanical relations of these conductors. All electrical experiments of a static nature are made with various forms of conductors under different conditions and a knowledge of their mutual relations when charged is therefore essential to a correct interpretation of the results of such experiments.

Of course if we can determine the potential function of the field our problem is completely solved; but this is the difficulty. Whatever this function it must be continuous and regular at infinity and satisfy the equation

$$\nabla^2\phi = 0$$

at each point of space. In addition to this it must be constant throughout each conductor. The question of the existence of such a solution is not one we can enter into now, and we shall content ourselves by saying that from the physical point of view there is a certain amount of evidence in favour of at least one solution.

If we assume the existence of a solution it is easy to prove mathematically that if the charges or the potentials of all the conductors are specified the whole circumstances of the field are uniquely determinable*.

130. (i) In the first case when the charges are given the potential ϕ of the external field has to satisfy the Laplacian equation and the usual regularity conditions, and it must in addition be constant over each conductor and such that

$$-\frac{1}{4\pi}\int_{f_r}\frac{\partial\phi}{\partial n}\,df_r$$

taken over the surface of the rth conductor is equal to Q_r, the charge on that conductor.

* Cf. Maxwell, *Treatise*, I. ch. III.

Suppose that there are two solutions ϕ and $\phi + \phi_1$ of these conditions: then ϕ_1, the difference of these two solutions, must satisfy all the conditions as before except that

$$-\frac{1}{4\pi}\int_{f_r}\frac{\partial\phi_1}{\partial n}\,df_r$$

is zero for each conductor.

If therefore we consider the integral

$$\int\left[\left(\frac{\partial\phi_1}{\partial x}\right)^2+\left(\frac{\partial\phi_1}{\partial y}\right)^2+\left(\frac{\partial\phi_1}{\partial z}\right)^2\right]dv$$

taken throughout the whole of space, it is equal to

$$\sum_{r=1}^{n}\int_{f_r}-\phi_1\frac{\partial\phi_1}{\partial n}\,df_r-\int\phi\frac{\partial\phi}{\partial n}\,df,$$

the sum Σ denoting the sum of integrals relating to the surfaces of the separate conductors: the last term refers to the infinite boundary. This latter term tends to vanish, ϕ_1 is regular at infinity; and also since ϕ_1 is constant over the surface of each conductor

$$\int_{f_r}\phi_1\frac{\partial\phi_1}{\partial n}\,df_r=\phi_1\int_{f_r}\frac{\partial\phi_1}{\partial n}\,df_r=0$$

for each of them. The original volume integral is therefore zero, but it consists of essentially positive elements, which must therefore separably be zero or

$$\frac{\partial\phi_1}{\partial x}=\frac{\partial\phi_1}{\partial y}=\frac{\partial\phi_1}{\partial z}=0$$

at all points of space: this means that ϕ_1 is constant, and being regular at infinity, it must be zero. There is only one solution of the conditions.

(ii) A similar proof applies to this second case: there is no need to detail it out in full.

131. We can now establish the important principle of superposition.

If for a system of conductors the potentials are $\phi_1, \phi_2, \dots \phi_n$ when the charges are $Q_1, Q_2, \dots Q_n$ and $\phi_1', \phi_2', \dots \phi_n'$ when the charges are $Q_1', Q_2' \dots Q_n'$, then when the charges are $Q_1 + Q_1', Q_2 + Q_2', \dots Q_n + Q_n'$ the potentials will be $\phi_1 + \phi_1', \phi_2 + \phi_2', \dots \phi_n + \phi_n'$.

Let ϕ, ϕ' be the potentials of the electrostatic fields in the first two cases and Φ the potential when the charges are superposed.

Then besides the usual conditions of regularity the first two functions satisfy

(i) $\nabla^2\phi = 0$, $\nabla^2\phi' = 0$ at all points.

(ii) $\phi = \phi_1, \phi_2, \dots \phi_n$, $\phi' = \phi_1', \phi_2', \dots \phi_n'$ on the respective conductors.

(iii) $\int_{f_r}\frac{\partial\phi}{\partial n}\,df_r = -4\pi Q_r$, $\int_{f_r}\frac{\partial\phi'}{\partial n}\,df_r = -4\pi Q_r'$ for each conductor.

The function Φ satisfies, besides the regularity conditions, also

$$\nabla^2\Phi = 0,$$

and it is constant over each conductor and such that

$$\int_{f_r} \frac{\partial\Phi}{\partial n}\, df_r = -4\pi(Q_r + Q_r')$$

for each of them. It is completely determined by these conditions; but an obvious solution of them all is

$$\Phi = \phi + \phi',$$

and since there can be but one solution this is *the* solution. But then the potentials of the various conductors are

$$\phi_1 + \phi_1', \quad \phi_2 + \phi_2', \ \dots\ \phi_n + \phi_n'.$$

Thus statical electric fields are superposable.

132. Now suppose that the result of placing unit charge on the first conductor and leaving the others uncharged is to produce potentials

$$b_{11}, \quad b_{12}, \dots \quad b_{1n}$$

on the n conductors respectively; then the result of placing Q_1 on this same conductor and leaving the others uncharged is to produce potentials

$$b_{11}Q_1, \dots \quad b_{1n}Q_1.$$

Similarly if placing unit charge on the second conductor and leaving the others uncharged gives potentials

$$b_{21}, \dots \quad b_{2n},$$

then placing Q_2 on this conductor and leaving the others uncharged gives potentials

$$b_{21}Q_{21}, \dots \quad b_{2n}Q_2.$$

In the same way we can calculate the result of placing Q_3 on the third conductor, Q_4 on the fourth and so on.

If we now superpose the solutions thus obtained, we find that the effect of simultaneous charges $Q_1, Q_2, \dots Q_n$ is to give potentials $\phi_1, \phi_2, \dots \phi_n$ where

$$\phi_1 = b_{11}Q_1 + b_{12}Q_2 + \dots + b_{1n}Q_n,$$
$$\phi_2 = b_{21}Q_1 + b_{22}Q_2 + \dots + b_{2n}Q_n,$$
$$\phi_3 = b_{31}Q_1 + b_{32}Q_2 + \dots + b_{3n}Q_n,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

These equations give the potentials of each of the conductors as linear functions of the charges on all of them. The coefficients $b_{11}, b_{22} \dots, b_{12}, \dots,$ called the coefficients of induction do not depend on either the potentials or charges, being purely geometrical quantities, which depend on the size, shape

By linear solutions of these equations regarded as equations in the charges we obtain

$$\begin{aligned} Q_1 &= c_{11}\phi_1 + c_{12}\phi_2 + \dots && + c_{1n}\phi_n, \\ Q_2 &= c_{21}\phi_1 + c_{22}\phi_2 + \dots && + c_{2n}\phi_n, \\ Q_3 &= c_{31}\phi_1 + c_{32}\phi_2 + \dots && + c_{3n}\phi_n, \\ &\dots\dots\dots\dots\dots\dots \end{aligned}$$

The coefficients c_{rs}, the coefficients of capacity for the conductors are given by the system of equations

$$\frac{c_{11}}{\begin{vmatrix} b_{12}, & b_{13}, \dots b_{1n} \\ b_{22}, & b_{23}, \dots b_{2n} \\ b_{32}, \dots & \\ \dots\dots & \end{vmatrix}} = \frac{c_{12}}{\begin{vmatrix} b_{21}, & b_{31}, \dots b_{n1} \\ b_{23}, & b_{33}, \dots b_{n3} \\ b_{24}, & b_{34}, \dots b_{n4} \\ \dots\dots & \end{vmatrix}} = \dots = \frac{1}{\begin{vmatrix} b_{11}, & b_{12}, \dots b_{1n} \\ b_{21}, & b_{22}, \dots \\ b_{31}, \dots & \\ \dots\dots & \end{vmatrix}}$$

133. There are many important relations which exist between these coefficients in all cases, and which are in fact direct consequences of certain general properties of the field surrounding them.

Many of these can be derived directly by a simple application of the general reciprocal theorem due to Gauss which was given in the first chapter. This theorem takes a simple form when the only charges in the field are those on the conductors, which may be put in the words :

If two electrical systems are simply one system of conductors charged in two ways with systems of charges $Q_1, Q_2, \dots Q_n$ in the first case and $Q_1', Q_2', \dots Q_n'$ in the second and if the potentials of the conductors in the two cases are $\phi_1, \phi_2, \dots \phi_n$ and $\phi_1', \phi_2', \dots \phi_n'$ respectively, then

$$\Sigma Q_r\phi_r' = \Sigma Q_r'\phi_r.$$

This follows directly from the general theorem, for although the charges on each conductor are spread out over its surface, the positions it occupies are positions of constant potential in the other distribution.

Now take the system of conductors to be that discussed above and suppose the first distribution of charge is represented by zero charge on all the conductors except the rth which has a unit of charge; the potentials are then respectively $b_{r1}, \quad b_{r2}, \dots \quad b_{rr}, \dots \quad b_{rs}, \dots \quad b_{rn}$:

similarly if the second distribution is unit charge on the sth conductor and zero on all the others the potentials are

$$b_{s1}, \quad b_{s2}, \dots \quad b_{sr}, \dots \quad b_{ss}, \dots \quad b_{sn},$$

and the application of the theorem at once shows that

$$b_{rs} = b_{sr},$$

which is probably the most important property of these coefficients : it will be shown in the next paragraph that it is an immediate consequence of the principle of energy as applied in these matters.

In a similar manner we can deduce that

$$c_{rs} = c_{sr},$$

or this may be inferred as an algebraical consequence of the equality of b_{rs} and b_{sr}.

134. Next suppose the rth conductor carries a charge Q_r but that all the others are uncharged; the potentials are then

$$b_{r1}Q_r, \quad b_{r2}Q_r, \ldots \quad b_{rr}Q_r, \ldots \quad b_{rn}Q_r.$$

The corresponding charge $-Q_r$ must be at infinity where the potential is zero. Thus the rth conductor is the only one on which the charge is entirely positive, and infinity (or to be precise, the earth) is the only one with an entirely negative charge: thus $Q_r b_{rr}$ and 0 are respectively the greatest and least of the potentials in the field; we must therefore infer that all the coefficients b_{rr}, b_{rs}, ... are positive, but that

$$b_{rr} > b_{rs}. \qquad (s = 1, 2, \ldots n)$$

We may prove in a similar way that c_{rr} is positive, but that c_{r1}, c_{r2}, ... are all negative and such that

$$c_{r1} + c_{r2} + \ldots c_{rr} + \ldots c_{rn} > 0.$$

135. The particular case in which one of the conductors is enclosed in another one is of such great importance practically that it is worth detailed treatment here as an example exhibiting certain properties of the coefficients. We shall suppose the second conductor encloses the first. If $Q_1 = 0$ the second conductor becomes a closed conductor with no charge inside, so that the potential in its interior is constant or $\phi_1 = \phi_2$. Putting $Q_1 = 0$ the relation $\phi_1 = \phi_2$ gives

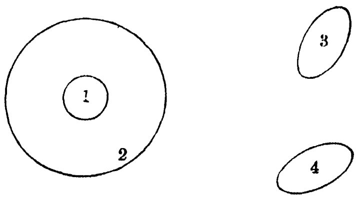


Fig. 34

$$(b_{12} - b_{22})\,Q_2 + (b_{13} - b_{23})\,Q_3 + \ldots = 0,$$

and this must be true for all values of Q_2, Q_3, ..., so that we must have

$$b_{12} = b_{22}, \qquad b_{13} = b_{23}, \ldots.$$

Now suppose that the second conductor is earthed so that $\phi_2 = 0$. Then if $Q_1 = 0$ it follows that $\phi_1 = 0$ also. Hence from the equation

$$Q_1 = c_{11}\phi_1 + c_{12}\phi_2 + \ldots c_{1n}\phi_n,$$

we obtain in this special case that

$$c_{13}\phi_3 + c_{14}\phi_4 + \ldots c_{1n}\phi_n = 0,$$

a relation which must be satisfied whatever the values of ϕ_3, ϕ_4, ... may be. This means that

$$c_{13} = c_{14} = \ldots = 0.$$

The second system of equations therefore reduce to, still with $\phi_2 = 0$,

$$Q_1 = c_{11}\phi_1,$$
$$Q_2 = c_{12}\phi_1 + c_{23}\phi_3 + c_{24}\phi_4 + \dots,$$
$$Q_3 = c_{33}\phi_3 + c_{34}\phi_4 + \dots,$$
$$Q_4 = c_{43}\phi_3 + \dots.$$

These equations show that the relations between the charges and potentials outside the second conductor are quite independent of the electrical conditions which obtain inside this conductor. So also the conditions inside the second conductor are not affected by those outside this conductor. These results become obvious when we consider that no lines of force can cross the second conductor either from inside or from out and that there is no way except by crossing this conductor for a line of force to pass from the first conductor to any other one outside the second.

An electric-system which is completely surrounded by a conductor at zero potential is said to be *electrically screened* from all systems outside the conductor; for charges outside this screen cannot affect the screened system. This principle of screening is often used in electrostatic instruments to shield the instrument from action by extraneous fields.

136. These properties may be illustrated by the calculation of the coefficients in a special case.

We consider the simple case of two concentric spherical conductors, the inner one of radius a_1 and the outer one of radius a_2. The equations connecting the potentials and charges are

$$\phi_1 = b_{11}Q_1 + b_{21}Q_2,$$
$$\phi_2 = b_{12}Q_1 + b_{22}Q_2.$$

A unit charge placed on the outer sphere raises both to the potential $\frac{1}{a_2}$, so that on putting $Q_1 = 0$, $Q_2 = 1$ we must have $\phi_1 = \phi_2 = \frac{1}{a_2}$. Thus

$$b_{21} = b_{22} = \frac{1}{a_2}.$$

If the outer conductor is uncharged and the inner has unit charge the field of force is that investigated above in § 87. Hence

$$b_{11} = \frac{1}{a_1}, \quad b_{12} = \frac{1}{a_2},$$

results which verify that $\qquad b_{12} = b_{21},$

and the relation peculiar to electric screening

$$b_{12} = b_{22}.$$

We have then
$$\phi_1 = \frac{Q_1}{a_1} + \frac{Q_2}{a_2},$$
$$\phi_2 = \frac{Q_1}{a_2} + \frac{Q_2}{a_2}.$$

Solving for Q_1 and Q_2 in terms of ϕ_1 and ϕ_2 we obtain
$$Q_1 = \frac{a_1 a_2}{a_2 - a_1}\phi_1 - \frac{a_1 a_2}{a_2 - a_1}\phi_2,$$
$$Q_2 = -\frac{a_1 a_2}{a_2 - a_1}\phi_1 + \frac{a_2^2}{a_2 - a_1}\phi_2,$$

so that
$$c_{11} = \frac{a_1 a_2}{a_2 - a_1}, \quad c_{12} = c_{21} = -\frac{a_1 a_2}{a_2 - a_1}, \quad c_{22} = \frac{a_2^2}{a_2 - a_1}.$$

These verify the general properties of the coefficients of capacity given above.

137. The case when the spheres are eccentric or external to one another is much more difficult to analyse. Approximate results can however be obtained by using certain results obtained in the section of the last chapter dealing with the general theory of images. We first suppose the spheres carry charges Q_1 and Q_2.

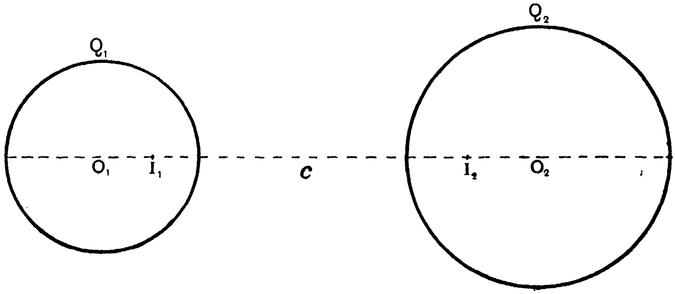


Fig. 35

If the spheres are at a great distance c apart and of radii a_1 and a_2 their charges will be uniformly distributed, the field of the one not disturbing the distribution on the other. This is the first approximation. The next approximation is obtained by taking account of the influence of one charge on one sphere on the distribution on the other. To a first approximation the charge Q_1 on the sphere a_1 will influence the second sphere just as if it were collected at its centre: the distribution on the second sphere is then such that the outside field is the same as that of the images $Q_2 + \frac{Q_1 a_2}{c}$ at the centre and $-Q_1 \frac{a_2}{c}$ at the inverse point I_2 of O_1 in the sphere O_2: this

is at a distance $\frac{a_2^2}{c}$ from O_2 towards O_1. The distribution of charge on the first sphere will to the same order of approximation give a field outside it just like images

$$Q_1 + \frac{Q_2 a_1}{c}$$

at its centre and
$$-Q_2 \frac{a_1}{c}$$

at the inverse point I_1 of O_2 in the sphere, i.e. at a distance $\frac{a_1^2}{c}$ from O_1. This gives the second approximation: to correct it and obtain the third approximation we calculate the full effect of the two images in Q_1 in the second sphere, i.e. we insert their images just as before in order to keep the potential of the sphere constant: this leads to a second image point I_1' and I_2' in each sphere at distances respectively

$$\frac{a_1^2}{c - \frac{a_2^2}{c}}, \quad \frac{a_2^2}{c - \frac{a_1^2}{c}}$$

from the centres; the charges there being

$$Q_2 \frac{a_1 a_2}{c^2 - a_1^2}, \quad Q_1 \frac{a_1 a_2}{c^2 - a_2^2}.$$

The charges at the centre are now

$$Q_1 + Q_2 \frac{a_1}{c} - Q_1 \frac{a_1 a_2}{c^2 - a_2^2}$$

and
$$Q_2 + Q_1 \frac{a_2}{c} - Q_2 \frac{a_1 a_2}{c^2 - a_1^2}$$

respectively: the other images are now

$$\frac{a_1}{c}\left(Q_2 + \frac{a_2 Q_1}{c}\right) \quad \text{and} \quad \frac{a_2}{c}\left(Q_1 + \frac{a_1 Q_2}{c}\right).$$

By calculating and compensating the effect of the new images we can proceed to the next approximation.

The potential of each sphere will then be equal to the sum of the images which fall at the centre of the sphere divided by its radius and will be therefore respectively

$$\phi_1 = Q_1\left(\frac{1}{a_1} - \frac{a_2}{c^2 - a_2^2} + \dots\right) + Q_2\left(\frac{1}{c} - \dots\right),$$

$$\phi_2 = Q_1\left(\frac{1}{c} - \dots\right) + Q_2\left(\frac{1}{a_2} - \frac{a_1}{c^2 - a_1^2} + \dots\right),$$

so that the coefficients of potential of the spheres are

$$b_{11} = \frac{1}{a_1} - \frac{a_2}{c^2 - a_2^2} + \dots, \qquad b_{22} = \frac{1}{a_2} - \frac{a_1}{c^2 - a_1^2} + \dots;$$

$$b_{12} = b_{21} = \frac{1}{c} - \dots.$$

It is more usual however to assume that the potentials are ϕ_1 and ϕ_2, given quantities, and to calculate the charges. This is easily done in the same way: to a first approximation the charges on the spheres are $a_1\phi_1$, $a_2\phi_2$ respectively: on inserting the images of these charges we get the next approximation just as above. In this way it is found that the coefficients of induction and capacity are

$$c_{11} = a_1 + \frac{a_1^2 a_2}{c^2 - a_2^2} + \frac{a_1^3 a_2^2}{(c^2 - a_2^2)^2 - a_1^2 c^2},$$

$$c_{12} = c_{21} = -\frac{a_1 a_2}{c} - \frac{a_1^2 a_2^2}{c(c^2 - a_1^2 - a_2^2)} + \dots,$$

$$c_{22} = a_2 + \frac{a_2^2 a_1}{c^2 - a_1^2} + \frac{a_1^2 a_2^3}{(c^2 - a_1^2)^2 - a_2^2 c^2},$$

as far as $\frac{1}{c^3}$*.

138. The energy and ponderomotive forces in the field of a system of conductors. We have next to consider the mechanical relations of the systems of conductors whose electrical conditions have just been examined. It is ultimately by the mechanical forces which these conductors exert on one another that we are able to examine and measure their state of electrification, and in fact it was only in this way that such a state was discovered at all. A precise formulation of the connection between the electrical and mechanical conditions of the conductors is therefore essential to a complete theory, in as far as it indicates the ultimate means of testing that theory.

We consider the case of n conductors in the field carrying charges Q_1, Q_2, ... Q_n and at potentials $\phi_1, \phi_2, \dots \phi_n$. Now the work required to bring up small increments of charge $\delta Q_1, \delta Q_2, \dots \delta Q_n$ to the respective conductors from a remote distance is

$$\Sigma (\phi_r - 0)\, \delta Q_r = \Sigma \phi_r \delta Q_r.$$

This follows directly from the work definition of the potential ϕ at any point of the field.

The work required to increase the charges on the conductors by the specified amounts is thus $\quad W = \Sigma \phi_r \delta Q_r,$

which is the fundamental differential form of the characteristic equation of energy for the system. If it is integrable, i.e. if a potential energy function W exists, then we deduce at once that

$$\frac{\partial W}{\partial Q_r} = \phi_r, \qquad (r = 1, 2, \dots n)$$

* The first determination of the problem of two spheres was given by Poisson, *Mém. de l'inst.* XII. 1, p. 1 (1811), 2, p. 163 (1811). The present solution is due to Kelvin, *Phil. Mag.* 1853 (Reprint, p. 86). In Maxwell's *Treatise* (ed. 1892), J. J. Thomson discusses a solution by Kirchhoff and gives further references.

but the differential form contains all these n partial equations in one expression. Thus if we know the energy we also know the potentials. These n equations involve the following relations

$$\frac{\partial \phi_r}{\partial Q_s} = \frac{\partial \phi_s}{\partial Q_r} = \frac{\partial^2 W}{\partial Q_r \partial Q_s}, \qquad \begin{pmatrix} r = 1, 2, \dots n \\ s = 1, 2, \dots n \end{pmatrix}$$

and conversely, if all these relations are true then δW is a complete differential, the function W exists and then the partial equations are also true. The truth of these reciprocal relations is thus the analytical criterion for the existence of the potential of the system and then the rest is a direct expression for the energy principle.

As a matter of fact we know that in the case under consideration the potentials $\phi_1, \phi_2, \dots \phi_n$ are linear functions of the charges $Q_1, Q_2, \dots Q_n$. Therefore W is a quadratic function of these quantities and by Euler's theorem

$$2W = \frac{\partial W}{\partial Q_1} Q_1 + \frac{\partial W}{\partial Q_2} Q_2 + \dots.$$

But
$$\frac{\partial W}{\partial Q_r} = \phi_r,$$

so that
$$W = \tfrac{1}{2}\Sigma \phi_r Q_r\text{*}.$$

If we multiply each charge by half its potential and add up over the whole system we get the general value for the potential energy of the system relative to the configuration in which all the conductors are at zero potential.

139. Having thus determined the potential energy of the system we can at once proceed to an examination of the mechanical forces between the conductors, the mere existence of which is involved in the idea of the energy of the system. To obtain these forces we need only give the conductors small virtual displacements and include the virtual work in these displacements in the general expression of the work done on the system during a general virtual change in its configuration.

If the positions of the various conductors are determined by the generalised coordinates $\theta_1, \theta_2, \dots \theta_n$, in the usual Lagrangian sense and if the force components corresponding to the coordinates are $\Theta_1, \Theta_2, \dots \Theta_m$ then the work done on the system during a general virtual displacement is

$$-\sum_{s=1}^{m} \Theta_s \delta\theta_s.$$

The work done in bringing up small additional charges to the conductors is

$$\sum_{r=1}^{n} \phi_r \delta Q_r,$$

if the conductors are fixed; if however in addition the conductors receive a small virtual displacement we must add the work done against the forces

* Kelvin, *Glasgow Phil. Soc. Proc.* III. (1853), *Math. and Phys. Papers*, I. § 61. Helmholtz, *Über die Erhaltung der Kraft* (1847).

acting in these displacements. The general formula for the work done on the system during the most general virtual change in its configuration is

$$\delta W = \sum_{r=1}^{n} \phi_r \delta Q_r - \sum_{s=1}^{m} \Theta_s \delta\theta_s,$$

and the principle of the conservation of energy asserts that this δW must be an exact differential, or that the function W exists; otherwise we should have perpetual motion.

Thus when W is obtained as a function of the Q's we have as before

$$\frac{\partial W}{\partial Q_r} = \phi_r, \qquad (r = 1, 2, \ldots n)$$

and also

$$\frac{\partial W}{\partial \theta_s} = -\Theta_s, \qquad (s = 1, 2, \ldots m)$$

from which a whole series of reciprocal relations of the type

$$\frac{\partial \phi_r}{\partial \theta_s} = -\frac{\partial \Theta_s}{\partial Q_r}$$

can be deduced. The principle of the conservation of energy leads to all these relations; in fact the experimental test of the conservative system is that all these reciprocal relations hold. If they are true then we know that the energy in the system is conserved, i.e. none of it is frittered away into heat. The real property which indicates conservation of the energy is of course the reversibility of changes in the system. If all the operations for changing the system from one configuration to another can be reversed the system is conservative. The above relations are merely the analytical form of this property.

140. The relations so far discussed depend on an expression for W in terms of the charges on the conductors: the form of results is therefore that suitable for the discussion of the relations in the system when the conductors are insulated and their charges are constant. We want however to be able to extend the results to a system in which some or all of the bodies are maintained at constant potential, perhaps by connecting them to batteries. The most suitable form of W is then its expression in terms of the ϕ's.

What now are the relations when W is expressed in terms of the ϕ's? To obtain them we proceed by a universal method, viz. to get an expression for δW involving $\delta\phi_1, \delta\phi_2, \ldots$ instead of $\delta Q_1, \delta Q_2, \ldots$. The procedure is to write

$$\phi_1 \delta Q_1$$

in another form; it is in fact

$$\delta(\phi_1 Q_1) - Q_1 \delta\phi_1,$$

so that we have

$$\delta\left(W - \sum_{r=1}^{n} \phi_r Q_r\right) = -\sum_{r=1}^{n} Q_r \delta\phi_r - \sum_{s=1}^{m} \Theta_s \delta\theta_s,$$

and this is the required transformation.

The usual argument shows that

$$\delta W' = \delta (W - \Sigma\phi_r Q_r)$$

is an exact differential and so we can deduce another set of reciprocal relations of type

$$\frac{\partial Q_r}{\partial \theta_s} = \frac{\partial \phi_r}{\partial \Theta_s}. \qquad \begin{pmatrix} r = 1, \ 2, \ldots n \\ s = 1, \ 2, \ldots n \end{pmatrix}$$

This set of reciprocal relations must of course be equivalent to the first set because they both are the analytical test of the principle of the conservation of energy. It is in fact an easy example in the calculus to show that the one set can actually be deduced from the other. Either set is in itself a complete expression for the conservation of the energy.

141. In the general case the new function W' introduced above is not the energy; it is an entirely new function, but in the present case it so happens since

$$2W = \Sigma Q_r \phi_r,$$

that

$$W' = - W,$$

and so we have

$$\delta W = \Sigma Q_r \delta\phi_r + \Sigma \Theta_s \delta\theta_s.$$

If then we use the two symbols W_Q, W_ϕ to represent the energy, when this is expressed as a function of the Q's and ϕ's respectively we have the two following expressions for δW_Q and δW_ϕ, which are the characteristic equations representing the principle of energy

$$\delta W_Q = \Sigma\phi_r \delta Q_r - \Sigma \Theta_s \delta\theta_s,$$

$$\delta W_\phi = \Sigma Q_r \delta\phi_r + \Sigma \Theta_s \delta\theta_s.$$

The first is equivalent to a series of relations of the types

$$\text{(i)} \quad \frac{\partial W_Q}{\partial Q_r} = \phi_r, \qquad (r = 1, \ 2, \ldots n)$$

$$\text{(ii)} \quad \frac{\partial W_Q}{\partial \theta_s} = - \Theta_s, \qquad (s = 1, \ 2, \ldots m)$$

which are to be compared with the equivalent series for the second

$$\text{(i)} \quad \frac{\partial W_\phi}{\partial \phi_r} = Q_r, \qquad (r = 1, \ 2, \ldots n)$$

$$\text{(ii)} \quad \frac{\partial W_\phi}{\partial \theta_s} = + \Theta_s. \qquad (s = 1, \ 2, \ldots m)$$

The striking contrast is in the last (ii) in each set. When the bodies are insulated the forces exerted by the system tend to diminish the potential energy; the equation of energy being

$$\delta W_Q = - \Sigma\Theta_s \delta\theta_s,$$

the charges being constant. This is of course right as the work of these forces has to come out of the potential energy of the system.

On the other hand when the conductors are maintained at constant potential (by connection with batteries) they exert forces tending to increase their energy

$$\delta W_\phi = \Sigma \Theta_s \delta \theta_s.$$

The forces are of course still the same as in the previous case, but the regions into which the conductors tend to move are now regions of greater electrical energy.

The batteries have now to supply not only the increased electrical energy δW_ϕ but also the equal amount of energy for the work done by the mechanical forces $\Sigma P_s \delta p_s$. Energy flows out of the batteries and half of it goes in mechanical work and the other half in increasing the intrinsic electrical energy.

Thus everything is summed up in the doctrine of energy, as long as we confine ourselves to static systems.

142. In the case of a mixed system in which some of the conductors are maintained at constant potential and others are insulated the relations can be obtained by a partial transformation of δW_Q, of the form

$$\delta\,(W - \sum_{r=1}^{h} Q_r \phi_r) = -\sum_{r=1}^{h} Q_r \delta\phi_r + \sum_{r=h+1}^{n} \phi_r \delta Q_r - \sum_{s=1}^{m} \Theta_s \delta\theta_s,$$

and the right-hand side is again an exact differential of some function W'. But now this function is something quite new and different from W.

An interesting point to notice is the analogy between the analysis here exemplified and that which occurs in the examination of the Hamiltonian transformation of the equations of dynamics.

143. Of course if the full details of the charge distribution on the conductors are known we need not resort to these general methods of treatment, as the forces and energy themselves can be immediately deduced by simple integrations when the results of § 67, chapter I are taken into account. Let us consider for example the simple case of the forces on a single conductor in the field. The force inside the conductor is zero and outside it is normal to the surface so that the average force on each element of charge on the surface is a normal one of amount $\frac{1}{2}\mathbf{E}_n$ per unit charge at the point. There is thus a resultant force on each element of a charged conductor which is normal to the surface at the place and of amount

$$\tfrac{1}{2}\mathbf{E}_n \sigma$$

per unit area; σ is the density of the charge distribution at the point on the surface. This force is an outward pull. If the conductor is surrounded by air then

$$\sigma = \frac{\mathbf{E}_n}{4\pi},$$

so that the normal force per unit area on its surface amounts to

$$\frac{E_n^2}{8\pi} = 2\pi\sigma^2$$

at any point.

We can thus evaluate the force on the conductor by mere integration of the forces on each element as thus specified.

This result was first given by Coulomb.

144. On electrical condensers. The most important applications of the foregoing principles are to the construction of instruments for the exact measurement of the electrical conditions of conductors, or in fact of any other body: such measurements have not only a theoretical interest in that they provide us with the ultimate basis for our theory, but they are of great practical importance in the technical applications which have been made of the principles of this subject.

The method is to work out the theory as exactly as possible for some simple type of conductor system and then to construct these conditions in an actual case so that the essential consequences of the theory may be tested as to their validity. In all simple systems of conductors the circumstances are always more or less complicated by the presence in the surrounding field of other conducting bodies, and the protection afforded by partial screening is not always theoretically satisfactory. Means have therefore to be adopted for restricting the fields under examination, and for such purposes the condensing arrangement of conductors is most effective. In this arrangement the field of the charges on the conductors is always more or less confined within certain simple and easily defined spaces, so that the circumstances in it are susceptible of exact mathematical specification.

The theory of the action of a condenser may be treated as an example of the foregoing general theory; but we can with certain advantages discuss it from a more elementary standpoint. The subject is best approached by the consideration of a simple example of the ordinary type of electrostatic field surrounding conductors as discussed above.

145. Let us consider the electric field surrounding two charged conductors so placed as to have a large part of both surfaces very close together and as nearly parallel as possible (see diagram). We shall first consider that the conductors have equal and opposite charges so that all the lines of force from one conductor go to the other and they cut each conductor normally.

Thus at a point between the near surfaces the lines of force are practically straight portions of the common normal to the two surfaces. The tubes of force are therefore practically cylindrical and have the same cross-section all the way across. The force is therefore uniform between the surfaces and

the surface densities on the opposed faces are equal but opposite. Thus if $\mathbf{E}$ is the intensity of force in any tube whose length is t and if σ is the density of the charge at each end of this tube ($+\sigma$ on one end and $-\sigma$ on the other) then

$$\mathbf{E} = 4\pi\sigma,$$

and since $\mathbf{E}$ is uniform all across the difference of potential $\phi_1 - \phi_2$ between the two conductors is given by

$$\phi_1 - \phi_2 = \int \mathbf{E}dt = \mathbf{E}t,$$

and thus

$$\sigma = \frac{\phi_1 - \phi_2}{4\pi t},$$

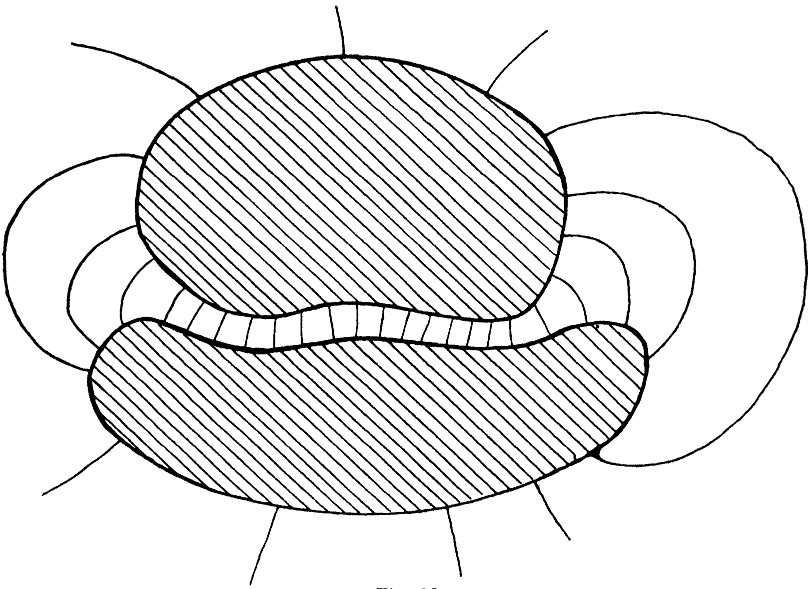


Fig. 36

and is positive on one side and negative on the other. The density of the distribution on the surfaces is thus inversely as the distance across. If the surfaces are very close together and the difference of potential is finite, σ must be very large.

Now consider any other line of force not between the near surfaces. Such a line is much longer than those between the surfaces but the total fall in potential is just the same, the rate of fall, i.e. the force intensity along such lines must therefore be very small and therefore also the charges at each end on the conductor are also small. Thus nearly the whole of the charge is concentrated on the near opposing faces of the conductors; the two equal

charges as it were bind one another together there and prevent spreading out. At the edges and outer parts of the conductors the field and charges are uncertain and difficult to estimate, but in ordinary cases when the opposing areas are big we can neglect the field and charges beyond that in the space between the conductors.

The total charge on either face is

$$Q = \int \sigma df,$$

and this is also practically the charge on the conductors: thus

$$Q = \int \frac{\phi_1 - \phi_2}{4\pi t} ds = \frac{\phi_1 - \phi_2}{4\pi} \int \frac{df}{t}.$$

Thus Q is proportional to $\phi_1 - \phi_2$; the constant of proportionality being

$$C = \frac{1}{4\pi} \int \frac{df}{t} * .$$

If the surfaces are very close together, or at least if C is large, and there is a finite difference of potential, the charges are equal and opposite and also very large. The arrangement is therefore called a condenser; it accumulates a large charge for an ordinary difference of potential. The constant C is called the capacity of the condenser; so that

$$Q = C(\phi_1 - \phi_2).$$

The capacity is equal to the charge when the difference of potential is unity.

146. The commonest form of condenser in actual use is known as the Leyden jar and is illustrated diagrammatically in the figure. It consists of a vessel made of thin glass; the inside and outside surfaces of this vessel are coated with tin foil. An electrode is connected to the inside of the jar in order that electrical connection can easily be made with it. If A is the area of each coat of tin foil, t the thickness of the glass, i.e. the distances between the surfaces of the tin foil, then, neglecting the effect of the glass as a dielectric medium the capacity is approximately

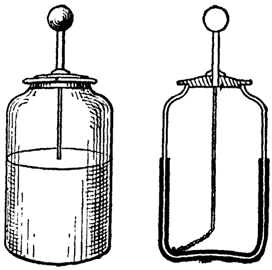


Fig. 37

$$\frac{A}{4\pi t}.$$

For purposes of more exact measurement however it is usual to adopt forms of conductors more readily susceptible of mathematical treatment and in such cases the simpler types indicated below are adopted.

* Green, *Essay etc.*, pp. 43–45. The second approximation is also obtained by this author.

147. *The parallel plate condenser.* We are given two parallel conducting plates with equal and opposite surface densities σ. The distance between the two plates is so small compared with their dimensions that we may regard them as very large and neglect any irregularities arising from the edges.

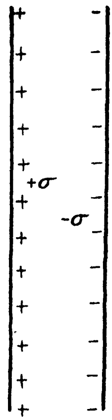


Fig. 38

From the symmetry of the arrangement we may assume that the lines of force are normal to both planes and go straight across between them. The tubes of force are cylindrical and thus the electric force $E = 4\pi\sigma$ is uniform all across. From the previous general considerations we may conclude that the field exists only between the plates; in reality there is a field in the surrounding space and also charges on the backs of the plates, but when the capacity of the arrangement is large these are all negligible in comparison with the field and charges between the plates.

If the plates are at a distance t apart and the potentials are ϕ_1 and ϕ_2

$$Et = \phi_1 - \phi_2,$$

so that

$$\phi_1 - \phi_2 = 4\pi\sigma t.$$

Thus if A is the area of a plate the charge on it is

$$A\sigma = \frac{A}{4\pi t}(\phi_1 - \phi_2).$$

The capacity of this arrangement is therefore

$$\frac{A}{4\pi t},$$

or $\frac{1}{4\pi t}$ per unit area; in agreement with our general formula $\frac{1}{4\pi}\int\frac{df}{t}$.

148. *The spherical condenser.* The two spherical radii are a and $b > a$. The sphere a carries the charge $+Q$ and the sphere b the charge $-Q$. The whole space is divided into three parts.

(i) Inside the inner sphere a there is no electric intensity, it is internal to both spherical distributions $E_1 = 0$.

(ii) In the space between the spheres the electric intensity is due to the inner sphere alone and is

$$E_2 = \frac{Q}{r^2}.$$

(iii) In the space outside the two spheres the field is due to both spheres

$$E_3 = \frac{Q}{r} - \frac{Q}{r} = 0.$$

The electrical field is therefore entirely confined to the space between the spheres and then

$$-\frac{\partial\phi}{\partial r} = \frac{Q}{r^2},$$

and thus
$$\phi_1 - \phi_2 = \int_a^b \frac{Q}{r^2}\,dr = Q\left(\frac{1}{a} - \frac{1}{b}\right),$$

where ϕ_1 and ϕ_2 are the potentials of the spheres. The capacity is

$$C = \frac{ab}{b-a}.$$

If the sphere b becomes infinitely large then

$$\frac{ab}{b-a} = \frac{a}{1-\dfrac{a}{b}} = a,$$

so that the capacity of the sphere a alone is equal to a, a result which we might have deduced from previous results.

149. *The cylindrical condenser.* By proceeding in a manner exactly analogous to the above we may find the capacity per unit length of a condenser formed of infinite coaxial cylinders of radii a and b. If the charges are Q and $-Q$ per unit length the field between the cylinders is radial and at a distance r from the axis is $\dfrac{2Q}{r}$; thus integration for the potential gives

$$\phi_1 - \phi_2 = \int_a^b \frac{2Q}{r}\,dr = 2Q\left(\log\frac{b}{a}\right),$$

so that the capacity per unit length is

$$C = \frac{1}{2\log\dfrac{b}{a}}.$$

150. These results are the particular cases of a more general theorem.

If we know the equi-potentials in the field of any freely charged conductor C, then we can determine the capacity of any condenser formed by conducting surfaces in the shape and relative position of any two equi-potentials in this field say A and B.

Suppose ϕ is the potential function of the conductor C with a charge Q and let the values of ϕ on A and B be denoted by ϕ_A and ϕ_B.

Now consider the problem of the determination of the potential for the two conductors A and B charged in any manner with quantities Q_A and Q_B. We have to find a regular potential function ϕ' which will be constant on A and B and give the right charges.

Now supposing the surface B encloses the surface A; then a possible form for the potential of this new distribution is obviously

$$\phi_1 = a_1 \phi_A,$$
$$\phi_2 = a_2 \phi + b\phi_B,$$
$$\phi_3 = a_3 \phi,$$

the suffices 1, 2, 3 referring respectively to the three regions the first inside A, the second between A and B and the third outside B.

These potentials must of course be continuous at the surfaces, requiring that

$$\phi_A a_1 = a_2 \phi_A + b\phi_B,$$
$$a_2 \phi_B + b\phi_A = a_3 \phi_B,$$

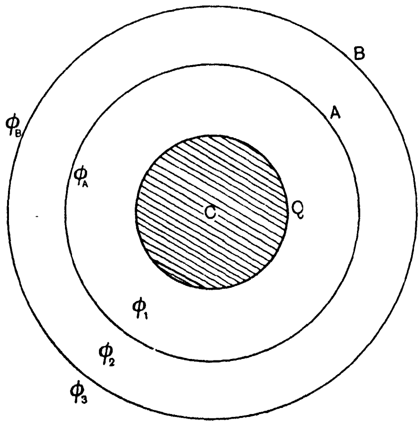


Fig. 39

and they must give the charges right, requiring that

$$Q_A = -\frac{a_2}{4\pi}\int_{f_A} \frac{\partial \phi}{\partial n}\, df_A,$$

$$Q_B = +\frac{a_2}{4\pi}\int_{f_B} \frac{\partial \phi}{\partial n}\, df_B - \frac{a_3}{4\pi}\int_{f_B} \frac{\partial \phi}{\partial n},$$

but remembering that A and B are surfaces surrounding C in the first case we see that each of the surface integrals on the right of these equations is equal to $-4\pi Q$. Thus

$$Q_A = a_2 Q,$$
$$Q_2 = (a_3 - a_2)\, Q,$$

and there are therefore enough equations to determine a_1, a_2, a_3 and b; solving them we find

$$\phi_1 = \frac{Q_A\phi_A + Q_B\phi_B}{Q},$$

$$\phi_2 = \frac{Q_A\phi + Q_B\phi_B}{Q},$$

$$\phi_3 = \frac{Q_A + Q_B}{Q}\phi.$$

In the particular case when $Q_A = -Q_B = Q$ these potentials are

$$\phi_1 = \phi_A - \phi_B,$$

$$\phi_2 = \phi - \phi_B,$$

$$\phi_3 = 0,$$

and the field is confined to the space between the conductors. The system then acts as a condenser whose capacity is

$$Q/(\phi_A - \phi_B).$$

Numerous results can be deduced as special cases of this general formula.

151. *Combinations of condensers.* It often happens that the capacities of the separate condensers at our disposal are not sufficiently large for the purposes in hand, or they may be too large to produce the required potential with a given charge. It is however possible to connect a number of such condensers so as to secure these advantages. We consider n condensers in space, so distant from one another that they do not mutually influence each other.

(i) Parallel condensers: we connect the positive plates of all the condensers together and all the negative ones as well. The total charge in the system

$$Q = \Sigma Q_r = \sum_{r=1}^{n} C_r(\phi_1 - \phi_2),$$

and ϕ_1 and ϕ_2 are the same for all the positive and negative plates respectively, so that if we write

$$Q = C(\phi_1 - \phi_2),$$

$$C = \sum_{r=1}^{n} C_r,$$

the capacity of this arrangement is the sum of the capacities of the separate condensers.

(ii) Condensers in series: we connect the negative plate of each condenser with the positive plate of the next succeeding condenser and charge the first plate of the first condenser to potential ϕ_1 and the second plate of the last to potential ϕ_2; the potential of the positive plate of the $(r+1)$th condenser is then equal to that of the negative plate of the rth or

$$\phi_2^{(r)} = \phi_1^{(r+1)}.$$

Moreover the charges on all of the plates are equal to Q, so that

$$\phi_1^{(r)} - \phi_2^{(r)} = \frac{Q}{C_r},$$

and consequently

$$\phi_1 - \phi_2 = Q\Sigma\frac{1}{C_r} = \frac{Q}{C}.$$

The reciprocal of the capacity of the combination is obtained by adding the reciprocals of the separate capacities.

Thus if we want to get a higher potential difference than that usually at our disposal we need only charge a battery of condensers in parallel and then rearrange them in series. In the first place, if the condensers are equal in all respects

$$nQ = nC\phi,$$

where ϕ is the directly accessible potential difference and C the capacity of a single condenser of the battery, and then afterwards

$$Q = \frac{C\phi'}{n},$$

so that $\phi' = n\phi$*.

152. On electrometers and their use. Having now secured certain simple conductor systems whose behaviour can be accurately specified, we can compare the actual mechanical behaviour of these systems with that which might be expected on theoretical grounds and thus secure a decisive test for the theory. Having then convinced ourselves of its general validity we may reverse the argument and regard the mechanical relations of the system as determining its electrical conditions, and then the arrangement is called an *electrometer*. The principal types of such instruments, practically all due to Lord Kelvin, will now be briefly described.

If two parallel metal plates of area A at potentials V_1 and V_2 form a condenser whose capacity is

$$\frac{A}{4\pi t},$$

if we disregard the complications at the edges as being too small to affect matters.

The energy of this combination is

$$W = \tfrac{1}{2}\Sigma QV = \tfrac{1}{2}A\sigma(V_1 - V_2),$$

where σ is the surface density of the charge on the plates; but

$$4\pi\sigma = \frac{V_1 - V_2}{t},$$

so that

$$W = 2\pi A\sigma^2 t$$

$$= \frac{A}{8\pi}\frac{(V_1 - V_2)^2}{t}.$$

* Kelvin, *Proc. R. S* (1867), *Reprint*, § 352, also §§ 401–426 and §§ 427–9.

The force tending to increase t is

$$-\frac{\partial W_Q}{\partial t} = -\frac{\partial W_V}{\partial t} = -2\pi A\sigma^2 = -\frac{A}{8\pi}\left(\frac{V_1 - V_2}{t}\right)^2,$$

so that there is really an attraction per unit area of amount $2\pi\sigma^2$.

This is the idea of Kelvin's absolute electrometer (trap door)*. The one plate of the condenser is hung on a balance and the force of attraction towards the other plate measured by compensating it by a weight. To eliminate the irregularities at the edges of the plates which are not taken account of in the theory Thomson extends the plates by surrounding them by so-called *guard-rings*. In this way the field between the effective parts of the plates is made to conform to that theoretically described.

153. *The cylindrical electrometer* †. Consider now another similar case: two coaxal cylinders one of which (the outer) is fixed vertically and the other is suspended inside it but partly projecting at the upper end. If both cylinders are charged the inner cylinder will be sucked down into the other; with what force?

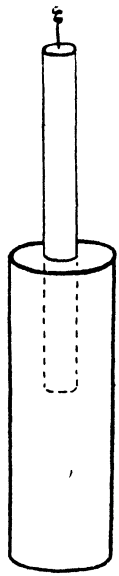


Fig. 40

In the middle the charge distribution is uniform and even; it is only near the ends that the unknown irregularity in the distribution occurs. The unknown distributions are however not pertinent because they do not alter to any appreciable extent when the inner cylinder drops down a short distance; provided of course that there is a good length of either cylinder projecting at each end. The unknown distributions remain the same whatever the depth of immersion and they therefore do not matter. The energy W of the arrangement may therefore be calculated in the form

$$W = W_r + C,$$

where W_r is the energy calculated as if the distribution to the actual depth z of immersion were uniform and there were nothing else and C is the constant correction to be added for the irregularities at the ends. If the cylinders are circular (radii a, b) and the difference of potential is ϕ

$$W = \frac{z\phi^2}{4\log\frac{b}{a}} + C,$$

the capacity per unit length being

$$\frac{1}{2\log\frac{b}{a}}.$$

* W. Thomson, *Reprint of Papers on Electricity and Magnetism*, §§ 358, 360, *Phil. Mag.* (4), VIII. (1854), p. 42.

† W. Thomson, *Reprint etc.* p. 38. Cf. also Maxwell, *Treatise*, I. § 129. Bichat and Blondlot, *Jour. de Physique* (2), V. (1886), p. 325.

There is therefore a force

$$\frac{\phi^2}{4\log\dfrac{b}{a}},$$

drawing the inner cylinder down; a knowledge of which in a particular case would enable a determination of ϕ to be made.

The importance of this type of electrometer is that its capacity can be varied at will by simply lowering or raising the cylinders.

154. *The quadrant electrometer**. This is probably the most suitable instrument for determining relative potential differences. It consists of a flat cylindrical metal case divided into four equal quadrants by perpendicular central sections. The two opposite quadrants (diagonally) are in each case metallically connected and are at potentials ϕ_1 and ϕ_2: a flat paddle shaped needle with its plane parallel to the top and bottom of the box and which is capable of turning about the vertical axes of symmetry of the box, is at a potential ϕ_3. If $\phi_1 = \phi_2 = \phi_3$ the paddle lies symmetrically between the quadrants. The whole apparatus is enclosed in a metal case at zero potential. The charges on the three portions are

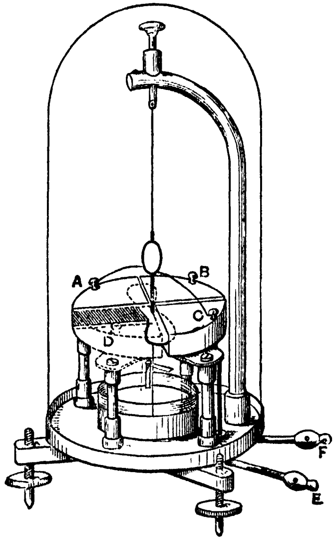


Fig. 41

$$Q_1 = c_{11}\phi_1 + c_{12}\phi_2 + c_{13}\phi_3,$$
$$Q_2 = c_{21}\phi_1 + c_{22}\phi_2 + c_{23}\phi_3,$$
$$Q_3 = c_{31}\phi_1 + c_{32}\phi_2 + c_{33}\phi_3,$$

or since $c_{rr} > 0$ and $c_{rs} < 0$ we can write this in the form

$$Q_1 = a_{12}(\phi_1 - \phi_2) + a_{13}(\phi_1 - \phi_3) + a_1\phi_1,$$
$$Q_2 = a_{21}(\phi_2 - \phi_1) + a_{23}(\phi_2 - \phi_3) + a_2\phi_2,$$
$$Q_3 = a_{31}(\phi_3 - \phi_1) + a_{32}(\phi_3 - \phi_2) + a_3\phi_3,$$

where all the a's are positive and $a_{rs} = a_{sr}$.

The energy is

$$W = \tfrac{1}{2}\{a_{12}(\phi_1 - \phi_2)^2 + a_{13}(\phi_1 - \phi_3)^2 + a_{23}(\phi_2 - \phi_3)^2 + a_1\phi_1{}^2 + a_2\phi_2{}^2 + a_3\phi_3{}^2\}.$$

The couple on the needle is

$$G = \frac{\partial W}{\partial \theta},$$

* W. Thomson, *Reprint etc.* § 345

where θ is the angle turned through from the position of rest. Now the construction of the needle enables us to deduce the dependence of the α's on θ. The broad form of the paddle and the smallness of the sections between the quadrants show that very few, if any, lines of force get away from the edges of the paddle to the case, so that in an infinitely small displacement $d\theta$ the number of lines of force which run from the paddle to the second pair of quadrants increases by a quantity proportional to $d\theta$. It thus follows by symmetry that*

$$\frac{\partial \alpha_{23}}{\partial \theta} = -\frac{\partial \alpha_{13}}{\partial \theta} = k, \text{ say,}$$

whilst all the other constants are independent of θ, because the number of lines of force which go from the quadrants or needle to the case or from one quadrant to the other, do not vary during the small displacement of the paddle.

Thus
$$G = \frac{k}{2}[(\phi_2 - \phi_3)^2 - (\phi_1 - \phi_3)^2]$$
$$= k[\phi_1 - \phi_2]\left[\phi_3 - \phi_1 + \frac{\phi_1 - \phi_2}{2}\right].$$

155. Different arrangements enable us to determine various potential differences. The couple is always independent of the position of the needle provided k is constant and is determined by balancing it against the torsion couple of its suspension. If it is not too large the angular deflection of the needle is proportional to the couple which balances it in the final position of equilibrium of the needle. If $|\phi_2 - \phi_3| > |\phi_1 - \phi_3|$ the couple is positive and the needle is thus always drawn towards the quadrants whose potential differs most from its own.

If the needle is connected to one pair of quadrants, i.e. if $\phi_1 = \phi_3$, then

$$G = \frac{k}{2}(\phi_2 - \phi_3)^2$$

is proportional to the square of the potential difference $(\phi_2 - \phi_3)$. This arrangement is adopted for the accurate measurement of large differences of potential.

If on the other hand $(\phi_1 - \phi_3)$ is always very large compared with $\phi_1 - \phi_2$, then we can write approximately

$$G = k(\phi_1 - \phi_2)(\phi_1 - \phi_3),$$

and thus if $(\phi_1 - \phi_3)$ is maintained constant G is proportional to $\phi_1 - \phi_2$. This furnishes a very convenient arrangement for comparing small differences of potential and for this purpose the paddle is very highly charged, so that ϕ_3 is large, so that we may even write

$$G = -k(\phi_1 - \phi_2)\phi_3.$$

* The positive sign for k is chosen to correspond to the positive direction of quadrantal order 1–2 for increasing θ.

If as is often the case the second pair of quadrants is earthed then $\phi_2 = 0$ and then

$$G = -k\phi_1\phi_3.$$

156. This instrument may also be used for measuring a charge of electricity and the arrangement adopted is similar to that last described for measuring small potential differences. The paddle is highly charged whilst both pairs of quadrants are originally earthed: a quantity Q_0 of electricity will then be induced symmetrically; and if ϕ_3 is the potential of the paddle

$$Q_0 = c\phi_3,$$

where c is the common value of c_{13} and c_{32} when the paddle is in the symmetrical position. If now one pair of quadrants is insulated and given a charge Q the paddle will be deflected and in the final position we shall have

$$Q_1 = Q + Q_0 = c_{11}\phi_1 + c_{13}\phi_3,$$

so that

$$Q = c_{11}\phi_1 + (c_{13} - c)\phi_3.$$

The couple on the paddle is still very approximately given by

$$G = k\phi_1\phi_3,$$

and if the final deflection of the paddle from the symmetrical position is through the angle θ

$$G = \int_0^\theta K d\theta = K\theta,$$

and

$$c_{13} = c - \int_0^\theta k d\theta = c - k\theta,$$

if k and K are independent of θ as is usually the case with sufficient approximation. We have then

$$Q = \theta\left(\frac{c_{11}K + k^2\phi_3}{k\phi_3}\right),$$

and is proportional to θ if ϕ_3 is kept constant.

It is interesting to notice that when the potential of the needle is increased beyond a certain point the deflection of the needle due to a given charge Q on the quadrants diminishes as the potential of the needle increases, hence to obtain the greatest sensitiveness when measuring electrical charges we must be careful not to charge the needle too highly.

157. Now let us consider the practical application of this instrument to determine the potential of any conductor which exists in the field of a given system of charges. By connecting this conductor by a long thin wire to one pair of quadrants of the electrometer and the other pair of quadrants to earth, the difference in the potential between the two quadrants then measures the potential of the conductor under investigation relative to the earth*. It is

* Assuming that the capacity of the quadrants is sufficiently small compared with that of the conductor, unless this latter is maintained at a constant potential.

assumed that the wire is so thin that its presence does not affect the conditions of the system and that the electrometer is so far removed from the system so as not to affect it or be affected by it.

The charges on the conductors could be determined by transferring them to one pair of quadrants and determining them in the manner described above, but this is in general not very practical and it is usual to determine what is known as the capacity of the conductor whence the charge can be inferred from its potential. This is easily done provided three condensers are available and the capacity of one of them can be varied. The three condensers may be of any of the simple theoretical types illustrated above.

We need only discuss the problem of determining the capacity of a condenser consisting of two conductors: the capacity of a single conductor is in reality the capacity of the condenser formed by it and the earth. Let therefore A, B be the plates of the condenser whose capacity is required:

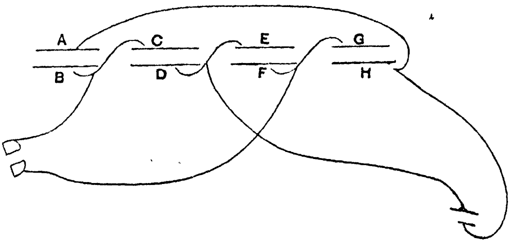


Fig. 42

C, D; E, F; G, H the plates of three condensers whose capacities are known. Connect plates BC together and to one pair of quadrants of an electrometer*; also connect F and G together and to the other quadrants. Connect DE together and to one pole of an apparatus for producing a potential difference, and connect AH together and to the other pole of this apparatus. In general this will cause a deflection of the electrometer; if there is a deflection then we must alter the capacity of the condenser whose capacity is variable until the vanishing of this deflection shows that the plates BC, FG are at the same potential. When this is the case a simple relation exists between the capacities.

Let b_1, b_2, b_3, b_4 be the capacities of the respective condensers. Let ϕ be the potential of AH, ϕ' that of BC and therefore also of FG when there is no deflection in the galvanometer, ϕ_0 that of D and E.

The charge on B is then $\quad b_1(\phi' - \phi)$,

* It is again assumed that the capacity of the quadrants is very small; otherwise it has to be eliminated or allowed for by additional measurements. In actual practice different methods are employed depending on the theory of electric current flow. Cf. Pidduck, *A Treatise on Electricity* (Cambridge, 1916), p. 143.

whilst that on C is $$b_2(\phi' - \phi_0);$$
these two must be equal and opposite so that
$$b_1(\phi' - \phi) = b_2(\phi_0 - \phi').$$
Again, the charges on F and G must be similarly equal and opposite and they are respectively
$$b_3(\phi' - \phi_0), \qquad b_4(\phi' - \phi),$$
so that we may conclude that
$$b_1 b_3 = b_2 b_4,$$
and thus $$b_1 = \frac{b_2 b_4}{b_3},$$
and is known as soon as b_2, b_4 and b_3 are known.

Thus if we have standard condensers whose capacities are known, we can measure the capacity of other conductors and condensers.

158. On the Proofs of the Inverse Square Law. In the experiments described in the first chapter it was found that on inserting an electrified body into the closed metal vessel and then bringing it into contact with the vessel the electrification entirely left the body and appeared outside the vessel; and on the withdrawal of the body there were no signs of electrification either on it or on the interior of the vessel.

This fact was found to follow as a general principle from Gauss' theorem on normal induction which depends essentially on the validity of the inverse square law for the action between the electric charge elements. Cavendish reversed the argument* and by a direct experimental determination of the limits to the probable truth of the fact that the force in the interior of a charged conductor is zero, he was able to conclude that the law of force between electrical charges can only differ very slightly from the inverse square law. Cavendish's work although carried out some time before Coulomb's determination of the law of force however remained unpublished until Maxwell saw it.

Maxwell improved slightly on the apparatus used by Cavendish and repeated the experiments and his measurements provide us with the most exact experimental verification of law which has ever been given.

In this experiment two concentric spherical conductors of which the inner one is charged are put into connection and the residual charge of the inner one is measured.

159. The first problem to solve is that of finding the potential at any point due to a uniform spherical shell, the repulsion between two units of electricity being any given function of the distance.

* Cf. Maxwell, *Treatise*, I. pp. 80–86.

Let $\phi(r)$ be the repulsion between two units at a distance r, and let $f(r)$ be such that

$$\frac{d}{dr}f(r) = f'(r) = r\int_r^\infty \phi(r)\,dr.$$

Let the radius of the shell be a, and its surface density σ, which by symmetry is uniform over the whole surface; then if Q denote the whole charge of the shell

$$Q = 4\pi a^2\sigma.$$

Now let b denote the distance of the given point from the centre of the shell and r its distance from any point of the shell. If we refer to spherical polar coordinates with centre of shell as pole and the radius b as axis then

$$r^2 = a^2 + b^2 - 2ab\cos\theta.$$

The charge on the element of shell is

$$\sigma a^2 \sin\theta\, d\theta\, d\phi,$$

and the potential due to this element at the given point is

$$\sigma a^2 \sin\theta \frac{f'(r)}{r}\, d\theta\, d\phi,$$

and since $r\,dr = ab\sin\theta\, d\theta$ this is

$$\sigma a^2 f'(r)\, dr\, d\phi,$$

which has to be integrated over the whole surface of the sphere: the result is that

$$\phi = 2\pi\sigma\frac{a}{b}\{f(r_1) - f(r_2)\},$$

where r_1 is the greatest value of r, which is always $(a + b)$ and r_2 is the least value, which is $(b - a)$ when the given point is outside the shell and $(a - b)$ when it is inside.

The potential at an external point is

$$\phi = \frac{Q}{2ab}\{f(b + a) - f(b - a)\},$$

and at a point on the shell

$$\phi = \frac{Q}{2a^2}\{f(2a)\},$$

and inside $\qquad \phi = \frac{Q}{2ab}\{f(a + b) - f(a - b)\}.$

160. Now let us determine the potentials of the shells in Cavendish's experiment on the supposition that the one has a charge Q_1 and the other a charge Q_2; the potentials ϕ_1 and ϕ_2 are given by

$$\phi_1 = \frac{Q_1}{2a^2}f(2a) + \frac{Q_2}{2ab}\{f(a + b) - f(a - b)\},$$

$$\phi_2 = \frac{Q_2}{2b^2}f(2b) + \frac{Q_1}{2ab}\{f(a + b) - f(b - a)\}.$$

In the Cavendish experiment the two spheres are in connection and so

$$\phi_1 = \phi_2 = \phi,$$

so that the charge on the inner sphere

$$Q_2 = 2b\phi \frac{bf(2a) - a\{f(a+b) - f(a-b)\}}{f(2a)f(2b) - \{f(a+b) - f(a-b)\}^2}.$$

The hemispheres forming the outer sphere are then removed to a distance and discharged and the potential of the inner shell is determined; it would be

$$\phi' = \frac{Q_1}{2b^2} f(2b).$$

If now we assume with Cavendish that the law of force is some inverse power of the distance, not differing much from the law of inverse squares, we can put

$$\phi(r) = r^{q-2},$$

and then

$$f(r) = \frac{1}{1-q^2} r^{q+1},$$

and if we suppose q small we can expand this in the form

$$f(r) = \frac{r}{1-q^2}\left\{1 + q\log r + \frac{(q\log r)^2}{2} + \dots\right\},$$

so that to a first approximation in q

$$Q_1 = \frac{a}{2(a-b)}\phi q\left[\log\frac{4a^2}{a^2-b^2} - \frac{a}{b}\log\frac{a+b}{a-b}\right],$$

from which q may be determined if Q_1 is measured. Experiment will thus determine an upper limit to q and it has in fact been shown that q must certainly be less than 10^{-5}. This is probably the best experimental test of the law that there is in existence.

161. *Laplace's proof of the inverse square law**. A second proof was given by Laplace and is based on the two definite experimental facts that there is no electric force inside a conductor in electrostatic equilibrium and that no free charge resides in the interior of the conductor, it is all on the surface. It is quite easy to show that these two results require that the law of action should be the inverse square law.

We prove that no function of the distance except the inverse square satisfies the condition that a uniform spherical distribution of surface charge exerts no force on a particle within it.

If in the previous example we suppose that Q_2 is always zero we can apply Laplace's method to determine $f(r)$. For then we have

$$bf(2a) - af(a+b) + af(a-b) = 0,$$

for all values of b: differentiating twice with respect to b we get

$$f''(a+b) = f''(a-b),$$

* *Mec. céleste*, I. p. 163.

which is only satisfied for all values of $b < a$ if

$$f''(r) = c_0 \text{ constant},$$

i.e.
$$f'(r) = c_0 r + c_1.$$

Hence
$$\int_r^\infty \phi(r)\,dr = \frac{f'(r)}{r} = c_0 + \frac{c_1}{r}.$$

Thus
$$\phi(r) = \frac{c_1}{r^2}.$$

We may observe however that though the assumption of Cavendish, that the force varies as some power of the distance, may appear less general than that of Laplace, who supposes it to be any function of the distance, it is the only one consistent with the fact that similar surfaces can be electrified so as to have similar electrical properties.

CHAPTER IV

THE FARADAY-MAXWELL THEORY OF ELECTROSTATIC ACTION

162. Action at a distance or transmission through a medium*? The fact that certain bodies after being rubbed appear to attract other bodies at a distance from them was known to the ancients. In modern times a great variety of other phenomena have been observed and found to be related to these phenomena of attraction at a distance. The first really definite notion of this interaction of two bodies at a distance apart however first arose in Newton's law of gravitational attraction which states that one body attracts another according to a simple law, an essential point being that any intermediate matter does not affect the action. This law of gravitation proved so successful in astronomy that it was made the pattern for the solution of more abstract problems in physics and attempts were made to model the whole of natural philosophy on this one principle, by expressing all kinds of material interaction in terms of laws of direct mechanical attraction across space. Of course if material systems are constituted of discrete atoms, separated from each other by many times the diameter of any one of them, this simple plan of exhibiting their interactions in terms of direct forces between them would probably be exact enough to apply to a wide range of questions, provided we could be certain that the laws of force depended only on the positions and not also on the motions of the atoms†.

Then when it was discovered that the laws of electric and magnetic actions were of the same type as that of gravitation it was only too natural to suppose that they were direct distance actions, without any reference to any medium which may occupy the space between.

This doctrine of action at a distance which thus expresses the view that our knowledge in such cases *may* be completely represented by means of laws of action at a distance expressible in terms of the positions (and possible motions) of the interacting bodies without taking any heed of the intervening space, was specially favoured by the French and German scientific schools

* Cf. the article "Aether" by Sir J. Larmor in the *Encyclopaedia Britannica* (1911). Also the book *Aether and Matter* by the same author.

† The most successful of these applications has been in the theory of capillary action elaborated by Laplace, though even here it appears that the definite results attainable by the hypothesis of mutual atomic attractions really reposed on much wider and less specia principles—those, namely, connected with the modern doctrine of energy.

and in W. Weber's hands electric theory was built up on it to a most wonderful perfection*. The doctrine was however strongly repudiated by Newton himself and hardly ever became influential in the English school of abstract physicists.

The modern view of these things, according to which the hypothesis of direct transmission of physical influences expresses only part of the facts, is that all space is filled with physical activity and that while an influence is passing across from a body A, to another body B, there is some dynamical process in action in the intervening region, though it appears to the senses to be mere empty space. The problem is of course whether we can represent the facts more simply by supposing the intervening space to be occupied by a medium which transmits physical actions, after the manner that a continuous material medium, solid or liquid, transmits mechanical disturbances. The object of the following pages is to answer this question in the affirmative following closely along the lines laid down by Faraday in his wonderful experimental researches and followed by Maxwell in his mathematical discussion of the results of these researches from which he evolved a definite justification† of the fundamental notion.

163. We are thus at the outset confronted with these two opposite notions, pure action at a distance and transmission through and by the '*aether*' of space. The two notions are to be contrasted but are in reality not so much opposed to each other as at first sight appears. Both theories can and have been logically developed and up to a certain stage give a very good account of the phenomena. There is however one question which can decide between the two theories. Are the effects of gravitational and electrical attractions propagated in time? If time is taken to transmit a condition there must be something to transmit it. If the change appears to be instantaneously set up then it is either actually instantaneous or is too fast to be observed; in either case we can neglect the medium as it is of no use talking of a medium which acts too quickly to be followed.

In electrical theory this question has been definitely settled. The machinery underlying electric and magnetic attractions is the same as that in light phenomena. Oersted was the first to state this but before his time it was an obvious guess. Ampère treated the question very fully but no one constructed a theory of it till Maxwell put Faraday's ideas into mathematics and thereby arrived at a definite formulation of the connection. Experimental confirmation of Maxwell's theory was provided twenty years after its

* An historical account of the developments of electrical theory on this basis is given with full references by Reiff and Sommerfeld in *Ency. der math. Wissench.* v. 12, pp. 1–62.

† The "justification" is not absolute. In fact in the view of the more modern "reletavists" the existence of a material aether is denied, although the mathematical form of Maxwell's theory is accepted in its entirety.

publication by Hertz, who then succeeded in producing and examining the long electric waves, whose existence was an essential consequence of Maxwell's fundamental hypothesis on the electrodynamic properties of the aether.

As far as gravitational theory is concerned the fundamental question is however still an open one. No velocity of gravitational action has yet been detected experimentally.

The great success achieved by Maxwell's theory has now made it very improbable that electric and magnetic forces (and probably also gravitational ones) acting in distances large compared with molecular dimensions are pure distance actions. The question of molecular forces is still an open one.

164. The theory of electric actions developed in the previous chapters is essentially a distance action theory, inasmuch as it depends merely on the concept of the electric charge with a definite law of action between point charges. No reference whatever is made to the medium between the charges and in fact our theory is true only if one single homogeneous medium pervades the whole of space. Such is in general not the case and the theory thus needs generalisation on the lines suggested by Faraday and worked out mathematically by Maxwell.

165. Faraday's* ideas on the nature of the electric action between charged bodies. Faraday firmly believed that the action between two charged bodies was transmitted through and by the medium between them. To test his ideas experimentally he tried to alter the medium by interposing between the charges different dielectric substances. The procedure actually adopted to obtain the exact effect was to use two equal spherical condensers, one with an air dielectric and the other with some other substance, such as sulphur. By connecting the two inner spheres of the condensers together and the outer ones to earth any charge is divided between them. If the condensers are of exactly the same size the charge should be divided equally if the presence of the sulphur makes no difference. Faraday found however that they were not equally divided; but were in a definite ratio $\epsilon : 1$; the presence of the sulphur increases the capacity of the one condenser ϵ times. This constant ϵ was found to be typical of the substance used in the second condenser and is therefore called the specific inductive capacity (S. I. C.) of the substance or simply its dielectric constant†.

In attempting to find some cause or theory of this new effect Faraday was induced to a closer investigation of the electric field in the cases when the otherwise simple circumstances are complicated by the presence of some

* Cf. *Experimental Researches*, especially I. 1231, 1613–16; III. 3070–3299.

† Dielectric constants were independently determined by Cavendish in 1773. Cf. Maxwell, *Treatise*, I. p. 54.

dielectric material, his method being to trace out experimentally the lines of force and to form them into tubes of force in the manner indicated in the previous chapter. He intuitively got the idea that the amount of electricity at one end of the tube was the same as that at the other and he then succeeded in verifying it experimentally even in the case when dielectric media of great complexity were present in the field. Moreover he discovered that the other important property of tubes of force, viz. that the product of the force intensity at any point in a tube by the cross-section at that point is constant all along, still held in the general case.

Such obviously fundamental results naturally led Faraday to think that there must be some physical cause for this equality and he put it down to some physical action in the tube. His idea was that the tube was full of an incompressible fluid and the behaviour is just as if the liquid were pushed along the tube, whatever excess is produced at one end has an equivalent diminution at the other, both being equal to the amount crossing any section of the tube during the establishment of the displacement.

This notion of Faraday's was the ruin of his recognition by contemporary mathematicians. The theory of simple attractions gave satisfactory proofs and explained everything known at the time and no new idea seemed necessary. Faraday however went deeper and tried to explain the mechanism of the attractions but he could only offer suggestions which appeared rather crude against the completeness of the attraction theory.

The essential point in this idea of Faraday's is that an electric field arises through an electric displacement or induction along the curved lines of force resulting in an accumulation of positive charge at one end and negative at the other. The term 'electric displacement' thus introduced is however not to be taken too literally in this sense. The idea that survives is that it is some vector of the same nature as the displacement of a fluid which is related in the usual manner to the lines of force which experiment maps out.

166. In the mathematical formulation of this scheme it is first necessary to define the electric force independently of the idea of simple attraction. We define the electric force intensity $\mathbf{E}$ at a point in the field so that $\mathbf{E}\delta q$ is the force on a small charge δq placed there. Before however we put the charge there we must make room for it, we must remove the matter inside a small cavity surrounding it. The question as to whether this removal affects the force at the point is reserved for future consideration and for the present we shall assume that the force intensity thus defined is a mathematically definite vector quantity.

The force $\mathbf{E}$ as thus defined is still the gradient of a potential function ϕ. A simple justification of this statement could be based, in the manner previously indicated, on the perpetual motion idea, the argument being that

under steady conditions carrying a charge round a closed circuit ought to involve no work on the whole, assuming that the conditions are the same at the end as at the beginning. We shall therefore assume the existence of a potential function ϕ at each point of the field.

We must next define the electric displacement or something akin to an electric displacement. The intensity of the displacement $\mathbf{D}_s$ in any direction at a point is such that the total amount displaced across any small area df_s perpendicular to the direction during the establishment of the field is

$$\mathbf{D}_s df_s.$$

Suppose now we consider that the field is established by slowly increasing all charges proportionally: the configuration of the field at each instant as regards the lines of force and displacement will then be similar to the final one. Now consider a tube formed by the lines of displacement drawn through any small area df, which is perpendicular to its mean axis, and let df_s be any adjacent slanting cross-section making an angle θ with df. This tube is a tube of displacement at each instant during the establishment of the field and thus the totality of displacement across any section of it is the same all along: that is

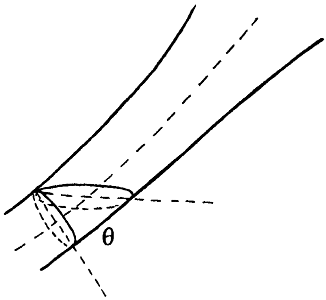


Fig. 43

$$\mathbf{D}_s df_s = \mathbf{D}\, df,$$

but $df_s \cos\theta = df$ and therefore

$$\mathbf{D}_s = \mathbf{D}\cos\theta,$$

and thus the quantity $\mathbf{D}$ so defined is an ordinary vector, its direction being at each point along the line of displacement through that point.

It is easily verified that the final result of any more complicated method of establishing the field would be expressible by the same quantity $\mathbf{D}$.

167. Thus in our electric field we have at each point two kinds of vectors:

(i) the force intensity $\mathbf{E}$ which is the gradient of a potential function and

(ii) the flux intensity $\mathbf{D}$.

Now our theory implies that $\mathbf{E}$ is the cause of $\mathbf{D}$; the displacement at a point is conditioned by the force intensity there. The simplest possible law of causality we can have is that in which there is a simple proportionality,

so that if we double the cause we double the effect. In this case the components ($\mathbf{D}_x$, $\mathbf{D}_y$, $\mathbf{D}_z$) of the displacement would be linear functions of the components ($\mathbf{E}_x$, $\mathbf{E}_y$, $\mathbf{E}_z$) of the electric force, i.e.

$$4\pi\mathbf{D}_x = \epsilon_{11}\mathbf{E}_x + \epsilon_{12}\mathbf{E}_y + \epsilon_{13}\mathbf{E}_z,$$
$$4\pi\mathbf{D}_y = \epsilon_{21}\mathbf{E}_x + \epsilon_{22}\mathbf{E}_y + \epsilon_{23}\mathbf{E}_z,$$
$$4\pi\mathbf{D}_z = \epsilon_{31}\mathbf{E}_x + \epsilon_{32}\mathbf{E}_y + \epsilon_{33}\mathbf{E}_z*,$$

or in symbolic vector form $$4\pi\mathbf{D} = (\epsilon)\,\mathbf{E}.$$

For isotropic media, i.e. for those having the same properties in all directions, this would be more simply expressed by the vector relation

$$\mathbf{D} = \frac{\epsilon}{4\pi}\mathbf{E}.$$

This is the simplest possible form of the theory. Subsequent developments will show that it is very approximately the correct one. Let us now follow it to some of its more important consequences, confining ourselves however to the case of isotropic media.

168. Draw out all the lines of displacement in the field and form them into tubes. As we have assumed the total displacement across any cross-section of a tube is constant along the tube. Moreover the displacement at one end of a tube where it abuts on a charge is equal to that along the tube and on the simplest assumption is measured by the charge there. If we are dealing only with conductors carrying charges and if a tube ends at a place where the surface density is σ and the cross-section there is df_1, then if df is the cross-section at any other part of the tube where the flux density is $\mathbf{D}$ $$\mathbf{D}\,df = \sigma\,df_1.$$

On this idea the surface density, or charge at the ends of a displacement tube, is merely the terminal aspect of the displacement in the tubes.' The way the displacement reveals itself is by piling up surface density.

In this theory the ordinary Gaussian surface integral theorem has a distinct physical significance. Our notion of the displacement means that if we take a closed surface of any kind in the field and integrate the normal displacement over it, then the total thus obtained must be equal to the total charge inside. Thus if $\mathbf{D}_n$ is the normal component of the displacement at the position of the element df of this surface

$$\int_f \mathbf{D}_n\,df = Q,$$

where Q is the total charge inside the surface f. This means that if the charge inside is a volume charge of finite density ρ then

$$\int_f \mathbf{D}_n\,df = Q = \int_f \rho\,dv,$$

* The 4π is introduced in conforming to the usual notation.

which by Green's lemma reduces to

$$\int_f \operatorname{div} \mathbf{D} \, dv = \int_f \rho \, dv,$$

and as this is true for any volume f we must have

$$\operatorname{div} \mathbf{D} = \rho$$

at each point of space: if $\rho = 0$, as is usually the case in all electrostatic problems, then

$$\operatorname{div} \mathbf{D} = 0,$$

and as much is displaced out of any region as flows into it, i.e. the displacement is like the flux of an incompressible fluid, as assumed by Faraday and Maxwell; $\mathbf{D}$ then satisfies the conditions for a *stream vector*.

169. We have already seen that the electric force intensity at each point of the field is a vector whose components are

$$(\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi;$$

moreover our assumptions imply that the displacement $\mathbf{D}$ is conditioned by the electric force $\mathbf{E}$ and the relation adopted was

$$\mathbf{D} = \frac{(\epsilon)\,\mathbf{E}}{4\pi} = -\frac{(\epsilon)}{4\pi} \operatorname{grad} \phi;$$

hence we have the characteristic equation of the field in the form

$$\operatorname{div}\left((\epsilon) \operatorname{grad} \phi\right) = -4\pi\rho,$$

which is the equation that replaces Poisson's equation for our theory. It is the characteristic equation of the potential function on the new generalised method of procedure invented by Faraday.

If the dielectric medium is isotropic this equation reduces to

$$\frac{\partial}{\partial x}\left(\epsilon \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\epsilon \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\epsilon \frac{\partial \phi}{\partial z}\right) = -4\pi\rho,$$

and if it is homogeneous as well it becomes

$$\nabla^2 \phi = -\frac{4\pi\rho}{\epsilon},$$

a modified form of Poisson's equation. This result stated in words implies that the result of having the dielectric throughout the region is that the same distribution of potential requires a distribution of charge ϵ times larger than in free space. Thus the constant ϵ of homogeneous isotropic medium which has been here introduced merely as the physical constant in the relation between the electric force and displacement, is identical with Faraday's dielectric constant ϵ.

170. As in the previous method of analysis of the electric field it is essential that we consider the alterations in the above analytical procedure necessitated by singularities in the distribution of charge, so that we may know how to deal with them when they turn up in any applications. The most important case is that in which ρ is infinite along a surface so that there is a surface distribution of density σ; we shall confine our attention to this example.

In the first place it is obvious that discontinuities can arise only in the neighbourhood of the surface distribution. Moreover ϕ must be continuous as we cross the surface as otherwise its gradient across would be infinite. If we now apply our generalised Gauss' theorem to a small flat cylinder enclosing a part of the surface of area df we can conclude at once that the normal component of the displacement across the surface is discontinuous by an amount $+\sigma$, i.e. if $\mathbf{D}_n$ is the component of $\mathbf{D}$ normal to the surface on which the surface density is situated and if also suffices 1 and 2 denote the values of the functions at near points on the same normal one at each side of the surface

$$\mathbf{D}_{n_1} - \mathbf{D}_{n_2} = \sigma,$$

but, as above, in the case of isotropic media

$$\mathbf{D}_{n_1} = -\frac{\epsilon_1}{4\pi}\frac{\partial \phi_1}{\partial n},$$

$$\mathbf{D}_{n_2} = -\frac{\epsilon_2}{4\pi}\frac{\partial \phi_2}{\partial n},$$

where we have also included for generality the possibility of different values of ϵ for the substance on the two sides of the surface. Thus

$$\frac{\epsilon_1}{4\pi}\frac{\partial \phi_1}{\partial n} - \frac{\epsilon_2}{4\pi}\frac{\partial \phi_2}{\partial n} = -\sigma.$$

If the surface is one of discontinuity in the isotropic dielectric medium, without any charge, $\sigma = 0$ and thus

$$\epsilon_1\frac{\partial \phi_1}{\partial n} - \epsilon_2\frac{\partial \phi_2}{\partial n} = 0,$$

and these equations give the form of the boundary conditions to which ϕ is subject in the present form of the theory.

171. It must now be noticed that the theory here given agrees perfectly with that deduced from the old attraction ideas for a vacuum. We have only to put $\epsilon = 1$ to get all the results of our previous theory. We have however gone deeper into the matter; instead of talking of mere attractions we have attempted to see what is going on and have generalised the theory to include the properties of the medium conveying the action. Instead of a simple theory of attractions we have now a theory of flux stimulated by

electric force. The exposition is of course merely of the nature of an explanation; no proof can be given that it is the correct view of the affair. We have merely invented a consistent scheme whose continued existence depends only on the test of its reality.

172. On electric displacement. The present scheme of course turns on the electric displacement. What sort of thing is this electric displacement? Why is it different when a dielectric substance is present? The first question presents one of the fundamental difficulties of the theory, whose solution has not yet been effected. The second question was answered by Kelvin by comparing the behaviour of dielectrics under electric force with the behaviour of magnetisable bodies under magnetic force. The magnetic behaviour of iron had just been worked out by Poisson* and Kelvin† simply transferred the theory to dielectric media and it fitted splendidly. In this theory electric excitement is always accompanied by separation; the old fashioned method is that there are practically infinite amounts of positive and negative electricity in each body and the electric force pulls on them and of course in opposite directions and thus tends to separate them. In a conductor the two are pulled right apart and as the supply is practically unlimited the separation will go on while the electric force exists. If on the other hand the body is an insulator the extent of the separation is limited by certain elastic resilience forces tending to hold the charge elements to the particles of matter. The result of the small separation which can be produced is to render each molecule (or perhaps molecular groups) of the substance bi-polar, like a little magnet. A fuller explanation will be given when we return to the subject of polarised media in a subsequent chapter. For the present we merely regard the electric force as actually doing two things:

(i) moving electric charges,

(ii) producing electric displacement in a dielectric.

It is part of a physical theory to explain how it is capable of acting in these two ways, why it is that one really involves the other.

There are two further remarks respecting the electric displacement which it might be convenient to mention before concluding. Firstly notice that the direction of the vector **D** in isotropic media is in the positive direction of the lines of force and thus since the lines of force go outwards from a positive charge the displacement is also away from the charge. This rule of signs is opposite to that which one might expect and thus makes it clear that the 'displacement' is not a displacement of electricity in the usual sense of those words.

* *Mémoires de l'Académie*, 5 (1826), pp. 247, 488; 6 (1827), p. 441. Cf. also Maxwell, *Treatise*, II. § 385.

† *Camb. and Dublin Math. Journ.* (1845). *Reprint*, § 447. The physical idea of polarisation in dielectrics was introduced by Faraday, *Experimental Researches*, 1295 (1837).

Secondly it is important to emphasise that the vector **D** represents a totality of displacement. If the rate of change of the displacement at any time t be denoted by a vector **R**, then the amount displaced across any surface element df during the small time dt is $\mathbf{R}_n\, dt\, df$, and during the finite interval since the initial instant t_0 required in setting up the field the total displacement across the surface element is

$$\mathbf{D}_n\, df = df \int_{t_0}^{t} \mathbf{R}_n\, dt,$$

and this is the vector **D** of our theory.

173. The inverse electrostatic problem with dielectrics. The considerations of this chapter have slightly modified the conditions governing the problem of the determination of the circumstances in any electrical system under given conditions.

The potential function is now a regular function ϕ satisfying the following conditions:

(i) $\operatorname{div}(\epsilon \operatorname{grad} \phi) = -4\pi\rho$ at any point in the dielectric which for the present purposes is assumed to be isotropic; in a vacuum or practically speaking in air we shall have still

$$\nabla^2\phi = -4\pi\rho;$$

(ii) at any surface of discontinuity in the field (excluding double sheet distributions which we need not here consider) the normal component of the induction is discontinuous by the amount $4\pi\sigma$ so that

$$\epsilon_1 \frac{\partial \phi_1}{\partial n} - \epsilon_2 \frac{\partial \phi_2}{\partial n} + 4\pi\sigma = 0,$$

but the potential and tangential force must be continuous.

The problems of any practical interest mainly concern the estimation of the modification of the field of given charges by the introduction into their field of one or more dielectric masses. We must attempt to obtain a solution of the above conditions subject to the restrictions imposed by the conditions implied in the special data for the problem. The essential features of such problems are illustrated by the following simple cases.

174. (*a*) *To find the effect of introducing a uniform dielectric sphere into a uniform electric field.*

The potential of the original undisturbed field can be written in the form

$$\phi = -Ex,$$

if the x-axis of coordinates is parallel to the uniform direction of the field near the sphere. The potential ϕ of the final field must agree with this field at remote distances, where the disturbance due to the sphere is inappreciable: it must also satisfy $\nabla^2\phi = 0$

both inside and outside the sphere (ϵ being now constant), and be continuous at the boundary of the sphere, i.e. between the two dielectric media. These conditions determine ϕ completely and following the hints suggested by the analogous problem of the conducting sphere already examined in detail, we choose the origin of coordinates to coincide with the centre of the sphere, whose radius is a, and then try solutions for ϕ in the two regions

(i) $$\phi_i = Ax$$

inside the sphere and

(ii) $$\phi_0 = -Ex + \frac{Bx}{r^3}$$

outside. These satisfy all but the surface conditions; they also satisfy these if

$$A = -E + \frac{B}{a^3}$$

and $$\epsilon A = -E - \frac{2B}{a^3};$$

the first of these equations expresses the continuity of the potential and the second the continuity of normal induction $\left(\epsilon \frac{\partial \phi}{\partial r}\right)$. We have therefore

$$A = -\frac{3E}{\epsilon + 2},$$

$$B = \frac{\epsilon - 1}{\epsilon + 2} Ea^3,$$

and thus the problem is completely solved by

$$\phi_i = -\frac{3}{\epsilon + 2} Ex,$$

$$\phi_0 = -Ex + \frac{\epsilon - 1}{\epsilon + 2} E \frac{a^2 x}{r^3}.$$

The force inside the sphere is uniform but diminished in the ratio $3 : \epsilon + 2$.

175. (*b*) *The effect of introducing a homogeneous dielectric ellipsoid in a uniform field.*

If we take first the case when the original field is parallel to an axis of the ellipsoid, its potential in the undisturbed state is

$$\phi = -Ex.$$

Now, noticing that the various forms obtained for the potentials of charge distributions on ellipsoids in the first chapter must be solutions of the potential equation, we are induced to try solutions in this case in the form

(i) for outside space

$$\phi = \phi_0 - Ex + Lx \int_\lambda^\infty \frac{dt}{(a^2 + t)\sqrt{(a^2 + t)(b^2 + t)(c^2 + t)}},$$

(ii) for inside space $\phi = \phi_0 + L'x,$

and we must then try and find L, L' to satisfy the continuity conditions at the boundary.

Continuity of potential requires

$$-E = L' - L\int_0^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}}.$$

Continuity of induction requires

$$-E + L\int_0^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}} - \frac{2L}{a^3bc} = \epsilon L',$$

whence using, as in the first chapter,

$$A = \tfrac{1}{2}a^3bc\int_0^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

we have

$$L = \frac{\frac{a^3bc}{2}(\epsilon-1)E}{1+A(\epsilon-1)},$$

$$L' = -\frac{E}{1+A(\epsilon-1)},$$

so that the correct forms of the potential for the disturbed field are: outside the ellipsoid

$$\phi = \phi_0 - Ex + \frac{a^3bc}{2}\frac{(\epsilon-1)Ex}{1+A(\epsilon-1)}\int_\lambda^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

and inside

$$\phi = \phi_0 - \frac{Ex}{1+A(\epsilon-1)}.$$

We see at once that we can generalise this result. If the original field were given by

$$\phi = \phi_0 - E_x x - E_y y - E_z z,$$

then the outside potential would be

$$\phi = \phi_0 - E_x x - E_y y - E_z z + \frac{abc}{2}(\epsilon-1)\left[\frac{a^2E_x x}{1+A(\epsilon-1)}A_\lambda + \frac{b^2E_y y}{1+B(\epsilon-1)}B_\lambda + \frac{c^2E_z z}{1+C(\epsilon-1)}C_\lambda\right],$$

whilst the inside one is

$$\phi = \phi_0 - \frac{E_x x}{1+A(\epsilon-1)} - \frac{E_y y}{1+B(\epsilon-1)} - \frac{E_z z}{1+C(\epsilon-1)},$$

wherein

$$A_\lambda = \int_\lambda^\infty \frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

and similarly for B_λ and C_λ.

176. (c) *The effect of introducing an infinite slab of dielectric with one plane face into the field of a single point charge.*

Let the point charge be q at A_1 and at a distance c from the slab: take A_2 on the normal A_1N to the plane face of the slab so that $A_1A_2 = 2c$.

Now try a solution of the conditions for the potential function

$$\phi_1 = \frac{q}{A_1P} + \frac{q'}{A_2P},$$

$$\phi_2 = \frac{q''}{A_2P},$$

ϕ_1, ϕ_2 being the potentials in the respective regions. These forms satisfy everything but the conditions of continuity at the boundary.

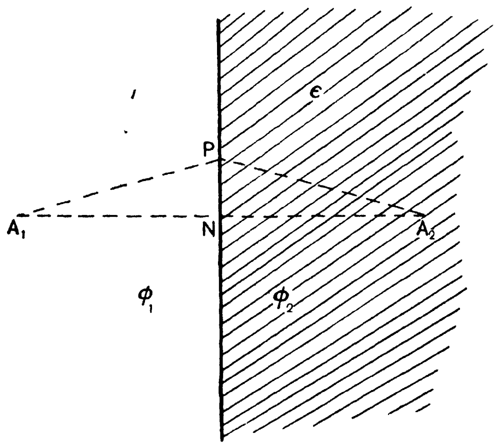


Fig. 44

Continuity of potential gives

$$q + q' = q''.$$

Continuity of normal displacement gives

$$q - q' = \epsilon q'',$$

whence with $$q' = \frac{\epsilon - 1}{\epsilon + 1} q, \quad q'' = \frac{2q}{\epsilon + 1},$$

we satisfy all the conditions for ϕ_1 and ϕ_2. Thus if we use r_1 and r_2 for A_1P and A_2P we have

$$\phi_1 = \frac{q}{r_1} - \frac{\epsilon - 1}{\epsilon + 1} \frac{q}{r_2},$$

$$\phi_2 = \frac{2}{\epsilon + 1} \frac{q}{r_1}.$$

177. The effect of a dielectric substance on the capacity of a condenser. We have seen how Faraday investigated the modifying action of dielectric media in electric fields by comparing condensers in which the air space is either wholly or partially occupied by such substances. In order to interpret the results of these experiments we must examine on our theoretical basis the details of the circumstances in some practical cases, where the air space of a condenser is occupied as in Faraday's experiments by some dielectric substance.

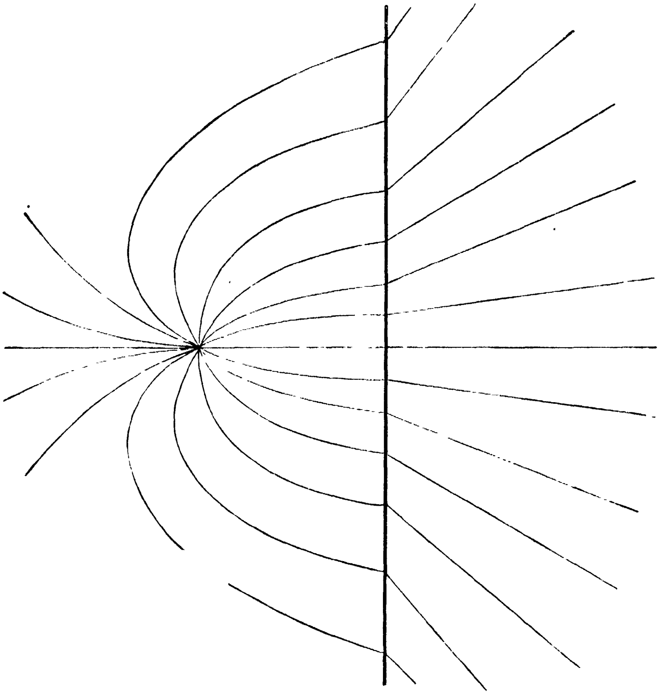


Fig. 45

(*a*) In the parallel plate condenser of the previous chapter we introduce a parallel slab of dielectric of constant ϵ and thickness t_2. The densities of the charges on the plates are still σ and $-\sigma$ and by symmetry the lines of force go straight across and the tubes are cylindrical, and so the electric displacement along them is constant. At the positive plate the electric force is

$$E = 4\pi\sigma,$$

the dielectric being air. In the dielectric slab the force is constant across it

owing to its homogeneity and from the equality of the normal displacement at its surface on both sides we must have

$$E' = \frac{E}{\epsilon},$$

and then again in the air near the other plate the force is again

$$E = 4\pi\sigma.$$

The difference of potential between the plates is easily obtained,

$$\phi_1 - \phi_2 = \int_0^t E\,dt$$
$$= t_1 E + \frac{t_2 E}{\epsilon} + (t - t_1 - t_2)\,E,$$

t_1 being the thickness of the first air slab. The total charge on a plate is

$$Q = A\sigma = \frac{AE}{4\pi},$$

and thus the capacity of the arrangement as a condenser is

$$C = \frac{A}{4\pi\left(t - t_2 + \dfrac{t_2}{\epsilon}\right)}.$$

We might describe t_2 as the 'reduced thickness' of the dielectric slab, the thickness of an equivalent layer of air.

This result can be generalised for several slabs; the potential is

$$\phi_1 - \phi_2 = 4\pi\sigma\left[t - t_1 - t_2 + \int_{t_1}^{t_2}\frac{dt}{\epsilon}\right].$$

178. (*b*) There is a more general theorem* of which the above is a particular case.

Supposing C is any conductor for which the equi-potentials are known when it is freely charged: if A and B are any two of these equi-potentials we can determine the capacity of a condenser formed by conducting surfaces A and B between which there is a dielectric stratified so that ϵ is constant along the equi-potential surfaces of C.

Supposing ϕ is the potential function of C with a charge Q when alone in the field, and let us assume that the potentials of the conductors are ϕ_A and ϕ_B.

When the dielectric is absent the capacity of the condenser formed by the surfaces ϕ_A and ϕ_B is

$$C = \frac{Q}{\phi_A - \phi_B}.$$

* This theorem is virtually due to Kelvin, *Reprint*, § 45.

Now suppose that the dielectric substance is inserted so that at any point in it

$$\epsilon = f(\phi),$$

then the potential in the new field is obviously of the form

$$\phi' = a\int_{\phi_c}^{\phi} \frac{d\phi}{\epsilon} + b$$

and the constants a, b can be determined to make this satisfy all conditions.

The charge on the conductor C is still Q and thus

$$Q = -\frac{1}{4\pi}\int_{f_c} \frac{\partial \phi'}{\partial n}\, df_c = -\frac{a}{4\pi\epsilon_c}\int_{f_c} \frac{\partial \phi}{\partial n}\, df_c$$
$$= +\frac{Qa}{\epsilon_c}.$$

Thus
$$a = \epsilon_c,$$

ϵ_c denoting the value of ϵ at the surface of C which we can assume is unity.

The difference of potential between the surfaces A and B is now

$$\phi_A' - \phi_B' = \int_{\phi_B}^{\phi_A} \frac{d\phi}{\epsilon},$$

and thus the new capacity of these two surfaces with the dielectric between them is

$$C' = C\frac{\phi_A - \phi_B}{\int_{\phi_B}^{\phi_A} \frac{d\phi}{\epsilon}}.$$

If the dielectric were uniform this would give

$$C' = \epsilon C,$$

which is Faraday's result.

179. As a particular case of the general theorem we may also quote the results for a spherical condenser (radii of surfaces a, b). Here we have

$$\phi = \frac{Q}{r},$$

and thus in the new case

$$\phi' = a\int_0^{\phi} \frac{d\phi}{\epsilon} + b$$
$$= -aQ\int_0^{\phi} \frac{dr}{\epsilon r^2} + b.$$

Thus
$$\phi_A - \phi_B = -aQ\int_b^a \frac{dr}{\epsilon r^2}$$

and the capacity is
$$C' = \frac{1}{\int_b^a \frac{dr}{\epsilon r^2}}.$$

In the particular case when $\epsilon = \epsilon_1$ for all values $b < r < c$ and $\epsilon = \epsilon_2$ for all values $c < r < a$ the capacity is

$$C' = \frac{1}{\int_b^c \frac{dr}{\epsilon_1 r^2} + \int_c^a \frac{dr}{\epsilon_2 r^2}}$$

$$= \frac{1}{\frac{1}{\epsilon_1}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{1}{\epsilon_2}\left(\frac{1}{c} - \frac{1}{b}\right)}.$$

180. On the refraction of the lines of force in an electrical field on passing across a discontinuous dielectric surface. It is interesting to notice how the lines of force are suddenly bent on crossing an interface between two dielectrics. The conditions of transition tell us at once what the law of refraction is.

If the forces are E_1, E_2 on either side of the interface and their directions make angles i_1, i_2 with the normal at the point of incidence, then we have :

(i) the tangential component of the force is continuous, if there is no double sheet distribution, thus

$$E_1 \sin i_1 = E_2 \sin i_2;$$

(ii) the electric displacement across the surface is continuous as there is no surface charge, so that

$$\epsilon_1 E_1 \cos i_1 = \epsilon_2 E_2 \cos i_2.$$

These are the two conditions to be satisfied; from them we deduce at once

$$\frac{\tan i_1}{\epsilon_1} = \frac{\tan i_2}{\epsilon_2}.$$

The lines of force are refracted according to a law of tangents (Maxwell). In entering a denser medium (ϵ bigger) the line is bent away from the normal.

This can be generalised by considering a double sheet distribution to be present on the surface as well.

181. The distribution of the electrostatic energy. In the previous chapters we found that the total energy in the electrostatic field was expressible in terms of the charge distribution in such a way that the amount of energy required to increase the volume density of charge at any point by $\delta\rho$ and the surface density by $\delta\sigma$ was given in the form

$$\delta W = \int \phi \delta\rho \, dv + \int_f \phi \delta\sigma \, df,$$

the first integral being taken throughout the whole of the field and the second over all those surfaces where there is a surface density.

Although the previous discussions have entirely ignored the dielectric medium and its effect on the electrical conditions of the system the result just quoted is perfectly correct whatever the complexity of this dielectric medium; this results from the general definition of the potential function ϕ which we have given above.

Now the simple hypothesis of action through a medium regards the electric charges merely as manifestations of a varied condition in the whole of the medium throughout the field. In such a theory therefore the energy of the electrical system is distributed throughout the space of the field so that each element of volume furnishes a part to the total amount, which part depends solely on the electrical conditions existing in the element. Now on the Faraday-Maxwell form of the theory the essential specification of the conditions at any point of the field is involved in the vector $\mathbf{D}$, the electric displacement, so that the alteration of the energy produced in the system by a small arbitrary charge in its specification should be expressible in terms of the alteration of the conditions of the medium, viz. by $\delta\mathbf{D}$, at each point. But in the form of the theory adopted, this increase of displacement $\delta\mathbf{D}$ at each point of the field is effected by the agency of the electric force $\mathbf{E}$, which produces it and is alone effective in altering it, and if the analogy with material phenomena implied in our choice of names is valid the work done by the force intensity $\mathbf{E}$ in producing an additional virtual displacement $\delta\mathbf{D}$ throughout the small volume element dv would be

$$(\mathbf{E} \cdot \delta\mathbf{D})\, dv,$$

so that the total increase in the energy of the system should be expressible by the integral

$$\int (\mathbf{E} \cdot \delta\mathbf{D})\, dv$$

taken throughout the whole field.

182. Owing however to the great indefiniteness in our knowledge of the true nature of the vector $\mathbf{D}$, this deduction of an expression for the energy cannot be regarded as anything more than a mere analogy. It can however be shown that the result obtained on this analogy agrees exactly in the total amount with the former estimate. In fact we have at each point of space

$$\operatorname{div} \mathbf{D} = \rho,$$

and at points on surfaces on which there is a charge density σ

$$\mathbf{D}_{1_n} - \mathbf{D}_{2_n} = \sigma,$$

and thus the variations $\delta\mathbf{D}$ are connected with the variations $\delta\rho$ and $\delta\sigma$ by the conditions

$$\operatorname{div} \delta\mathbf{D} = \delta\rho$$

at each point of space and

$$\delta\mathbf{D}_{1_n} - \delta\mathbf{D}_{2_n} = \delta\sigma$$

at each surface distribution. Thus the first estimate of W is equivalent to

$$\int \phi \operatorname{div} \delta\mathbf{D}\, dv + \int_f \phi\, (\delta\mathbf{D}_{1_n} - \delta\mathbf{D}_{2_n})\, df,$$

and a simple transformation of the first integral by integration by parts shows that this is equal to

$$\delta W = -\int (\delta\mathbf{D} \,.\, \nabla)\, \phi\, dv$$

$$= \int (\mathbf{E} \,.\, \delta\mathbf{D})\, dv^*.$$

Thus if the change in the total energy of the whole system is distributed throughout the field with a density $(\mathbf{E} \,.\, \delta\mathbf{D})$ at each point the total amount is consistent with our other theories. For the above mentioned reason we have however no really definite theoretical basis for regarding this distribution of the energy as the actual one. Maxwell assumed that it was† and there are certainly many points in favour of his view. It is the simplest distribution which suits the case, and for this reason alone it might be regarded as the correct one. More we cannot say except that future developments confirm the assumption.

183. Now let us examine the equation of energy in this latter form a little more closely:

$$\delta W = \int (\mathbf{E} \,.\, \delta\mathbf{D})\, dv.$$

Now δW is the work supplied by external agency to the system during the arbitrary change of its configuration specified by $\delta\mathbf{D}$ (a virtual displacement in the ordinary mechanical sense), and the principle of the conservation of energy asserts that it is an exact differential, or, in technical terms, a function W exists; otherwise we should have perpetual motion. The argument, as we had it before, is that if we suppose the system to be taken through a definite series of changes from a general configuration which we shall denote by 1 to any other configuration which we shall denote by 2, then the work done on the system during the change, which is expressed by

$$\int_1^2 \delta W = \int_1^2 \int (\mathbf{E} \,.\, \delta\mathbf{D})\, dv,$$

must be independent of the actual series of changes through which the system is taken from the one configuration to the other; provided of course that each series of changes is reversible, so that any work gained to the external agency in traversing it one way would be turned into an equal loss in going

* To make this argument quite rigorous it is necessary to include a bounding surface in the field at a great distance from the origin and to examine the integral over it which results in the integration by parts. In any real case however the field will be always regular at infinity and this surface integral tends to vanish.

† *Treatise*, I. p. 167.

the other way. If this were not so we could pass from configuration 1 to configuration 2 through one series of changes and back again through another series reversed, and could thus gain work on the whole in the complete cycle. The system being finally in precisely the same state as before we should have gained work from nothing and by repeating the process as often as we like we could get an indefinite amount of energy out of nothing, a fact which is highly improbable if not actually impossible. Thus under such conditions

$$\int_1^2 \delta W$$

depends only on the initial and final configurations 1 and 2 and not at all on the series of changes through which the system is taken from one to the other. This can only be true when δW is an exact differential when it is

$$\int_1^2 \delta W = W_2 - W_1.$$

We thus see that under ordinary conditions

$$\delta W = \int (\mathbf{E} \,.\, \delta\mathbf{D})\, dv$$

is an exact differential. Owing to the arbitrariness of $\delta\mathbf{D}$ this means that $(\mathbf{E} \,.\, \delta\mathbf{D})$ must be a complete differential of some function of $\mathbf{D}$ or rather of its components.

184. Writing this out in full we have, denoting the function by w,

$$\mathbf{E}_x \delta\mathbf{D}_x + \mathbf{E}_y \delta\mathbf{D}_y + \mathbf{E}_z \delta\mathbf{D}_z = \delta w,$$

so that w is a function of $\mathbf{D}_x$, $\mathbf{D}_y$, $\mathbf{D}_z$ whose partial differential coefficients with respect to these variables are respectively equal to the corresponding components of $\mathbf{E}$. But by hypothesis the various components of $\mathbf{D}$ are functions of the various components of $\mathbf{E}$ and thus by substitution we can interpret the function w as a function of $\mathbf{E}_x$, $\mathbf{E}_y$, $\mathbf{E}_z$ and this is the more fundamental form since $\mathbf{E}$ is the independent variable in the physical theory. But having made this transformation we see at once that

$$(\mathbf{D}\delta\mathbf{E}) = \delta\{w - (\mathbf{ED})\} = \delta w',$$

say; thus $(\mathbf{D}\delta\mathbf{E})$ is also a complete differential of a function w' which is a function of the variables $\mathbf{E}_x$, $\mathbf{E}_y$, $\mathbf{E}_z$; its partial differential coefficients will be similarly equal to the corresponding components of $\mathbf{D}$. Moreover this function w' must remain unaltered if the direction of the electric force $\mathbf{E}$ is simply reversed*, so that it can only involve even powers of the components $(\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z)$ and for weak fields would be practically quadratic. If we imply this restriction to small fields then we can write the function w' in the form

$$\frac{1}{8\pi}(\epsilon_{11}\mathbf{E}_x^2 + \epsilon_{22}\mathbf{E}_y^2 + \epsilon_{33}\mathbf{E}_z^2 + 2\epsilon_{12}\mathbf{E}_x\mathbf{E}_y + 2\epsilon_{23}\mathbf{E}_y\mathbf{E}_z + 2\epsilon_{31}\mathbf{E}_z\mathbf{E}_x),$$

* Provided there is no permanent displacement, as is generally assumed to be the case.

and thence we deduce that

$$4\pi\mathbf{D}_x = \epsilon_{11}\mathbf{E}_x + \epsilon_{12}\mathbf{E}_y + \epsilon_{13}\mathbf{E}_z,$$
$$4\pi\mathbf{D}_y = \epsilon_{21}\mathbf{E}_x + \epsilon_{22}\mathbf{E}_y + \epsilon_{23}\mathbf{E}_z,$$
$$4\pi\mathbf{D}_z = \epsilon_{31}\mathbf{E}_x + \epsilon_{32}\mathbf{E}_y + \epsilon_{33}\mathbf{E}_z,$$

in agreement with the results of our previous speculations. We see here however that the doctrine of energy requires that

$$\epsilon_{12} = \epsilon_{21}, \quad \epsilon_{13} = \epsilon_{31}, \quad \epsilon_{23} = \epsilon_{32},$$

a fact not involved in the former arguments.

In this case the energy in the field is

$$W = \frac{1}{8\pi}\int(\epsilon_{11}\mathbf{E}_x{}^2 + \ldots + 2\epsilon_{12}\mathbf{E}_x\mathbf{E}_y + \ldots)\,dv = \tfrac{1}{2}\int(\mathbf{ED})^*.$$

For isotropic media these relations simplify very considerably, for then

$$4\pi\mathbf{D} = \epsilon\mathbf{E},$$

so that

$$W = \frac{1}{8\pi}\int\epsilon\mathbf{E}^2 dv\dagger,$$

and this is the real form of Maxwell's result. The energy of an electrical system may in a restricted sense be considered as distributed throughout the field with a volume density $\frac{\epsilon\mathbf{E}^2}{8\pi}$ at each point.

185. A simple application of Green's analysis will show that in the general case of a linear relation between **E** and **D** the total energy of the system is given also by

$$W = \tfrac{1}{2}\int\rho\phi\,dv + \tfrac{1}{2}\int_f \sigma\phi\,df,$$

which is the form suitable for applications based on the distance action theory. In fact in such a case

$$W = \tfrac{1}{2}\int(\mathbf{ED})\,dv$$
$$= -\tfrac{1}{2}\int(\mathbf{D}\nabla)\,\phi\,dv,$$

and this transforms by the analytical theorem to

$$W = -\tfrac{1}{2}\int\phi\,\mathrm{div}\,\mathbf{D}\,dv - \tfrac{1}{2}\int_f\phi\,(\mathbf{D}_{n_1} - \mathbf{D}_{n_2})\,df,$$

the first integral being taken throughout the whole of the field and the second over those surfaces where **D** is discontinuous as regards its normal component.

* Maxwell, *Treatise*, I. p. 147.

† Kelvin, *Math. and Phys Papers*, I. § 61.

The integral over the infinite boundary as usual tends to zero and is neglected altogether. But

$$\operatorname{div} \mathbf{D} = -\rho,$$

and

$$\mathbf{D}_{n_1} - \mathbf{D}_{n_2} = -\sigma,$$

so that

$$W = +\tfrac{1}{2}\int \rho\phi\, dv + \tfrac{1}{2}\int_f \sigma\phi\, df,$$

which is identical with the form obtained in the more restricted theory given in the first chapter. The present analysis indicates the necessary restrictions which must be placed on the method of argument used on the former occasion*.

186. On the transmission of force through the dielectric medium. We must next turn to another fundamental problem in the present mode of formulation of electric theory: this concerns the explanation of the ponderomotive forces between charged bodies whose existence is implied in the determination of the potential function just found and which can be derived from this function, suitably expressed, in the manner elaborated in detail in the first chapter.

The transmission hypothesis underlying the Faraday-Maxwell theory of electric action here under discussion regards all electrical phenomena merely as the result of a certain state of affairs established in the surrounding dielectric field. It would therefore be necessary on such a theory to regard the ponderomotive forces resulting from the attractions and repulsions between the charges as the terminal aspects of some state of stress in the medium between. We must now enquire as to a possible representation of the manner in which these forces are transmitted across the space between the bodies. The problem reduces itself to finding the state of stress in the elastic medium which agrees with the known boundary values. If we knew the nature of the elasticity of the medium we could solve this problem completely; but this is just what we do not know in the case of the aether. We can only guess and of course there can be any number of guesses. Faraday divined a very simple scheme which has high claims on account of its simplicity, but as we shall see later, it is not general enough for our purposes.

After experimentally investigating the nature of the electric fields around conductors Faraday came to the conclusion that the forces between them could be accounted for by a pull along the tubes of force, i.e. as if they were tending to contract like stretched elastic bands. This would obviously account for the attraction, but with it alone the elements of the transmitting medium could not be in equilibrium. He then saw somehow that in addition there must be an equal pressure in all directions perpendicular to the tubes:

* It must however be emphasised that the restrictions are of theoretical interest only. The simple linear relation satisfies all the requirements of actual fact.

the tube tends not only to contract itself along its length but also to expand against a normal pressure all round. Under such a stress system the medium would be in equilibrium as regards its own parts but would transmit the force from one body to another. Although Faraday invented this scheme he was unable to prove its reality : it was Maxwell who formulated it mathematically and put it in a very precise form.

187. As we are here going rather deeply into this question it might be as well to indicate how the stress in any medium is analysed*. Consider the medium in the neighbourhood of any point separated over a small interface there. Forces would then be required to be applied at each of the exposed surfaces to hold the medium in equilibrium : the same forces would be required for each of the two interfaces since they represent the action and reaction which were exerted between the parts of the medium on the two sides of the slit before they were separated. We shall consider these forces measured as so much per unit area. If for the slit made of area δf we know that the force required to hold the medium on either side of the interface in equilibrium is $\mathbf{T}\delta f$, then the vector $\mathbf{T}$ defines the stress for this particular direction of df. If we knew so much for every direction of the slit we should have a complete knowledge of the state of the stress at the point. It can however be shown that it is quite sufficient to know it for three directions only.

If in fact we consider the equilibrium of a portion of the medium included in any closed surface f we see that it is in equilibrium under the actions of (i) the rest of the substance across the surface f, and (ii) the external bodily forces applied to the material in the tetrahedron such as the force of gravity. If we denote these latter forces generally as a force $\mathbf{F}$ per unit volume at any point then the linear equations of equilibrium of the mass are vectorially

$$\int \mathbf{F}\,dv + \int \mathbf{T}\,df = 0,$$

the latter integral being taken over the surface f enclosing the volume over which the former integral is extended. The angular equations are similarly of type

$$\int (\mathbf{F}_z y - \mathbf{F}_y z)\,dv + \int_f (\mathbf{T}_z y - \mathbf{T}_y z)\,df = 0\dagger,$$

the regions of integration being the same.

From the form of these two equations we can deduce a result of great importance. Let the volume of integration be very small in all its dimensions,

* The general theory is discussed by Love, *Mathematical Theory of Elasticity* (2nd Ed. 1906, Cambridge).

† The analysis is of course referred to a convenient system of rectangular axes which also form the base for the reduction of the forces.

and let l^3 denote this volume. If we divide both terms in each of these equations by l^2 and then pass to the limit by diminishing l indefinitely we find that

$$\underset{l=0}{\mathrm{Lt}}\, l^{-2} \int \mathbf{T} df = 0,$$

and three equations of the type

$$\underset{l=0}{\mathrm{Lt}}\, l^{-2} \int (\mathbf{T}_z y - \mathbf{T}_y z)\, df = 0,$$

which show that the tractions on the elements of area of the surface of any portion of a body which is very small in all its dimensions are ultimately, to a first approximation, in equilibrium among themselves.

188. Now let us suppose that the stress components per unit area for directions of the small interface perpendicular to the axes of coordinates at any point to be respectively

$$\begin{array}{llll} \text{(i)} & T_{xx}, & T_{xy}, & T_{xz}, \\ \text{(ii)} & T_{yx}, & T_{yy}, & T_{yz}, \\ \text{(iii)} & T_{zx}, & T_{zy}, & T_{zz}, \end{array}$$

then the force on any small area $dy\,dz$ perpendicular to the x-axis has components

$$T_{xx} dy\,dz, \qquad T_{xy} dy\,dz, \qquad T_{xz} dy\,dz.$$

Now consider a very small cube of the material with its edges parallel to the coordinate axes. To a first approximation the resultant attractions exerted upon the forces perpendicular to the x-axis are

$$T_{xx}\Delta, \qquad T_{xy}\Delta, \qquad T_{xz}\Delta$$

for the face on the positive side and

$$-T_{xx}\Delta, \qquad -T_{xy}\Delta, \qquad -T_{xz}\Delta$$

for the other face, Δ being the area of a face. Similar expressions hold for the other faces. The value of the integral

$$\iint (y\mathbf{T}_z - z\mathbf{T}_y)\, df$$

for the cube can be taken to be simply

$$l\Delta\,(T_{yz} - T_{zy}),$$

where l is the length of any edge; but the cube being to a first approximation in equilibrium under these forces we must have

$$T_{yz} - T_{zy} = 0,$$

and therefore $$T_{yz} = T_{zy},$$

similarly $$T_{xz} = T_{zx}, \qquad T_{yz} = T_{zy},$$

so that the nine quantities just given reduce to six.

189. We next consider the equilibrium of a portion of the medium included in a small tetrahedron of which three edges are parallel to the axes of coordinates and of lengths (dx, dy, dz): let us suppose that the fourth face is of area df and is normal to the direction whose cosines are (l, m, n). Since the surface tractions over the four faces of this tetrahedron form to a first approximation a self-equilibrating system of forces we must have for the component action on the fourth face of the tetrahedron parallel to the x-axis

$$\begin{aligned} T_x df &= \left\{\mathbf{T}_{xx}\frac{dy\,dz}{2} + T_{yx}\frac{dz\,dx}{2} + T_{zx}\frac{dx\,dy}{2}\right\} \\ &= (lT_{xx} + mT_{yx} + nT_{zx})\,df, \end{aligned}$$

because it must balance the forces in the same direction on the other faces.

Thus the components of the stress across df are

$$\begin{aligned} \mathbf{T}_x &= lT_{xx} + mT_{xy} + nT_{xz}, \\ \mathbf{T}_y &= lT_{yx} + mT_{yy} + nT_{yz}, \\ \mathbf{T}_z &= lT_{zx} + mT_{zy} + nT_{zz}, \end{aligned}$$

and therefore the stress at any point in the medium is completely specified by the six quantities T_{xx}, T_{yy}, T_{zz}, $T_{yz} = T_{zy}$, $T_{zx} = T_{xz}$, $T_{xy} = T_{yx}$.

The stress across this fourth face will be normal to the face if these three quantities be proportional to l, m, n; that is, if

$$\frac{lT_{xx} + mT_{xy} + nT_{xz}}{l} = \frac{lT_{yx} + mT_{yy} + nT_{yz}}{m} = \frac{lT_{zx} + mT_{zy} + nT_{zz}}{n}.$$

These equations we know determine three directions at right angles, viz. the axes of the ellipsoid

$$T_{xx}x^2 + T_{yy}y^2 + T_{zz}z^2 + (T_{xy} + T_{yx})\,xy + (T_{xz} + T_{zx})\,xz + (T_{zy} + T_{yz})\,yz = 1.$$

These directions are the directions of principal stress at the point and were we to refer the ellipsoid to these axes it would become

$$T_1x^2 + T_2y^2 + T_3z^2 = 1,$$

where T_1, T_2, T_3 are the principal stresses.

190. If we examine the matter a little more closely we shall find that the tractions over the surface of any element of the medium do not exactly balance among themselves as assumed above. There is a small outstanding part which must balance the bodily force. To find this part we have only to consider the statical equilibrium of the medium enclosed in any surface f: the forces acting on it are the bodily forces $\mathbf{F}$ per unit volume at any point and the tractions over the surface f. Resolving along the x-axis the condition for equilibrium is

$$\int \mathbf{F}_x dv = \int_f (lT_{xx} + mT_{xy} + nT_{xz})\,df,$$

(l, m, n) now denoting the direction cosines of the normal to the surface element df. A transformation by Green's lemma and the usual line of argument shows that

$$\mathbf{F}_x = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z},$$
$$\mathbf{F}_y = \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z},$$
$$\mathbf{F}_z = \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z}.$$

It is the small differences or space gradients of the stress components that balance the extraneous forces.

Reversing the argument we see that if the applied forcive $\mathbf{F}$ can be determined in such a manner as will enable us to express its x-component in the form

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z},$$

and obtain similar forms for the other components, then we may say that the bodily forces on that part of the system enclosed in any surface f can be represented as the result of an elastic stress traction over the surface f: in fact

$$\int \mathbf{F}_x dv = \int (lX_x + mX_y + nX_z)\, df,$$

and thus we can at once write

$$T_{xx} = X_x, \quad T_{xy} = X_y, \quad T_{xz} = X_z,$$

as providing a sufficient solution of the problem.

191. Returning now to the consideration of the stresses in the dielectric medium of an electrical system, we know that the x-component of the bodily forcive acting on the part of the system inside f is

$$\int \mathbf{F}_x dv = \int \rho \mathbf{E}_x dv + \tfrac{1}{2}\int_{f'} \sigma\,(\mathbf{E}_{x_1} + \mathbf{E}_{x_2})\, df',$$

the first integral being taken throughout the volume and the latter over all surface charges. But $\rho = \operatorname{div}\mathbf{D}$ and $\sigma = \mathbf{D}_{n_1} - \mathbf{D}_{n_2}$ so that

$$\int \mathbf{F}_x dv = \int \mathbf{E}_x \operatorname{div}\mathbf{D}\, dv + \tfrac{1}{2}\int_{f'} (\mathbf{D}_{n_1} - \mathbf{D}_{n_2})\,(\mathbf{E}_{x_1} + \mathbf{E}_{x_2})\, df'.$$

But we can easily verify that

$$\mathbf{E}_x \operatorname{div}\mathbf{D} = -\tfrac{1}{2}\left[\frac{\partial}{\partial x}(-2\mathbf{E}_x\mathbf{D}_x + (\mathbf{E}, \mathbf{D})) - \frac{\partial}{\partial y}(\mathbf{E}_x\mathbf{D}_y) - \frac{\partial}{\partial z}(\mathbf{E}_x\mathbf{D}_z)\right]$$
$$+ \tfrac{1}{2}\left(\mathbf{E}, \frac{\partial \mathbf{D}}{\partial x}\right) - \tfrac{1}{2}\left(\mathbf{D}, \frac{\partial \mathbf{E}}{\partial x}\right),$$

so that

$$\int \mathbf{F}_x dv = \tfrac{1}{2}\int_{f'} \sigma\,(\mathbf{E}_{x_1} + \mathbf{E}_{x_2})\, df'$$

$$+ \tfrac{1}{2}\int_{f'} \Big| l\,(-2\mathbf{E}_x\mathbf{D}_x + (\mathbf{ED})) - m\,(2\mathbf{E}_x\mathbf{D}_y) - n\,(2\mathbf{E}_x\mathbf{D}_z)\Big|_1^2 df' *$$

$$- \tfrac{1}{2}\int \{l\,(-2\mathbf{E}_x\mathbf{D}_x + (\mathbf{ED})) - m2\mathbf{E}_x\mathbf{D}_y - n2\mathbf{E}_x\mathbf{D}_y\}\, df$$

$$- \tfrac{1}{2}\left\{\left(\mathbf{D}\frac{\partial \mathbf{E}}{\partial x}\right) - \left(\mathbf{E}\frac{\partial \mathbf{D}}{\partial x}\right)\right\},$$

where in the integrals f' are now included also all surfaces inside f over which the dielectric medium is discontinuous.

The surface integral over f' is

$$\tfrac{1}{2}\int\left[(\mathbf{D}_{n_1} - \mathbf{D}_{n_2})\,(\mathbf{E}_{x_1} + \mathbf{E}_{x_2}) - \Big|2\mathbf{E}_x\mathbf{D}_n + l\,(\mathbf{E}, \mathbf{D})\Big|_1^2\right] df',$$

which reduces to

$$\tfrac{1}{2}\int_{f'} [l\,\{(\mathbf{E}, \mathbf{D})_1 - (\mathbf{E}, \mathbf{D})_2\} + (\mathbf{E}_{x_2} - \mathbf{E}_{x_1})\,(\mathbf{D}_{n_2} + \mathbf{D}_{n_1})]\, df'.$$

Hence we have in all

$$\int \mathbf{F}_x dv = -\tfrac{1}{2}\int_f \{l\,(-2\mathbf{E}_x\mathbf{D}_x + (\mathbf{E}, \mathbf{D})) - m\,(2\mathbf{E}_x\mathbf{D}_y) - n\,(2\mathbf{E}_x\mathbf{D}_z)\}\, df$$

$$+ \tfrac{1}{2}\int_{f'} \{(\mathbf{E}_{x_2} - \mathbf{E}_{x_1})\,(\mathbf{D}_{n_2} - \mathbf{D}_{n_1}) + l\,((\mathbf{ED})_1 - (\mathbf{ED})_2)\}\, df'$$

$$- \tfrac{1}{2}\int\left\{\left(\mathbf{D}\frac{\partial \mathbf{E}}{\partial x}\right) - \left(\mathbf{E}\frac{\partial \mathbf{D}}{\partial x}\right)\right\} dv.$$

This is the most general result for which no further reduction is possible. We must therefore conclude that in the most general possible cases we are unable to reduce the forces on the system inside any surface to a representation by means of a system of tractions over that surface and they cannot therefore be regarded as the result of an applied stress system throughout the medium inside f. The outstanding terms representing a bodily forcive whose x-component is

$$-\tfrac{1}{2}\int\left\{\mathbf{D}\frac{\partial \mathbf{E}}{\partial x} - \mathbf{E}\frac{\partial \mathbf{D}}{\partial x}\right\} dv$$

and the surface forces on the various surfaces of discontinuity shew that a stress system which would otherwise be determined by the first surface integral above in the final expression for $\int \mathbf{F}_x dv$ would not in general be self-balanced when applied to the system; it could not of itself keep the elements of volume of the medium of the dielectric in equilibrium when acting alone.

* l, m, n are the direction cosines of the normal to the surface f at the position of the element df.

192. The failure to obtain the desired result in the most general case, although of theoretical importance, in reality affects the usefulness of the underlying ideas but little, because in the cases of practical importance most of the outstanding terms vanish and the attempt to obtain the representative stress is entirely successful.

Electrostatic problems are in general concerned mainly with charged conductors, no other charge distribution being generally possible. In this case whatever the complexity of the dielectric medium the resultant electric force at any point of a conductor is always normal to the surface outside and zero inside and thus the part of the integral

$$\int_{f'} [(\mathbf{E}_{x_2} - \mathbf{E}_{x_1})(\mathbf{D}_{n_2} - \mathbf{D}_{n_1}) + l\{(\mathbf{ED})_1 - (\mathbf{ED})_2\}]\, df'$$

corresponding to the surfaces of the conductors vanishes, for the integrand in such parts is

$$-\mathbf{E}_{x_1}\mathbf{D}_{n_1} + l\,(\mathbf{ED})_1 = -l\mathbf{E}_{n_1}\mathbf{D}_{n_1} + l\mathbf{E}_{n_1}\mathbf{D}_{n_1} = 0,$$

so that the outstanding terms arise solely from discontinuities in the dielectric medium; if there are no sudden discontinuities in the dielectric medium (and any such could be considered as a rapid but continuous transition) these terms are completely represented by the volume integral

$$-\tfrac{1}{2}\int\left(\mathbf{D}\frac{\partial \mathbf{E}}{\partial x}\right) - \left(\mathbf{E}\frac{\partial \mathbf{D}}{\partial x}\right) dv,$$

which in the case of isotropic media reduces to

$$-\tfrac{1}{2}\int \mathbf{E}^2 \frac{\partial \epsilon}{\partial x}\, dv.$$

This would mean that the stress system, determined by the first surface integral over f and more fully specified below, when applied to the medium between the conductors would leave an outstanding pull on each element of volume which would have to be balanced somehow. This requires an additional finite force per unit volume and if the change in the character of the medium as specified by ϵ is very rapid this force is very large. No such force has however ever been contemplated.

This integral however also vanishes for homogeneous media, however anisotropic, provided only that the linear relation between **D** and **E** holds; this is easily verified by the substitution of the usual linear functions for the components of **D**, a reference to the principal crystalline axes having previously been made.

193. The conclusion is therefore that for homogeneous dielectric media the mechanical force acting on the part of an ordinary electrical system inside any surface f drawn in the field can be represented in the main by a system of interfacial tractions over the surface. These surface tractions

themselves are the representatives of a stress system specified at any point by nine components *

$$
\begin{aligned}
T_{xx} &= \tfrac{1}{2}(\mathbf{E}_x\mathbf{D}_x - \mathbf{E}_y\mathbf{D}_y - \mathbf{E}_z\mathbf{D}_z),\\
T_{yy} &= \tfrac{1}{2}(-\mathbf{E}_x\mathbf{D}_x + \mathbf{E}_y\mathbf{D}_y - \mathbf{E}_z\mathbf{D}_z),\\
T_{zz} &= \tfrac{1}{2}(-\mathbf{E}_x\mathbf{D}_x - \mathbf{E}_y\mathbf{D}_y + \mathbf{E}_z\mathbf{D}_z),\\
T_{xy} &= \mathbf{E}_x\mathbf{D}_y, \qquad T_{yx} = \mathbf{E}_y\mathbf{D}_x,\\
T_{yz} &= \mathbf{E}_y\mathbf{D}_z, \qquad T_{zy} = \mathbf{E}_z\mathbf{D}_y,\\
T_{zx} &= \mathbf{E}_z\mathbf{D}_x, \qquad T_{xz} = \mathbf{E}_x\mathbf{D}_z.
\end{aligned}
$$

But in the general case of crystalline medium, for which

$$\mathbf{D}_x : \mathbf{D}_y : \mathbf{D}_z \neq \mathbf{E}_x : \mathbf{E}_y : \mathbf{E}_z,$$

this stress is not self-conjugate. It does not therefore represent a stress system of the ordinary mechanical type which is always self-conjugate, as we proved before. The difference between the cross-terms in the matrix show in fact that there is a torque on the element of volume of amount

$$[\mathbf{ED}]\, dv,$$

and therefore the medium cannot be in equilibrium.

We thus see that, except in the very special case of isotropic homogeneous media, it is impossible to obtain an ordinary elastic stress system to represent the mechanical actions between electrified conductors. The specification of the necessary stress system even in homogeneous media is more general than is possible in simple ordinary elastic solids and necessitates a modification of our views on the constitution of dielectrics. As we shall see in the next chapter the specification of the stress in media with electric or magnetic polarised molecules is much more general and involves a solution of most of the difficulties here encountered.

194. In homogeneous isotropic media for which the ordinary law of induction

$$\mathbf{D} = \frac{\epsilon\mathbf{E}}{4\pi}$$

holds, the actions between electrified conductors may be completely represented as the result of an imposed stress system whose six components are represented in the matrix

$$\frac{\epsilon}{4\pi}\begin{Vmatrix} \mathbf{E}_x^2 - \mathbf{E}_y^2 - \mathbf{E}_z^2, & 2\mathbf{E}_x\mathbf{E}_y, & 2\mathbf{E}_x\mathbf{E}_z \\ 2\mathbf{E}_x\mathbf{E}_y, & -\mathbf{E}_x^2 + \mathbf{E}_y^2 - \mathbf{E}_z^2, & 2\mathbf{E}_y\mathbf{E}_z \\ 2\mathbf{E}_x\mathbf{E}_z, & 2\mathbf{E}_y\mathbf{E}_z, & -\mathbf{E}_x^2 - \mathbf{E}_y^2 + \mathbf{E}_z^2 \end{Vmatrix},$$

which is Maxwell's self-conjugate stress system †; the cross terms are symmetrical. The stress quadric for this system is obviously

* Heaviside, *Phil. Trans.* 183A (1892), p. 423, or *Electromagnetic Theory*, I. p. 84.

† Cf. *Treatise*, I. p. 159. The only case dealt with by Maxwell is that in which $\epsilon = 1$.

$$2\,(\mathbf{E}_x{}^2x^2 + \ldots + 2\mathbf{E}_x\mathbf{E}_y xy + \ldots) - (\mathbf{E}_x{}^2 + \mathbf{E}_y{}^2 + \mathbf{E}_z{}^2)(x^2 + y^2 + z^2) = \frac{8\pi}{\epsilon},$$

or $$2\,(\mathbf{E}_x x + \mathbf{E}_y y + \mathbf{E}_z z)^2 - (\mathbf{E}_x{}^2 + \mathbf{E}_y{}^2 + \mathbf{E}_z{}^2)(x^2 + y^2 + z^2) = \frac{8\pi}{\epsilon}.$$

This quadric is one of revolution and its axis is in the direction

$$\mathbf{E}_x : \mathbf{E}_y : \mathbf{E}_z,$$

i.e. along the lines of force. Referred to the principal axes its equation is

$$2\mathbf{E}^2x^2 - \mathbf{E}^2(x^2 + y^2 + z^2) = \frac{8\pi}{\epsilon},$$

so that the principal stress system consists of the three components

$$\frac{\epsilon\mathbf{E}^2}{4\pi} - \frac{\epsilon\mathbf{E}^2}{8\pi}, \quad -\frac{\epsilon\mathbf{E}^2}{8\pi}, \quad -\frac{\epsilon\mathbf{E}^2}{8\pi},$$

and in this form we see that it consists of a uniform hydrostatic pressure $\frac{\epsilon\mathbf{E}^2}{8\pi}$ combined with a tension along the lines of force $\frac{\epsilon\mathbf{E}^2}{4\pi}$, and thus we can identify it with the system given by Faraday.

The important conclusion is however that unless the dielectric medium is uniform and isotropic this Maxwellian stress system could not of itself keep the elements of volume of the dielectric medium in equilibrium when acting alone. This fact renders the system quite impossible for any real case because the only realisable homogeneous medium is the unattainable empty space.

195. The Maxwellian stress system is not the only one obtainable under the specified conditions. Special emphasis has however been placed on it on account of its particular disadvantages. It is however possible to obtain two different stress specifications by a slight transformation of the outstanding terms obtained in the general reduction given in § 191. The surface terms arising on account of the discontinuities are susceptible of the same treatment as before but it is possible to reduce the outstanding volume integral in the expression for $\mathbf{F}_x$ to the form

$$\int\left(\mathbf{D} - \frac{\mathbf{E}}{4\pi},\ \frac{\partial\mathbf{E}}{\partial x}\right) dv,$$

which in the simplest type of media for which

$$4\pi\mathbf{D} = \epsilon\mathbf{E},$$

reduces to

$$\frac{1}{8\pi}\int(\epsilon - 1)\frac{\partial\mathbf{E}^2}{\partial x}\,dv.$$

In this case the stress remaining is specified by the components

$$T_{xx} = \mathbf{E}_x\mathbf{D}_x - \frac{1}{8\pi}\mathbf{E}^2,$$

$$T_{yy} = \mathbf{E}_y\mathbf{D}_y - \frac{1}{8\pi}\mathbf{E}^2,$$

$$T_{zz} = \mathbf{E}_z\mathbf{D}_z - \frac{1}{8\pi}\mathbf{E}^2,$$

$$T_{xy} = \mathbf{E}_x\mathbf{D}_y, \qquad T_{yx} = \mathbf{E}_y\mathbf{D}_x,$$

$$T_{yz} = \mathbf{E}_y\mathbf{D}_z, \qquad T_{zy} = \mathbf{E}_z\mathbf{D}_y,$$

$$T_{zx} = \mathbf{E}_z\mathbf{D}_x, \qquad T_{xz} = \mathbf{E}_x\mathbf{D}_z,$$

and differs from Maxwell's system only as regards its uniform pressure constituent which is now $\frac{1}{8\pi}\mathbf{E}^2$ instead of $\frac{1}{2}(\mathbf{ED})$.

This particular type of stress is analogous to Maxwell's magnetic stress and will be more fully dealt with in the next chapter. When however the dielectric medium is regarded, as at present, as a mere transmitter of the forces between the electric charges, this stress is just as impossible as the former one, as it also requires the existence of body forces on the dielectric medium to maintain it in equilibrium. The type of force required is however very different to that discussed above and admits of a simple physical explanation.

196. Yet another type of stress system can be obtained by reducing the outstanding terms in the general reduction to

$$-\int\left(\mathbf{D}\frac{\partial \mathbf{E}}{\partial x}\right)dv.$$

In this case the stress is specified in the same way as above but with leading terms of the type

$$T_{xx} = \mathbf{E}_x\mathbf{D}_x.$$

This system is of course quite an impossible one as it would require a force on the elements of the dielectric medium to maintain it in equilibrium, even if that medium were free aether. It is however of interest because it is precisely the type of stress which should have been derived from the general theory of electrostatic dielectric forces originated by Korteweg†, formulated in general terms by von Helmholtz‡ and further developed by Lorberg, Kirchhoff§, Hertz|| and others. This general theory, based on the method of energy, is usually made to lead to the first type of stress discussed above,

* *Treatise*, II. p. 278.

† *Wied. Ann.* 9 (1880).

‡ *Wied. Ann.* 13 (1882); *Abhandlungen*, I. p. 798.

§ *Wied. Ann.* 24, 25 (1885); *Abhandlungen, Nachtrag.* p. 91.

|| *Wied. Ann.* 41 (1890).

but the analytical argument by which this is obtained has been criticised by Larmor*, who shows that, properly interpreted, it can only lead to this third type of stress, and must therefore involve some radical fallacy, as in fact is obvious on physical grounds. This criticism appears however to have been entirely overlooked and Helmholtz's procedure is still tacitly accepted and reproduced by all recent writers on the subject†.

* *Phil. Trans.* A. 190 (1897), p. 280. Cf. also Livens, *Phil. Mag.* xxxii. (1916), p. 162.

† Cf. Cohn, *Das electromagnetische Feld*, p. 87 (Leipzig, 1900); Abraham, *Die Theorie der Elektrizität*. i. p. 434 (2nd Ed. Leipzig, 1907): Jeans, *Electricity and Magnetism*, p. 172 (1st Ed. Cambridge, 1908); also the articles by Lorentz, Pockels and Gans in the *Encyclopädie der mathematischen Wissenschaften*, Bd. v.

CHAPTER V

THE THEORY OF POLARISED MEDIA

197. Introduction. We have already seen how the simple laws of action of electric charges are very considerably modified by the introduction into the field of certain substances which we called dielectrics. We have also seen how, up to a certain point, the general properties of the electric fields when such bodies are present can be mathematically expressed in terms of definite quantities definable in terms of the usual functions of the theory, the fundamental assumption being that the properties of electrostatic fields of any dielectric complexity whatever are expressible in terms of the assumed general properties of a continuous dielectric medium occupying the field, the continuity being the essence of the affair. We saw however that this simple theory led us into difficulties when we attempted to investigate the mode of action of the medium, the conclusion being that in general the non-homogeneous dielectric medium cannot transmit the actions between electric charges by means of a simple self-balanced stress system which leaves the elements of that medium in equilibrium.

We are therefore induced to start again on a new path and analyse more intimately and from a different standpoint the origin and cause of the modification of electric actions by the interposition of dielectric substances.

If there be brought near to a charged body A, a rod composed of some dielectric or conducting material, the usual phenomena of electric induction are observed: the ends of the rod near to and remote from the charged body behave just as if they carried respectively charges of the opposite and of the same kind as A. If the rod is made of conducting material it can be charged permanently in this way: on cutting the rod at any point between its two ends and removing it from the neighbourhood of A the separated fragments are found to retain the charges which they appeared to carry under the influence of the charge on A. But if the rod is made of insulating material the separated fragments will be without charge at whatever point the rod be cut. The old-fashioned explanation of this fact is that the neutral rod is supposed to contain in each of its elements or particles equal quantities (comparatively large) of electricity of opposite sign which normally counterbalance one another. When the body is brought into the neighbourhood of the charged body A the electricity on that body attracts that one of the two charges in each element of the neutral body which is of the opposite sign and

repels the other, the result being a separation of the charges. In a conductor the charges are quite free to move about as they like and the displacement may be of finite extent. In a dielectric on the other hand the charges appear to be bound to the molecules of the body by restraining forces of some kind*, quasi-elastic forces we may say, and the displacement is thereby limited to very small molecular dimensions, the charges each settling down into an equilibrium position where the electric forces of the field balance these quasi-electric forces. On such a view all electrification is merely electric separation, i.e. separation of positive and negative charges; this separation may be of finite amount in conductors, but in dielectrics it is confined within the molecule itself.

This theory has been completely substantiated by the discovery of the atomic structure of electricity and the consequent developments of experimental science in the elucidation of the 'electron theory' to which this discovery gave rise. According to this theory every atom contains as an essential element of its constitution a certain number of electrons more or less tightly bound in it, in addition to the necessary positive charge to make it neutral. The application of an electric field will then as above displace the negative electrons relative to the positive ones and thus render each atom bi-polar in the above sense.

198. The theory of polarised dielectric media is a molecular one; the polarisation is supposed to belong to the individual molecules of the dielectric substances or perhaps to molecular groups; the essence of the affair is that the physical element (the smallest thing we are concerned with) is a bi-polar one, i.e. it has two equal and opposite poles of electric charge. Each molecule of a substance contains one or more elements of electrical charge of each sign, which are originally practically coincident; the application of an electric field would then tend to pull them about until some elastic resilience (supposed for the present to be proportional to the displacement) balances the electric pull.

There is another theory of polarised media which assumes that the separate molecules are permanently polarised (i.e. the charges never coincide) but are usually arranged in all sorts of directions. It is owing to the fortuitous distributions of the directions that the total statistical effect is null in the normal condition. The application of an electric field would then turn each molecule round, all towards a definite direction, against elastic resilience and their separate fields would no longer cancel.

Either of these theories would do for the present purposes but we prefer the first for reasons which will afterwards appear. The statistical view of the two is the same, and this is, after all, all that we are concerned with at

* Mossotti assumes that the molecules are like small conductors insulated from one another. Cf. *Sur les forces qui régissent la constitution intime des corps* (Turin, 1836).

present. There are certain facts which suggest that there is, at least in some cases, a certain justification for the second view but these will be dealt with separately.

199. Mathematical formulation of the scheme*. Having thus formed a definite idea of the polarisation in the molecule we must now examine the field of a *polarised* medium regarded as a whole. The mathematical analysis to be here presented was started by Poisson in its application to the theory of magnetisation, but his notions were very elementary principles constructed on the distance action theory. Poisson's theory was transformed into a more physical theory by Kelvin, who added nothing to Poisson's results but explained his mathematical formulae with physical ideas. When Faraday discovered the corresponding phenomenon in dielectrics, Kelvin was at once able to explain it as the analogue of magnetisation in iron. We shall present the analysis in its application to dielectrics first and then transfer the results to the treatment of magnetism later on.

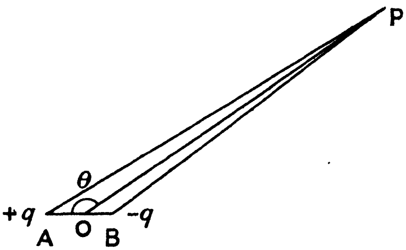


Fig. 46

We start by analysing the electric field of a polarised dielectric mass. The element of the analysis is the simple bi-pole consisting of two point charges $(+q, -q)$ placed at a small distance apart. The law of inverse squares is assumed for each constituent. If the pole $+q$ is at A and $-q$ at B, then the potential due to this element at any point P in the field is

$$\phi = \frac{q}{PA} - \frac{q}{PB}.$$

Let O be the mid-point of AB and θ be the angle AOP; also put $A = \delta s$, $OP = r$, then since δs is very small we have

$$PA = r - \frac{\delta s}{2}\cos\theta, \quad PB = r + \frac{\delta s}{2}\cos\theta,$$

so that
$$\phi = q\left(\frac{1}{r - \frac{\delta s}{2}\cos\theta} - \frac{1}{r + \frac{\delta s}{2}\cos\theta}\right)$$

$$= \frac{q\,\delta s\cos\theta}{r^2} \text{ practically.}$$

* The mode of presentation here adopted is due to Larmor. Cf. *Aether and Matter*, App. A, "On the principles of the theory of magnetic and electric polarisation."

In dealing with elements of this nature we do not as a rule know either q or δs, only the product $q\delta s$; but this is all we are concerned with in investigation of the field at distances large compared with the dimensions of the doublet; we introduce a single symbol m for it and we call it the *moment* of the element: the line AB is called the axis of the bi-pole.

If we put the bi-pole at the origin of rectangular coordinates and use $(\lambda, \mu, \nu)/e$ for the direction cosines of its axis, then the potential of the element at a point at a distance r in a direction (l, m, n) is

$$\phi = \frac{m}{r^2}(l\lambda + m\mu + n\nu)$$

$$= m\lambda \cdot \frac{l}{r^2} + m\mu \cdot \frac{m}{r^2} + m\nu \cdot \frac{n}{r^2},$$

but the first term is the potential of the x-component of the bi-pole. The potential of the element is the sum of those of its components. Thus if in the specification of a bi-pole by its moment we also imply a knowledge of the direction of its axis, we see that the moment can be regarded as a vector quantity, i.e. a directed quantity which is resolvable by the parallelogram law. We may therefore adopt our vector notation and use $\mathbf{e}$ for the moment of a bi-pole.

200. The molecule or element of a dielectric medium may consist of a whole system of simple doublets of the kind here examined. The molecule may contain a whole lot of positive and negative elements and if we group them in pairs (positive and negative) we have a system of bi-polar elements. We could find the resultant of the moments of all these separate elements and we should then define this as the moment of the molecule. If we are considering their effects at ordinary distances the different positions of the centres of all these bi-poles in the molecule will not concern us; for all practical purposes we can regard the centres as coincident on account of the extreme smallness of the molecule. It thus appears that we can treat each molecule, however complicated it may be, just as if it were a simple bi-pole with a definite moment, obtained perhaps by considering the positive charges as practically equivalent to a positive charge at its mean centre and similarly with the negative. The essential point is thus that we can treat the molecule as a simple element and we need not for the present trouble ourselves with details of how it is built up.

Now suppose we have a whole lot of these bi-polar molecules forming a finite mass. We must then treat the thing as a whole and take as the element of our analysis a volume element δv. The aggregate of the moments of all the bi-poles in this element is again obtained by combining them all vectorially without regard to their different positions in the element. The resulting moment must be proportional to δv if that volume is small*;

* 'Physically small.'

suppose it is $\mathbf{P}\delta v$, where of course $\mathbf{P}$ is a vector. This quantity $\mathbf{P}$ expresses the way in which the body is polarised; it is the polarisation *per* unit volume at the position, and in the language of physics is called the *intensity* of the polarisation. This vector expresses all that we can know or recognise experimentally about the polarisation of the body.

If the bi-polar elements or molecules are distributed anyhow in all different directions, $\mathbf{P} = 0$; but if there is any degree of convergence of their axes to a definite direction, then $\mathbf{P}$ has a definite value different from zero.

201. Having now defined this quantity $\mathbf{P}$ which completely specifies the electric state of the polarised body, we can by its means determine the electric field in the neighbourhood of the body, without at present stopping to consider the actual method by which this polarisation is produced.

Each volume element δv_1 of the body is like a little bi-pole of moment $\mathbf{P}_1\delta v_1$ and thus its potential at the point P is evidently

$$\phi = \frac{(\mathbf{P}\mathbf{r}_1)}{r^2}\,\delta v_1,$$

$\mathbf{r}_1$ denoting the unit vector along the direction of the radius r_1 from the position of the element δv_1 to P. The potential of the whole body at any point P is therefore

$$\phi = \Sigma\,\frac{(\mathbf{P}_1\mathbf{r}_1)}{r_1^{\,2}}\,\delta v_1,$$

wherein Σ denotes a sum taken over all the elements δv of the body. This may also be written in the form of an integral

$$\phi = \int \frac{(\mathbf{P}_1\mathbf{r}_1)}{r_1^{\,2}}\,dv_1,$$

which is the same as

$$-\int (\mathbf{P}_1\nabla)\,\frac{dv_1}{r_1},$$

the vectorial operator ∇ involving differentials with respect to the coordinates of P.

The intensity of force at the point P in the field can now be written down in an analogous manner. It appears as a vector—the negative gradient of the potential ϕ

$$\mathbf{E} = -\operatorname{grad}\phi.$$

One proviso, and an important one, has been missed out of the above statement: *the point P must be well outside the dielectric substance.*

202. The integral definitions here given necessarily involve some sort of continuity in the distribution of the polarisation intensity, and rather more than is actually the case in a real medium constituted of discrete molecules. To give them a definite sense in a strict mathematical theory

we can however replace as in the first chapter the real medium by a perfectly continuous distribution of polarisation with the proper intensity **P** at each point. This hypothetical distribution is effectively equivalent to the real one at all points of space which are not too near it.

We may next enquire as to how close to the distribution does this representation of the force and potential by integrals remain valid. In this connection it must first be noticed that since we have assumed the nucleus of the bi-pole which forms the basis of this theory to be entirely confined within a molecule or atom it may reasonably be supposed that the law of its action as defined above remains valid up to within a physically small differential distance from the molecule, which is a length defined so as to include a large number of molecular diameters; in other words the effective combination of positive and negative elements of charge in the molecule into doublets in the manner specified above is valid up to within this small distance from the molecule. But the substitution of an effectively continuous charge distribution for the distribution of both positive and negative elements thus combined is valid to the same extent, so that we may conclude that the hypothetical distribution of polarisation effectively replaces the actual discrete one as regards its field up to within a physically small differential distance from the polarised body, this being the distance at which the actual distribution of the charge in any small volume element ceases to be irrelevant.

203. The next process in the development of the mathematical theory is to specify the electric field at points in the interior of the polarised medium. Waiving for the present any other difficulties involved in the extension, let us assume that the hypothetical continuous distribution of polarisation effectively replaces the old one at all points of the field however near to the medium it may be and let us examine the force and potential at the internal point. We might jump to the conclusion that in this case the force and potential are correctly represented by the integrals as given above for external points; but any attempt to use these definitions for internal points for which the corresponding integrands both become infinite must be preceded by a justification of their convergence at such points. If the integrals representing them are convergent at internal points, then the force and potential so defined will have definite meanings at such points.

From a physical point of view we see that the contribution to the values of these functions at any point from adjacent parts of the medium involves a large factor in the integral and we want to know whether their aggregate is comparable with that from the rest of the body. If this is the case the integrals are at best semi-convergent and the definitions are almost useless, because we do not know anything about the local configuration of the elements; it may be anything for all we know.

But we have already seen that the integral expressing the potential is absolutely convergent, so that on the present basis, the local contribution due to the continuous distribution of polarity near the point under investigation is negligible : strictly speaking the effects of these adjacent parts involves the dimensions of their volume linearly and thus in the aggregate their effect is negligible compared with that of the rest of the body. The physical way of stating this is, as we had it before, that the scooping out of a vanishingly small cavity round P makes no difference to the integral; the shape of the cavity does not matter so long as it is indefinitely small.

It is however otherwise with the integral for the force at P. The x-component of this force is in fact represented by the integral

$$-\int \operatorname{grad}_x (\mathbf{P}_1 \nabla) \frac{dv_1}{r_1}$$

which, if we use (x_1, y_1, z_1) as the coordinates of the element dv_1 and (x, y, z) as those of P, can be written in the form

$$\int dv \left[-\frac{\mathbf{P}_x}{r_1^3} + 3\mathbf{P}_x \frac{\overline{x - x_1}^2}{r_1^5} + \frac{3(x - x_1)}{r_1^5} (\overline{y - y_1}\mathbf{P}_y + \overline{z - z_1}\mathbf{P}_z) \right],$$

which is precisely of the type which was stated in the introduction to be semi-convergent. The local parts of the polarisation even in the hypothetical continuous medium thus have an effective influence on the value of the force. Thus our definitions of the force, at least, in the internal field by means of the effectively continuous distribution of polarisation breaks down! This does not however seriously disturb our formulation of the theory because we can proceed as in the first chapter to modify the definitions to make them more consistent with a physical theory.

204. The necessary modification will come better if we first obtain Poisson's* transformation of the potential integral given above. We shall assume that the point P at which the field is investigated is well outside the dielectric substance, so that there is no doubt about an application of the above definitions and the consequent effectiveness of the continuous distribution of polarisation. If P is actually inside the medium we just remove a part of the medium inside a physically small cavity round P, so that it is still in free space. We can then adopt without any further hesitation the above definitions for the force and potential at P. The potential is in fact

$$\phi = -\int (\mathbf{P}_1 \nabla) \frac{dv_1}{r_1},$$

but since
$$\nabla \frac{1}{r_1} = -\nabla_1 \frac{1}{r_1},$$

* L.c. p. 164.

where ∇_1 denotes the same vector differential operator as ∇ but taken with respect to the coordinates (x_1, y_1, z_1) of dv_1 instead of (x, y, z) as above, this is also

$$\phi = \int (\mathbf{P}\nabla_1) \frac{dv_1}{r_1},$$

the integrals in each case being taken throughout the volume of the polarised medium, excluding the part removed from the small cavity about P if it is made, a simple transformation by Green's theorem shows that

$$\phi = \int dv_1 (\mathbf{P}\nabla_1) \frac{1}{r_1}$$

$$= -\int \frac{\operatorname{div} \mathbf{P}}{r_1} dv_1 + \int \frac{\mathbf{P}_n}{r_1} df_1,$$

where the surface integral is extended over the surface of the body (including the walls of the cavity if P is inside) and the volume integral over the volume of the body.

This means that the potential of this polarised body is the same as the gravitational potential of the mass distribution specified as:

(i) a volume density $\rho = -\operatorname{div} \mathbf{P}$

throughout the body excluding the cavity if made; although, as a matter of fact, the cavity may be filled in with the continuation of this distribution throughout its volume without making any appreciable difference to the integral for ϕ which is convergent at the point.

(ii) a surface density $\sigma = \mathbf{P}_n$*

over the surface of the body and cavity.

All this applies only to a point outside the substance of the polarised medium. If the point is right outside the medium the values of the force and potential due to such a mass distribution are quite definite and are in fact identical with those already given on the more direct definition; this distribution of attracting charges or masses effectively replaces the distribution of polarisation as regards its action at all external points. It is however in the analysis of the field at internal points that this mode of treatment helps us.

205. If the point P is inside the polarised medium we can draw round it a small surface whose linear dimensions are physically small. The distribution of polarisation in the medium outside this surface can then as regards its action at P be effectively replaced by the continuous distribution of attracting masses just described. We thus see that the total field at P can be separated into distinct components. Firstly the volume distribution ρ

* At an interface between two different dielectric media there is a surface distribution of density

$$\sigma = P_{n_1} - P_{n_2}.$$

outside the surface gives a definite force and potential at P, no matter what size or shape the cavity may be, provided only that it is physically very small. Similar remarks apply to the surface distribution σ on the outer surface of the body. But the distribution σ on the walls of and the distribution of molecules inside this cavity give a potential and force at P which depend entirely on the shape and size of the cavity even if it is very small and will in general be comparable with the other parts. This latter component of the field is however a purely local part depending entirely on the molecular configuration round the point and the conditions of polarisation existing in them: as we do not know the local molecular configuration, which may be changing rapidly, we cannot know what this local part really amounts to; but we have succeeded in separating it from the main part of the action due to the rest of the body.

We now adopt the arbitrary course of simply neglecting this local molecular part of the field, so that we can confine our discussions entirely to that definite part of the force which is due to the medium as a whole, i.e. the molar part. This is merely following a usual method in physics and involves but a simple extension of the ideas underlying the Young-Poisson principle of the mutual compensation of molecular forcives employed in their theory of capillary actions*. It requires that such local forcives shall set up a purely local physical disturbance of the molecular configuration in the material, until it is thereby balanced. Another example of this principle is provided in the ordinary theory of elasticity where in addition to the local strain forces in an elastic medium there are the comparatively very powerful cohesive forces, which are however presumed to form an equilibrating system and not to affect the phenomena as a whole. It is fortunate that we can in this way eliminate the influence of the neighbouring elements.

We can therefore define the electric field inside the body as that field when the effect of the local part is omitted and this definition will apply quite consistently to outside points as well. We have been able to separate the local part from the total, and the field of force of our subsequent discussion is that due to the rest. On this definition the integrals expressing the force and potential are always convergent and apply to inside as well as outside points as they are then just like the corresponding functions of our former theory involving only ordinary volume and surface distributions of charge. The electric force is then always the gradient of the potential.

206. Having thus fitted up these consistent definitions of the functions involved let us see how they work in the theory. The electric force vector $\mathbf{E}$

* Cf. Larmor, *Aether and Matter*, App. A; Young, "On the Cohesion of Fluids," *Phil. Trans.* (1805); Poisson, *Nouvelle Théorie de l'Action Capillaire* (Paris, 1831); Rayleigh, "On the theory of surface forces," *Phil. Mag.* 1883, 1890, 1892; especially 1892 (1), pp. 209–220; Van der Waals, *Essay on the continuity of the liquid and gaseous states.*

is still the gradient of the potential ϕ. Also since ϕ is due to the specified volume and surface distribution

$$\nabla^2\phi = -4\pi\rho$$
$$= +4\pi \operatorname{div} \mathbf{P} \quad \text{.................................(i)},$$

at each point of the field and at the boundary of the dielectric in air

$$\frac{\partial\phi_1}{\partial n_1} - \frac{\partial\phi_2}{\partial n_2} = -4\pi\sigma = -4\pi\mathbf{P}_n \quad \text{.....................(ii)},$$

but $$\mathbf{E} = -\operatorname{grad}\phi = -\nabla\phi,$$

so that $$4\pi \operatorname{div} \mathbf{P} = \nabla^2\phi = -\operatorname{div} \nabla\phi = -\operatorname{div} \mathbf{E},$$

or $$\operatorname{div}(\mathbf{E} + 4\pi\mathbf{P}) = 0.$$

The vector $(\mathbf{E} + 4\pi\mathbf{P})$ is therefore a streaming vector, it satisfies the usual equation of continuity of incompressible fluid flow. We call it the vector of *electric displacement** and denote it by $4\pi\mathbf{D}$; the factor 4π is introduced for a reason which will subsequently appear. This electric displacement is the important vector of the theory: its importance lies in the fact that the flux or displacement through any surface only depends on its boundary so that we can take the flux as estimated as so much through a circuit. For we have

$$\operatorname{div} \mathbf{D} = 0,$$

so that if we take any closed surface f in the field we get

$$\int (\operatorname{div} \mathbf{D})\, dv = 0$$

taken throughout the region bounded by f. But by Green's theorem this consists of

$$\int \mathbf{D}_n df$$

together with the surface integrals arising from discontinuities when we pass into the polarised medium; these are the integrals of

$$-(\mathbf{D}_{n_1} - \mathbf{D}_{n_2})$$

over the parts of the surfaces concerned which are inside f, or

$$-\frac{1}{4\pi}(\mathbf{E}_{n_1} - \mathbf{E}_{n_2} + 4\pi\mathbf{P}_{n_1} - 4\pi\mathbf{P}_{n_2}),$$

which is zero: we thus have

$$\int \mathbf{D}_n df = 0.$$

Now suppose we have an unclosed surface f abutting on the closed curve s, then we can take another surface f' with the same curve s as boundary and the two surfaces f and f' together form a closed surface. Thus if n

* Maxwell, *Treatise*, I. p. 64. The vector D is the displacement proper.

denote the normal to the element of either surface in a definite sense through the circuit, the above equation gives

$$\int_f \mathbf{D}_n\, df - \int_{f'} \mathbf{D}_n\, df' = 0,$$

and the result is as stated.

The real significance of these results does not however appear until we discuss the subject of electromagnetism, it is hidden by the more general circumstances under which the present theory has to be developed.

207. The general problem. We have so far merely discussed the fields of polarised media without any reference to the way the polarisation is created. As a rule however we cannot have dielectrics polarised unless they exist in an external electric field, i.e. in an imposed field due to an extraneous electrical system. The introduction of the dielectric into the field results in each element of it being turned into a little bi-pole in the manner above indicated and the total field of all these bi-poles has alone been under investigation, although as a matter of fact it merely represents the addition to the original field brought about by the introduction of the dielectric substance into it. For the general case therefore we must superpose on the field above investigated that original field which existed before the dielectrics were introduced and which we shall for the present suppose to be due to certain volume and surface densities ρ_0 and σ_0. The above discussion of the convergence of the force and potential integrals is not hereby affected, since the additional parts of these functions due to such a distribution are already known to have definite values at all points of space. The electric force is therefore still the gradient of a potential function. We now use ϕ_1 for the potential of the field above investigated, ϕ_0 for the potential of the original field, and ϕ for that of the total field; a similar suffix-notation is also adopted for the other quantities involved. We have now

$$\phi = \phi_0 + \phi_1,$$

and
$$\mathbf{E} = -\operatorname{grad}\phi = -\nabla\phi.$$

Thus
$$\nabla^2\phi = \nabla^2\phi_0 + \nabla^2\phi_1,$$

and
$$\nabla^2\phi_0 = -4\pi\rho_0, \quad \nabla^2\phi_1 = -4\pi\rho_1 = -4\pi \operatorname{div} \mathbf{P},$$

so that we now have
$$\operatorname{div}(\mathbf{E} + 4\pi\mathbf{P}) = 4\pi\rho_0.$$

The induction or displacement vector is no longer a stream or solenoidal as we say; if we again use $4\pi\mathbf{D}$ to represent it we have

$$\operatorname{div}\mathbf{D} = \rho_0.$$

208. At a boundary surface of the dielectric medium which also carries a surface charge of density σ_0 we have

$$\begin{aligned}\frac{\partial\phi}{\partial n_1} - \frac{\partial\phi}{\partial n_2} &= \left(\frac{\partial\phi_1}{\partial n_1} - \frac{\partial\phi_1}{\partial n_2}\right) + \left(\frac{\partial\phi_0}{\partial n_1} - \frac{\partial\phi_0}{\partial n_2}\right)\\ &= -4\pi\sigma_1 - 4\pi\sigma_0,\end{aligned}$$

or $$-\mathbf{E}_{n_1} + \mathbf{E}_{n_2} = -4\pi\mathbf{P}_n - 4\pi\sigma_0,$$

or $$-\mathbf{E}_{n_1} + (\mathbf{E}_{n_2} + 4\pi\mathbf{P}_n) = -4\pi\sigma_0.$$

In free space, i.e. on side 1 of general surface of discontinuity,

$$4\pi\mathbf{D} \equiv \mathbf{E};$$

in the dielectric medium however

$$4\pi\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P},$$

so that the above surface condition can be written as

$$\mathbf{D}_{n_1} - \mathbf{D}_{n_2} = \sigma_0.$$

The normal induction is discontinuous across the surface charge by an amount σ_0; if there is no surface charge

$$\mathbf{D}_{n_1} = \mathbf{D}_{n_2}.$$

Similar conditions are also found to hold at charged or uncharged surfaces of discontinuity in the dielectric medium itself, i.e. surfaces separating not the dielectric medium from a vacuum, but one medium from a second different one.

We have thus a complete specification of the field in terms of the electric force **E**, the electric induction or displacement **D** and the intensity of polarisation **P** where we know that at each point of the field

$$4\pi\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$$

in the vector sense. These three vectors give us the distribution of force induction and polarisation; but any two of them are sufficient as the third is determined when the other two are known. Although the vector **P** is perhaps the more fundamental physical one we shall regard the first two vectors as the independent variables; they turn out to be the more significant ones of the theory.

209. *The law of induction.* We have now examined the electrical field under the conditions of the dielectric being present and having induced in it a polarisation of intensity **P** at each place. But how do we know what polarisation will be induced in a given dielectric substance and of what use is the above analysis? We evidently want an additional physical principle to complete the scheme.

The electric force at any point in the dielectric medium is **E** (neglecting the local part) and the electric displacement induced is **D**, and as we have pictured the affair to ourselves the polarisation and therefore the displacement is conditioned by the electric field. Thus if there is to be any law about the matter at all one of these quantities is a function of the other. The simplest possible relation we could have is a simple proportionality so that if we double the cause we double the effect: this is moreover the only workable relation mathematically and so our procedure is to work out the results on it and see

how they are experimentally justified. Expressed mathematically a relation of this kind means that the components of the displacement are linear functions of the components of the electric force

$$4\pi\mathbf{D}_x = \epsilon_{11}\mathbf{E}_x + \epsilon_{12}\mathbf{E}_y + \epsilon_{13}\mathbf{E}_z,$$
$$4\pi\mathbf{D}_y = \epsilon_{21}\mathbf{E}_x + \epsilon_{22}\mathbf{E}_y + \epsilon_{23}\mathbf{E}_z,$$
$$4\pi\mathbf{D}_z = \epsilon_{31}\mathbf{E}_x + \epsilon_{32}\mathbf{E}_y + \epsilon_{33}\mathbf{E}_z,$$

or expressed shortly by a vector equation

$$4\pi\mathbf{D} = (\epsilon)\,\mathbf{E}.$$

For homogeneous media this relation would assume the simpler form

$$4\pi\mathbf{D} = \epsilon\mathbf{E}.$$

We might of course assume more generally that

$$4\pi\mathbf{D} = \epsilon\mathbf{E} + \epsilon_1\mathbf{E}^2 + \ldots,$$

but we presume that if ϵ is small the other terms beyond the first are negligible and we find that it fits the facts. In any case the simpler form is right for very small fields and anything more complicated is mathematically unworkable.

It might be thought that it would be better to take the polarisation as proportional to the total electric force *including* the local part. The local influences have however been regarded as equal and opposite actions and reactions occurring in and between the molecules concerned and cannot add anything to the total result in any definite direction. The presumption is that these local effects are erratic and cannot influence a vector effect at the place.

210. We can now complete our formulation of the scheme: we have in the vector sense

$$4\pi\mathbf{D} = (\epsilon)\,\mathbf{E}$$

as our physical relation; but

$$4\pi\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P},$$

and thus

$$\left(\frac{(\epsilon) - 1}{4\pi}\right).\,\mathbf{E} = \mathbf{P},$$

and $\mathbf{P}$ the electric polarisation intensity is thus also proportional to $\mathbf{E}$.

Also on this theory

$$4\pi\mathbf{D} = (\epsilon)\,\mathbf{E} = -\,(\epsilon)\,\mathrm{grad}\,\phi,$$

and since

$$\mathrm{div}\,\mathbf{D} = \rho,$$

we have

$$\mathrm{div}\,\{(\epsilon)\,\mathrm{grad}\,\phi\} = -\,4\pi\rho_0,$$

which is the characteristic equation of the theory; the potential function ϕ must always satisfy this equation when there are dielectrics about.

Notice that for isotropic media this becomes

$$\frac{\partial}{\partial x}\left(\epsilon\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\epsilon\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\epsilon\frac{\partial\phi}{\partial z}\right) = -\,4\pi\rho_0.$$

The boundary conditions to apply at a surface of discontinuity in the medium on which there is also a surface charge of density σ_0 are obtained in the usual way; they are

$$\phi_1 = \phi_2,$$

$$(\epsilon_1)\frac{\partial \phi_1}{\partial n} - (\epsilon_2)\frac{\partial \phi_2}{\partial n} = 4\pi\sigma_0,$$

or if $\sigma_0 = 0$ simply

$$(\epsilon_1)\frac{\partial \phi_1}{\partial n} = (\epsilon_2)\frac{\partial \phi_2}{\partial n}*.$$

211. We are thus enabled to express everything in terms of the potential function of the field which must satisfy the above characteristic equations and relations at each point of the field. We notice that these relations are identical in form with those obtained from the simple Faraday-Maxwell theory in the previous chapter, the constant ϵ being the same as that there adopted: the electrical conditions in the field are therefore the same on either theory, as of course they must be: the only difference lies in the physical basis for the mode of action of the dielectric medium in modifying the electric field by which it is surrounded. The advantage of the present theory lies however in another direction. When examining the energy and ponderomotive forces in the electric field containing dielectrics on the Maxwellian theory we got into difficulties. We found that an ordinary medium with the characteristic elasticity of solid bodies could not support the supposed state of strain between electrified bodies and be in equilibrium in its elements, the application of an extraneous couple of amount proportional to its surface being in general necessary to keep it balanced. If however we imagine an ordinary elastic medium full of elementary bi-poles of the type now under investigation with orientations distributed according to some law or even at random, and in internal equilibrium either in its own or an external field, then on rotational distortion a couple will be required to hold each element in equilibrium: the present mode of formulation of the theory will thus in all probability help us out of our former difficulties in this connection.

212. The energy relations of dielectric media. We can now analyse on the bases of the foregoing principles the distribution of energy and transmission of force in an electrical field throughout which dielectric media of the kind specified are distributed. The problem is however much more complex now, involving as it does all the difficulties and refinements of the statistical method in the mechanics of molecularly constituted media.

As an introduction it will be perhaps best to give a short sketch of the general principles underlying the simple dynamical problem of the motion of a body or system of bodies composed of an enumerable aggregate of molecules,

* These conditions are perhaps not so happily expressed as they might have been. In the general case of anistropic media the expression of the normal component of the induction vector in terms of the potential gradients requires great care.

each of which may be treated, for the purposes of theoretical dynamics, as a single mass particle. In such a problem we have to deal with the large number of mass-particles which we may typify by the mass m_r at the point whose coordinates at time t referred to any convenient rectangular axes are (x_r, y_r, z_r). Each of these masses will in the general case be subject to the action of forces which may be of different types; firstly there is the definitely controllable force exerted directly by external systems, which we may take to be equivalent to a resultant force $\mathbf{F}_r$ on the mass m_r; these are the so-called *external forces*: there is next the direct resultant forces of type $\mathbf{F}_{rr'}$ exerted on this same mass m_r due solely to and conditioned by the presence of the other mass particle $m_{r'}$ in the system; these are called the *internal* forces. Lastly there are the forces of constraint exerted indirectly as a consequence of some condition restricting the motion of the system, such as for example that certain points of it are forced to move along prescribed paths: these forces are typified by their resultant $\mathbf{F}_r'$ on the mass m_r and are called the reaction forces on the system. The equations of motion of the mass m_r are then of type

$$m_r \frac{d^2\mathbf{s}_r}{dt^2} = \mathbf{F}_r + \sum_{r'} \mathbf{F}_{rr'} + \mathbf{F}_r',$$

$\sum\limits_{r'}$ denoting a sum taken over all the mass points excluding the one m_r whose velocity is represented by the vector $\dfrac{d\mathbf{s}_r}{dt}$. There is one vector equation of this type for each particle in the system; they can however all be summed up into one equation, viz. the general differential equation of virtual work in the system: if $\delta\mathbf{s}_r$ denotes vectorially an arbitrary virtual displacement of the mass m_r, this equation is

$$\Sigma m_r \left(\frac{d^2\mathbf{s}_r}{dt^2}\,\delta\mathbf{s}_r\right) = \sum_r (\mathbf{F}_r\delta\mathbf{s}_r) + \sum_r\sum_{r'} (\mathbf{F}_{rr'}\delta\mathbf{s}_r) + \sum_r (\mathbf{F}_r'\delta\mathbf{s}_r).$$

The terms on the right represent respectively the total amounts of the work done during the displacement by the external forces, the internal forces and the reactions from the constraints. If the displacement system typified by $\delta\mathbf{s}_r$ is consistent with the implied geometrical restraints in the system, the total work of the reaction forces vanishes and the equation simplifies to

$$\Sigma m_r \left(\frac{d^2\mathbf{s}_r}{dt^2}\,\delta\mathbf{s}_r\right) = \sum_r (\mathbf{F}_r\delta\mathbf{s}_r) + \sum_r\sum_{r'} (\mathbf{F}_{rr'}\delta\mathbf{s}_r).$$

If the geometrical restrictions on the motion of the system are such that in the actual motion the reaction forces do no work, a still simpler form can be deduced from this equation; for a possible displacement system in the last equation is that actually taken by the system during the next succeeding small interval of time so that

$$\delta\mathbf{s}_r = \frac{d\mathbf{s}_r}{dt}\,\delta t,$$

and the equation becomes with these values of $\delta \mathbf{s}_r$

$$\Sigma m_r \left(\frac{d^2\mathbf{s}_r}{dt^2} \cdot \frac{d\mathbf{s}_r}{dt}\right) \delta t = \sum_r \mathbf{F}_r \delta \mathbf{s}_r + \sum_{r}\sum_{r'} (\mathbf{F}_{rr'} \delta \mathbf{s}_r).$$

We now use

$$\delta E_0 = \sum_r \mathbf{F}_r \delta \mathbf{s}_r$$

for the work done on the system by the external forces;

$$\delta E_i = \sum_{r}\sum_{r'} (\mathbf{F}_{rr'} \delta \mathbf{s}_r)$$

for the work done by the internal forces which might also with the reverse sign be counted as the increase of the mutual potential energy of the internal configuration of the system: and finally

$$T = \tfrac{1}{2} \Sigma m_r \left(\frac{d\mathbf{s}_r}{dt}\right)^2$$

for the total kinetic energy of the system. The equation of virtual work in the special form to be adopted is thus

$$\delta T = \delta E_0 + \delta E_i$$

which merely expresses that the increase in the kinetic energy of the system is effected partly at the expense of the external systems and partly at the expense of the store of potential energy in the medium.

213. But for purposes of the dynamical phenomena of material bodies, which we can only test by observation and experiment on matter in bulk, a complete atomic analysis of the kind here involved would (even if possible) be useless*; for we are unable to take direct cognisance of a single molecule of matter. The development of the theory which is to be in line with experience must instead concern itself with an effective differential element of volume, containing a crowd of molecules numerous enough to be expressible continuously as regards their average relations, as a volume density of matter.

It will therefore be necessary for us to interpret the above equation in terms of average quantities for the medium as a whole. As regards the kinetic energy of the motion we can interpret it as the kinetic energy of the average drifts or orientation of the molecules; but this always leaves out of account an average residue of energy concerned with the motion of the molecules devoid of any recognised regularity (which is superimposed on the regular motion sorted out) and of which we know nothing except its quantity: this part of the kinetic energy is the thermal part and is a function only of the temperature of the medium. It is the only part that exists if there is no visible motion of the medium as a whole.

Again the actual interactions between the molecules represented by forces of the type $\mathbf{F}_{rr'}$ are also necessarily presented to us divided into various

* Cf. Larmor, *Aether and Matter*, p. 87.

groups, which would be the subject of perception by different senses, so that their virtual work consists of several distinct parts. There is a part involving the interaction, with any molecule under consideration, of other molecules of the system at finite distances, which integrates into an energy function of the applied mechanical forces exerted between the various elements of the system. Of the remainder of this work, which arises from the mutual actions of neighbouring molecules, a regular or organised part can be separated out which represents the energy of elastic stress and is a function of the deformation of the volume treated as a whole : this stress arising from the immediate surroundings in part compensates for the element of mass under consideration, the applied mechanical forces aforesaid. The remaining usually wholly irregular part may be considered as compensated on the spot by other such forces of different origin that are not at present under review*.

If therefore we use T_0 for the kinetic energy of the organised motion of the medium, T_i for that of the irregular heat motion, W_i the potential function of the mechanical forces between the different elements of the system and W_s the energy of elastic stress, the virtual work equation can be written in the form

$$\delta T_0 + \delta T_i = \delta E_0 - \delta W_i - \delta W_s;$$

if we limit ourselves to the statical case with the dielectric media as a whole at rest, then $T_0 = 0$ and the equation is simply

$$\delta T_i = \delta E_0 - \delta W_i - \delta W_s,$$

and $\delta E_0 - \delta W_i$, which represents the work done by all the directly observable mechanical forces on the elements of the media, may be treated as a single quantity δW_0, which is the potential function of these forces

$$\delta T_i = \delta W_0 - \delta W_s;$$

an equation of fundamental importance in thermodynamics.

214. In the present theory of polarised media the problem is however rather more complex than that just discussed. The molecules of a dielectric medium are not simple mass particles, but each involves in its essential constitution a more minute system of electrically charged particles (the electrons), all more or less lightly bound to it by some quasi-elastic resilience. The application of an electric field would thus be effective in producing a strained condition within the molecules themselves, of a more fundamental type than that already discussed. The average stress in this intra-molecular deformation would be superimposed on that discussed above, so that δW_s now consists of two parts the first corresponding to the ordinary elastic stress which we shall still denote by δW_s and the second to the internal quasi-elastic stress in the molecules, which we can denote by $\delta W_s'$†.

* Cf. Larmor, *Phil. Trans.* A, 190 (1897), §§ 48, 49.

† It is fairly obvious that since the two strains here involved are absolutely independent their energies would be additive.

The kinetic energy T_i would also now contain a part due to the relative motions of the electrons in the molecules, we may call this part T_i' using T_i still for the thermal energy ordinarily associated with the irregular heat motion of the molecules*.

The general equation of virtual work is thus

$$\delta T_i + \delta T_i' = \delta W_0 - \delta W_s - \delta W_s'.$$

A simplification is now introduced into the theory by the fact that observation tends to confirm the view that processes directly concerned with the internal constitution of a molecule are, with few exceptions, almost entirely independent of the temperature of the substance in which the molecule exists: in other words the terms δT_i and $-\delta W_s$ and a part perhaps of δW_0 of the general equation above depending on and affecting the temperature balance naturally among themselves leaving the general equation of virtual work of the electrical system and its mechanical reactions in the form

$$\delta W_0' = \delta T_i' + \delta W_s',$$

$\delta W_0'$ being the part of the work of the mechanical forces directly concerned with the electrical attractions, it would not for instance include the part of δW_0, if such existed, corresponding to the external pressure balancing the fluid pressure at the boundary of a gaseous dielectric which depends essentially on the temperature, although of course it would in the main be represented by the work of applied boundary pressures superimposed on these thermal ones if they exist. This simplification is however not always possible and then the more general equation with the undashed letters must be retained. In any case the argument is the same.

215. Now let us apply these considerations to the case of a mass of some dielectric substance in an electric field. The applied forces on the mass will then be mainly of two kinds; the mechanical forces applied generally as pressures on the outer boundaries of the media and the electrical forces exerted on the charges rigidly connected therewith. We shall denote the work in these two parts by δW and δW_e respectively and then in the most general case our equation of virtual work assumes the form

$$\delta W_e = \delta E_i - \delta W,$$

where we use δE_i generally to denote the change of the total internal energy in the medium of both elastic and motional types

$$\delta E_i = \delta T_i + \delta W_s.$$

216. But δW_e is easily calculated: in fact the energy required to establish any constituent bi-polar element at any point in the medium can be regarded as mathematically equivalent to the work required to separate

* The part T_i would be the kinetic energy of motion of each molecule with its mass collected at its centre of gravity: T_i' would then be that of the motion relative to the centre of gravity: these two parts are additive in the ordinary way.

the positive pole $+q$ from coincidence with the negative pole $-q$. If the intensity of force at the point is $\mathbf{E}$, supposed uniform in the neighbourhood of the element, and $\mathbf{e}$ represents the vector moment of the doublet finally established, this work is easily seen to be equal to

$$(\mathbf{E}\mathbf{e}).$$

Thus summing for all the doublets in the element of volume δv we have the total work done in polarising the element equal to

$$\delta W_e = \Sigma\,(\mathbf{E}\mathbf{e}).$$

In the present theory of polarised media we saw that the force $\mathbf{E}$ at any point internal to the medium consists of a definite molar and an irregular molecular part which we succeeded in separating by means of the ideal volume and surface densities of Poisson; the method consisting essentially in computing the force by combining opposed poles of neighbouring elements, instead of taking the single polarised element as the unit. It appears that the adjacent poles nearly compensate each other except as regards a simple volume density whose attraction has no local or molecular part and a surface density partly at the outer surface and partly at the surface of the cavity which contains the point under consideration. The effect of the latter surface density depending as it does wholly on the immediate surroundings is the molecular or cohesive part of the average forcive. It is the irregular part of the forcive on the contained element of the dielectric which arises from the excitation of neighbouring molecules and is expressed in terms of them alone. It is not transmitted by a material stress; but forms a balance on the spot with cognate internal molecular forcives of other types. Thus in seeking for the mechanical relations for the dielectric as a whole we shall be justified in neglecting this local part of the total force and its associated energy. We can thus use $\mathbf{E}$ as the electric force as defined above, omitting the local part; and it is then clear that $\mathbf{E}$ will be practically constant throughout the small volume element δv and thus

$$\delta W_e = (\mathbf{E},\ \Sigma\mathbf{e}),$$

but

$$\Sigma\mathbf{e} = \mathbf{P}\delta v,$$

and thus the work done in polarising the element δv to intensity $\mathbf{P}$ is

$$\delta W_e = (\mathbf{E}\mathbf{P})\,\delta v,$$

and for the whole medium the work done is

$$W_e = \int (\mathbf{E}\mathbf{P})\,dv.$$

This is the energy required by the system as a whole on account of the polarisation induced in it.

217. As explained above this energy is to be regarded as consisting partly in the mechanical potential energy of the polarisations of the elements

of volume and partly in mechanical work done against internal quasi-elastic forces preventing displacement of the elementary charges ultimately constituting its polarisation. To effect a separation of the two parts thus involved we proceed by the method, usual in such problems, of varying the configuration of the system generally and calculating the coefficients of each part of the variation in the general expression for the virtual work thus obtained.

An infinitesimal displacement of the volume δv from a place where the field is $\mathbf{E}$ to where it is $\mathbf{E} + \delta\mathbf{E}$ involves a total change in W_e equal to

$$\delta W_e = [(\mathbf{E}\delta\mathbf{P}) + (\mathbf{P}\delta\mathbf{E})]\,\delta v$$

and then it is at once obvious that the second part with its sign reversed, viz.

$$-(\mathbf{P}\delta\mathbf{E})\,\delta v,$$

is the virtual work δW of the mechanical forces performed during the shifting of the element, for it is the part of δW_e which remains when the polarisation of the element is held rigid during the displacement so that no work is done in the internal degrees of freedom corresponding to the displacements involved in it.

The other part of the total energy δW_e expended during the displacement of the volume element δv corresponds to δE_ι and thus

$$\delta E_\iota = (\mathbf{E}\delta\mathbf{P})\,\delta v,$$

or integrated throughout the system for the total. This represents work done against the quasi-reactions to the setting up of the polarisation. It has nothing whatever to do with the mechanical forces on the dielectric as a whole, but is stored up in the polarisation of the medium as internal energy of intra-molecular strain*.

218. We may now draw some important conclusions respecting the relations between $\mathbf{P}$ and $\mathbf{E}$, which follow directly from a simple application of the energy principle with the forms for the energy above determined. Confining ourselves to the element δv we see that the work supplied by it to outside systems, which it drives, in traversing any path is

$$-\int \delta W = dv \int (\mathbf{P}\delta\mathbf{E}),$$

the integrals being taken along the path. If $\mathbf{P}$ is a function of $\mathbf{E}$ so that the operation is reversible this work must vanish for any closed cycle, otherwise energy would inevitably be created, either in the direct path or else in the reversed one, for the complete system of which dv is an element. The negation of perpetual motion therefore requires in this case that

$$(\mathbf{P}\delta\mathbf{E}) = d\phi$$

is a complete differential of some function of $\mathbf{E}$. Moreover this function ϕ

* The argument here employed is given implicitly by Larmor, *Phil. Trans.* A, 190 (1897); and in more detail for the cognate magnetic problem in *Proc. R. S.* 71 (1903), pp. 236–239.

can only involve even powers of $(\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z)$ and for weak fields is practically quadratic. Its coefficients are then the six electric constants for general anisotropic media, no rotational quality in the polarisation being thus allowable by the doctrine of energy;

$$\phi = \tfrac{1}{2}\left[\epsilon_{11}'\mathbf{E}_x^2 + \epsilon_{22}'\mathbf{E}_y^2 + \epsilon_{33}'\mathbf{E}_z^2 + 2\epsilon_{12}'\mathbf{E}_x\mathbf{E}_y + 2\epsilon_{23}'\mathbf{E}_y\mathbf{E}_z + 2\epsilon_{31}'\mathbf{E}_z\mathbf{E}_x\right],$$

and then

$$\mathbf{P}_x = \epsilon_{11}'\mathbf{E}_x + \epsilon_{12}'\mathbf{E}_y + \epsilon_{13}'\mathbf{E}_z,$$
$$\mathbf{P}_y = \epsilon_{21}'\mathbf{E}_x + \epsilon_{22}'\mathbf{E}_y + \epsilon_{23}'\mathbf{E}_z,$$
$$\mathbf{P}_z = \epsilon_{31}'\mathbf{E}_x + \epsilon_{32}'\mathbf{E}_y + \epsilon_{33}'\mathbf{E}_z.$$

This is the most general linear relation between the polarisation and polarising force which is true for crystalline bodies. We can simplify by taking the proper coordinates for ϕ so that it becomes

$$\phi = \tfrac{1}{2}(\epsilon_1'\mathbf{E}_x^2 + \epsilon_2'\mathbf{E}_y^2 + \epsilon_3'\mathbf{E}_z^2),$$

and this determines the principal electro-crystalline axis of the substance: the law of polarisation is now given by

$$(\mathbf{P}_x,\ \mathbf{P}_y,\ \mathbf{P}_z) = (\epsilon_1'\mathbf{E}_x,\ \epsilon_2'\mathbf{E}_y,\ \epsilon_3'\mathbf{E}_z),$$

which for isotropic media becomes simply

$$\mathbf{P} = \epsilon'\mathbf{E},$$

results agreeing with those deduced from our previous considerations.

219. If we assume this simple linear law of polarisation, viz.

$$\mathbf{P} = \epsilon'\mathbf{E}^*,$$

then the part of the energy expended by the medium, as it is displaced, in mechanical work against the electrical attractions is

$$\int dv \int_0^P (\mathbf{E}\delta\mathbf{P}),$$

which is

$$\tfrac{1}{2}\int dv\,\epsilon'\mathbf{E}^2 = \tfrac{1}{2}\int(\epsilon')\,\mathbf{E}^2 dv,$$

whilst the internal molecular stored energy is increased by

$$\int dv \int_0^{\mathbf{E}} (\mathbf{P}\,d\mathbf{E}) = \tfrac{1}{2}\int(\epsilon')\,\mathbf{E}^2 dv.$$

Thus the two parts are now equal and distributed throughout the medium with a density

$$\frac{\epsilon'\mathbf{E}^2}{2},$$

or at least may be considered as so distributed.

The whole of this analysis of course depends on the reversibility of the operations. As long as the polarisation is slowly effected against the resilience of reversible internal elastic forces the whole of the second part of the

* Notice that $\epsilon' = \frac{\epsilon - 1}{4\pi}$ when ϵ is the constant of our former theories.

work done is stored up in the medium as potential energy, but any want of reversibility involves degradation of some of it into internal molecular energy of another type, while if the field were instantaneously annihilated the particles would swing back and vibrate so that ultimately all of it would be lost. This is the principle underlying the hysteretic loss of energy which is of such great importance in the correlative subject of magnetism; we need not however dwell on it here as it is of but little importance in the present connection. When viscous and other hysteretic effects are practically absent the changes of polarisation keep step exactly with those of the polarising field and **P** is a function of **E**, so that the whole of the average organised energy involved in the polarisation is mechanically available.

220. The complete expression for the energy in any electrostatic field. So far we have tacitly assumed the existence of a rigid inducing electric field, in the discussion of its effects on polarising a body moving about in it. But in a complete analysis of any problem it is necessary for us to take all coexistent systems into account, since none of them are in reality rigid and the motion of one usually affects the distribution in the others. Thus in order to have a full account of the field it is necessary to specify it more closely. The simplest method would be to specify the distribution of free charge throughout the field; the other quantities being then defined by the relations established in the previous paragraphs. In this way results can be obtained equivalent, in the simpler cases, to those deduced in the previous chapter, but a more general and more fundamental interpretation can be attributed to them than was there possible.

We shall therefore suppose the electric field to arise from a continuous volume distribution of charge throughout the field of density ρ at any point: surface charges, if they exist, may be regarded as limiting cases of such a distribution and do not therefore need special treatment.

221. The total mechanical work done in establishing the field may now be calculated by building up the charge distribution ρ gradually in the presence of the dielectric media, the induced polarity simultaneously taking the appropriate value at each stage of the process. By the general definition of potential, the work done in bringing up small increments of charge $\delta\rho$ at each point of the field in which the potential is ϕ is

$$\delta W = \int \phi \, \delta\rho \, dv,$$

but generally $$\rho = \operatorname{div} \mathbf{D},$$

so that the small variation of **D** at any point of the field induced by the above change in ρ is defined by

$$\delta\rho = \operatorname{div} \delta\mathbf{D},$$

whence $$\delta W = \int \phi \operatorname{div} \delta \mathbf{D}\, dv,$$

and by Green's lemma this is

$$= -\int (\delta\mathbf{D} \cdot \nabla)\, \phi\, dv + \int \phi\, \delta\mathbf{D}_n\, df,$$

the latter integral being taken over an indefinitely extended surface bounding the field. For a finite charge system this integral vanishes and thus

$$\delta W = -\int (\delta\mathbf{D} \cdot \nabla)\, \phi\, dv,$$

but $$\mathbf{E} = -\nabla\phi,$$

so that $$\delta W = \int (\mathbf{E} \cdot \delta\mathbf{D})\, dv,$$

as in the previous theory of Maxwell-Faraday. But now **D** is a composite function

$$\mathbf{D} = \frac{\mathbf{E}}{4\pi} + \mathbf{P},$$

so that $$\delta\mathbf{D} = \frac{1}{4\pi}\, \delta\mathbf{E} + \delta\mathbf{P},$$

and thus $$\delta W = \frac{1}{4\pi} \int (\mathbf{E}\delta\mathbf{E})\, dv + \int (\mathbf{E}\delta\mathbf{P})\, dv,$$

and therefore the total work done in establishing the field can be calculated in the form

$$W = \int_0 \delta W = \frac{1}{4\pi} \int dv \int_0^E (\mathbf{E}\delta\mathbf{E}) + \int dv \int_0^P (\mathbf{E}\delta\mathbf{P})$$
$$= \frac{1}{8\pi} \int \mathbf{E}^2 dv + \int dv \int_0^P (\mathbf{E}\delta\mathbf{P}).$$

The first term in this expression represents the electrical potential energy stored up in the electrical field on account of the distribution of electricity involved in the charges and polarisations. It may be regarded, as in the ordinary theory, as the potential function of the mechanical forces equivalent to the electrical attractions between the various charged and polarised elements of matter in the field. The second part represents the energy stored up in the dielectric media as a consequence of the strained condition involved in its polarisation.

222. For a linear isotropic law of polarisation we have

$$\mathbf{P} = \epsilon'\mathbf{E},$$

and thus the total energy is

$$W = \frac{1}{8\pi} \int \mathbf{E}^2 dv + \tfrac{1}{2} \int \epsilon' \mathbf{E}^2 dv$$
$$= \frac{1}{8\pi} \int \mathbf{E}^2 (1 + 4\pi\epsilon')\, dv$$
$$= \frac{1}{8\pi} \int \epsilon \mathbf{E}^2 dv = \tfrac{1}{2} \int (\mathbf{E}\mathbf{D})\, dv,$$

and as before we may regard it as distributed continuously throughout the field with a density at any point equal to

$$\frac{\epsilon}{8\pi}\mathbf{E}^2,$$

the part

$$\frac{1}{8\pi}\mathbf{E}^2$$

belonging to the aether and the rest to the matter. This is the usual result with which our present theory therefore agrees. It is however shown to be restricted to the case when the polarisation is induced without hysteresis and follows a linear law of induction.

223. Of the total energy associated with the electrical conditions in the aether, viz.

$$\frac{1}{8\pi}\int \mathbf{E}^2 dv,$$

the part

$$-\int dv \int_0^E (\mathbf{P}\delta\mathbf{E})$$

corresponds to the part of the mechanical energy stored as electrical potential energy of the polarisations: it is concerned mainly with the electrical attractions, or their equivalent mechanical forces, exerted by the field on the polarised media as a whole, of which it may be considered as the potential function.

The remainder

$$\frac{1}{8\pi}\int \mathbf{E}^2 dv + \int dv \int_0^E (\mathbf{P}\delta\mathbf{E}) = \int dv \int_0^E \left(\frac{\mathbf{E}}{4\pi} + \mathbf{P},\ \delta\mathbf{E}\right) = \int dv \int_0^E (\mathbf{D}\delta\mathbf{E})$$

is concerned with the attractions exerted by the field on the elements of the free charge distribution, being balanced by the material elastic or other purely mechanical reactions necessary to maintain them in their specified distribution. This part may be transformed to

$$-\int dv \int_0^\phi (\mathbf{D}\nabla)\,\delta\phi,$$

and by Green's lemma, with the assumption of finite fields, this is easily shown to be

$$\int dv \int_0^\phi \operatorname{div}\mathbf{D}\,\delta\phi = \int dv \int_0^\phi \rho\,\delta\phi = \int dv \int_0^\infty \rho\mathbf{E}_s\delta s,$$

in which form it is at once recognised as the electrical potential energy stored up in the system of the free electric charges, or as the potential function of the reacting mechanical forces on the material masses in the field arising on account of their charges.

It is of course assumed in this argument that the free electric charges are in reality absolutely free to move about in the matter without restraint

except as far as they may be prevented by the direct action of mechanical forces of rigid restraint at the boundaries of conductors or in the interior of the dielectrics; so that they are completely effective in converting electrical attractions into mechanical bodily forcives by reaction.

224. It is important to notice that although the total work done on the system in building it up as aforesaid is equal to the sum of the electrical potential energy of the charges and of the polarisations together with the internal stored energy in the medium or in symbols

$$\int dv \int_0^P (\mathbf{E}\delta\mathbf{D}) = \int dv \int_0^E (\mathbf{D}\delta\mathbf{E}) + \left(-\int dv \int_0^E (\mathbf{P}\delta\mathbf{E})\right) + \int dv \int_0^P (\mathbf{E}\delta\mathbf{P}),$$

yet it is only in the case when the law of induction is linear when

$$\int_0^E (\mathbf{P}\delta\mathbf{E}) = \int_0^P (\mathbf{E}\delta\mathbf{P}),$$

that the total work done on the system is equal to the electrical potential energy of the charges in their specified distribution.

The fact that any want of reversibility in the elastic forces resisting the establishment of the polarisation is completely accounted for by a corresponding irreversibility in the purely electrical energy of the polarisation induced so that

$$(\mathbf{E}\delta\mathbf{P}) + (\mathbf{P}\delta\mathbf{E})$$

is a perfect differential under all circumstances verifies that the forces resisting the displacement of the free electric charges are reversible so that

$$(\mathbf{E}\delta\mathbf{D}) + (\mathbf{D}\delta\mathbf{E})$$

is a complete differential.

225. The mechanical relations of the polarised medium*. Having obtained in the previous paragraphs a definite value for the potential energy of the mechanical forces acting on the dielectric medium as a whole, we may at once proceed to investigate the nature of the forces of which it is the energy function.

To deduce the forces from their potential energy function we vary this function with regard to the geometrically possible displacements of the medium as a whole. In any such displacement however the polarisation must be kept constant, for it is determined wholly by the internal degrees of freedom of the medium. We see at once by giving the medium a small linear displacement that the force acting on it is the vector

$$-\operatorname{grad} W = \int \operatorname{grad} (\mathbf{P} \cdot \mathbf{E})\, dv,$$

the differentials however not operating on $\mathbf{P}$.

* Cf. Larmor, *Phil. Trans.* A, 190 (1897), p. 248.

This is the same as

$$\int \nabla (\mathbf{P} \,.\, \mathbf{E})\, dv,$$

the operator ∇ not affecting $\mathbf{P}$.

This determines the linear components of the force; there may also be a torque. To obtain its components give the body a small vectorial rotation $\delta\omega$: in this displacement the element dv goes into a position where $\mathbf{E}$ has the value $\mathbf{E} + [\mathbf{E} \,.\, \delta\omega]$ and so the variation of the energy is

$$\delta W = \int (\mathbf{P} \,.\, [\mathbf{E} \,.\, \delta\omega])\, dv = -\int (\delta\omega \,.\, [\mathbf{E} \,.\, \mathbf{P}])\, dv,$$

and thus the couple is

$$-\int [\mathbf{EP}]\, dv.$$

226. There is however also a simple alternative method of deducing these results from the ideas involved in the theory of the polarisation in the medium. The mechanical force acting on a single doublet of moment $\mathbf{e}$ at a point in the field where the electric force intensity is $\mathbf{E}$ is represented by the vector

$$(\mathbf{e} \,.\, \nabla)\, \mathbf{E}.$$

Thus by simple addition over all the doublets in the volume element dv of the polarised body, we obtain that the linear force on the element of the medium is equal to

$$\begin{aligned} \Sigma (\mathbf{e} \,.\, \nabla)\, \mathbf{E} &= (\Sigma \mathbf{e} \,.\, \nabla)\, \mathbf{E} \\ &= (\mathbf{P} \,.\, \nabla)\, \mathbf{E}\, dv \end{aligned}$$

or $(\mathbf{P} \,.\, \nabla)\, \mathbf{E}$ per unit volume.

In this calculation the value taken for $\mathbf{E}$ excludes the local part of the total forcive which acts on any pole at the place. In such cases when we are dealing with a summation throughout the element of volume, the local actions in each charge element, which really arise from the other elements in the volume, must all be cancelled by complementary reactions, so that in the aggregate such terms will not occur.

This expression for the linear component of the mechanical force per unit volume on the medium is slightly different to that obtained from the energy, the difference being actually

$$(\mathbf{P} \,.\, \operatorname{curl} \mathbf{E}),$$

which is however zero if

$$\operatorname{curl} \mathbf{E} = 0,$$

i.e. if the electrical forces have a potential which is true in the present statical theory: the difference should however be noticed for future reference.

The elementary theory also shows that the simple bi-pole above mentioned is subject to a torque of amount

$$[\mathbf{e} \,.\, \mathbf{E}]$$

and thus we obtain for the torque on the element of volume of the polarised medium

$$[\mathbf{P} \cdot \mathbf{E}]\, dv,$$

or in all a vector $[\mathbf{P} \cdot \mathbf{E}]$ per unit volume. This is the same as the previous result.

If the medium carries a volume charge of density ρ there is to be added to the linear constituent of the force on it the vector

$$\rho \mathbf{E} = \mathbf{E} \operatorname{div} \mathbf{D}.$$

227. In pursuance of our general plan we can now show that this bodily forcive can be represented by means of an applied stress system identical with one of the types found necessary in the previous chapter. The general method is to write down the bodily forcive on any element of the dielectric and then try and exhibit them as the result of an elastic stress acting over the surface of the element. To accomplish this we must as usual try and put the x-component of the bodily forcive $\mathbf{F}$ per unit volume into the form

$$\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z},$$

because then $\int \mathbf{F}_x dv$ taken throughout any volume can be expressed as a surface integral over the surface of that volume. Now in the general case

$$\begin{aligned} \mathbf{F}_x &= \nabla_x (\mathbf{P} \cdot \mathbf{E}) + \mathbf{E}_x \operatorname{div} \mathbf{D}^* \\ &= \mathbf{P}_x \frac{\partial \mathbf{E}_x}{\partial x} + \mathbf{P}_y \frac{\partial \mathbf{E}_y}{\partial x} + \mathbf{P}_z \frac{\partial \mathbf{E}_z}{\partial x} + \mathbf{E}_x \frac{\partial \mathbf{D}_x}{\partial x} + \mathbf{E}_x \frac{\partial \mathbf{D}_y}{\partial y} + \mathbf{E}_x \frac{\partial \mathbf{D}_z}{\partial z}. \end{aligned}$$

Now remembering that $\mathbf{E}$ is the gradient of a potential and using the substitution

$$\mathbf{P} = \mathbf{D} - \frac{\mathbf{E}}{4\pi},$$

we find that

$$\begin{aligned} \mathbf{F}_x &= \mathbf{D}_x \frac{\partial \mathbf{E}_x}{\partial x} + \mathbf{D}_y \frac{\partial \mathbf{E}_y}{\partial x} + \mathbf{D}_z \frac{\partial \mathbf{E}_z}{\partial x} - \frac{1}{8\pi} \frac{\partial}{\partial x} (\mathbf{E}_x^2 + \mathbf{E}_y^2 + \mathbf{E}_z^2) + \mathbf{E}_x \operatorname{div} \mathbf{D} \\ &= \mathbf{D}_x \frac{\partial \mathbf{E}_x}{\partial x} + \mathbf{D}_y \frac{\partial \mathbf{E}_x}{\partial y} + \mathbf{D}_z \frac{\partial \mathbf{E}_x}{\partial z} - \frac{1}{8\pi} \frac{\partial \mathbf{E}^2}{\partial x} + \mathbf{E}_x \operatorname{div} \mathbf{D} \\ &= \frac{\partial}{\partial x} (\mathbf{E}_x \mathbf{D}_x) + \frac{\partial}{\partial y} (\mathbf{E}_x \mathbf{D}_y) + \frac{\partial}{\partial z} (\mathbf{E}_x \mathbf{D}_z) - \frac{1}{8\pi} \frac{\partial \mathbf{E}^2}{\partial x}, \end{aligned}$$

thus

$$\mathbf{F}_x = \frac{\partial}{\partial x}\left(\mathbf{E}_x \mathbf{D}_x - \frac{\mathbf{E}^2}{8\pi}\right) + \frac{\partial}{\partial y} (\mathbf{E}_x \mathbf{D}_y) + \frac{\partial}{\partial z} (\mathbf{E}_x \mathbf{D}_z).$$

We have thus succeeded in our aim. We can write the other components in the same way and thus we see that this linear part of the bodily mechanical forcive on any element of the medium can be expressed by means of surface

* It is for the present always to be noticed that ∇ does not affect P.

tractions over its outer boundary. The complete specification of the stress system defining these tractions is given in the matrix form

$$\left\| \begin{matrix} \mathbf{E}_x\mathbf{D}_x - \dfrac{\mathbf{E}^2}{8\pi}, & \mathbf{E}_x\mathbf{D}_y \;, & \mathbf{E}_x\mathbf{D}_z \\ \mathbf{E}_y\mathbf{D}_x \;, & \mathbf{E}_y\mathbf{D}_y - \dfrac{\mathbf{E}^2}{8\pi}, & \mathbf{E}_y\mathbf{D}_z \\ \mathbf{E}_z\mathbf{D}_x \;, & \mathbf{E}_z\mathbf{D}_y \;, & \mathbf{E}_z\mathbf{D}_z - \dfrac{\mathbf{E}^2}{8\pi} \end{matrix} \right\| .$$

228. We notice again that in the general case this stress system is not self-conjugate, the diagonal or cross terms are not equal: this means that there is a bodily torque which is equal per unit volume to the differences in the cross terms; for instance the x-component is

$$\begin{aligned} T_{zy} - T_{yz} &= \mathbf{E}_y\mathbf{D}_z - \mathbf{E}_z\mathbf{D}_y \\ &= \mathbf{E}_y\mathbf{P}_z - \mathbf{E}_z\mathbf{P}_y \\ &= [\mathbf{EP}]_z . \end{aligned}$$

This is precisely the same torque which was determined from the elementary principles, so that this stress specification includes everything. It is moreover identical with the second type of stress examined in the previous chapter for the general dielectric medium with no discontinuities. But now there is no contradiction of principle: our material dielectric is now polarised and will thus be more than a mere medium of transmission as regards the mechanical force. In the Faraday-Maxwell theory the function of the dielectric, essentially continuous, is merely to transmit the forces without adding anything to them, but on the present theory the elements of the dielectric medium are subject to a definite additional type of strain the reactions to which are sufficient to account for the more general form of stress found necessary to transmit the electrical actions.

229. To obtain a simpler expression* of this stress let us choose convenient axes. Choose the (x, y) plane as the plane of D and E and the x-axis as the internal bisector of the angle 2θ between these two vectors. The z-axis is chosen to form the usual right-handed system. Now

$$\begin{aligned} (\mathbf{D}_x, \mathbf{D}_y, \mathbf{D}_z) &= (D\cos\theta,\ D\sin\theta,\ 0), \\ (\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z) &= (E\cos\theta,\ -E\sin\theta,\ 0), \end{aligned}$$

and thus the matrix is now

$$\left| \begin{matrix} ED\cos^2\theta - \dfrac{E^2}{8\pi}, & ED\sin\theta\cos\theta, & 0 \\ -ED\sin\theta\cos\theta, & -ED\sin^2\theta - \dfrac{E^2}{8\pi}, & 0 \\ 0, & 0, & -\dfrac{E^2}{8\pi} \end{matrix} \right|$$

which can be dissected into parts.

* Cf. Maxwell, *Treatise*, II. p. 280.

(i) The terms $-\frac{E^2}{8\pi}$ make a uniform hydrostatic pressure $\frac{E^2}{8\pi}$ throughout the medium.

(ii) The two terms in the diagonal give a torque per unit volume $ED \sin 2\theta$ and the remaining terms represent

(iii) a tension along the x-axis $ED \cos^2 \theta$,

(iv) a pressure $(-)$ along the y-axis $-ED \sin^2 \theta$.

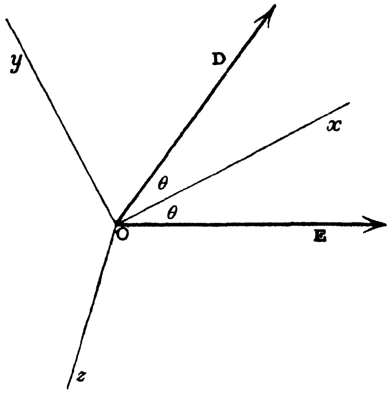


Fig. 47

In a diagram they are

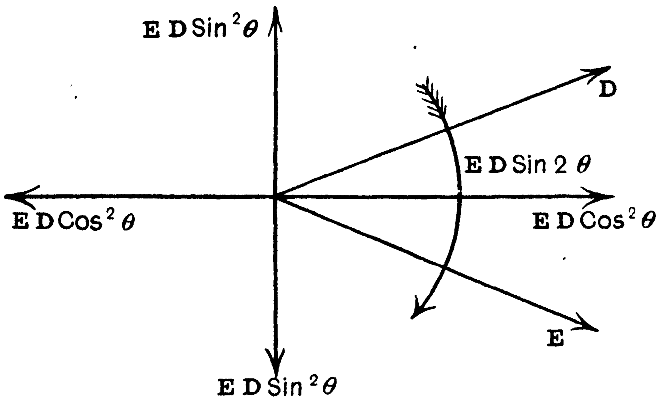


Fig. 48

This is the general result and is fairly complex but if we take the medium to be isotropic considerable simplification results. In this case $\theta = 0$ and the bodily torque vanishes. Moreover the other parts reduce to a pull or

tension along the lines of force or induction equal to ED and a hydrostatic pressure all round equal to $\frac{1}{8\pi}E^2$. This is identical with Maxwell's stress system, when the dielectric medium is free aether, which is the only case with which he deals.

230. The reduction of the bodily forcive on dielectric media to a representation by means of an imposed stress system, which we have just discussed, is valid only in so far as the medium is without discontinuities. When there are in the electric field interfaces of transition between different media, there will also exist surface tractions on them which may be evaluated either as the result of the Maxwellian tractions towards the two sides of the interface or by considering an actual, somewhat abrupt, interface to be the limit of a rapid continuous variation of the properties of the medium which takes place across a layer of finite thickness. Let then the total displacement $\mathbf{D}$ with its circuital characteristic where there is no free charge be made up of the dielectric material polarisation $\mathbf{P}$ and the displacement proper $\frac{\mathbf{E}}{4\pi}$. We have then

$$\operatorname{div} \mathbf{E} = 4\pi (\rho + \rho'),$$

wherein ρ' is the Poisson ideal volume density corresponding to the polarisation, and ρ is the volume density of free charge, surface distributions being now, by hypothesis, non-existent. Also

$$\operatorname{div} \mathbf{P} = -\rho'.$$

The mechanical forcive acting on the dielectric is per unit volume a force $\mathbf{F}$ and couple $\mathbf{G}$ where

$$\mathbf{F} = (\mathbf{P} \, . \, \nabla)\,\mathbf{E} + \rho\mathbf{E},$$

$$\mathbf{G} = [\mathbf{PE}].$$

The linear force acting on the whole transition layer is the value of $\int \mathbf{F}dv$ integrated through it. This integral is finite although the volume of integration is small, on account of the large values of the differential coefficients which occur in the expression of $\nabla\mathbf{E}$. To evaluate it we endeavour by integration by parts to reduce the magnitude of the quantity that remains under the sign of volume integration, so that in the limit we may be able to neglect that part: thus we obtain

$$\int \mathbf{F}dv = \int_f \mathbf{E} \, . \, \mathbf{D}_n df + \int (\rho + \rho')\,\mathbf{E}dv.$$

Now by the definition of the electric force $\mathbf{E}$ it is the force due to a volume distribution of density $\rho + \rho'$ and to extraneous causes; so that in the limit

when the transitional layer is indefinitely thin, we have, by Coulomb's principle.

$$\int (\rho + \rho') \mathbf{E} \, dv = \tfrac{1}{2} \int (\sigma + \sigma') (\mathbf{E}_1 + \mathbf{E}_2) \, df$$
$$= \frac{1}{8\pi} \int (\mathbf{E}_{1_n} - \mathbf{E}_{2_n}) (\mathbf{E}_1 + \mathbf{E}_2) \, df,$$

$\mathbf{E}_1$, $\mathbf{E}_2$ being the vectors of electric force on the two sides of the layer and $\mathbf{E}_{1_n}$, $\mathbf{E}_{2_n}$ the normal components of these forces, all measured towards the side 2, while σ and σ' are the surface densities constituted in the limit by the aggregates of ρ and ρ' respectively taken throughout the layer. Hence in the limit

$$\int \mathbf{F} \, dv = \left| \int \mathbf{P}_n \mathbf{E} \, df \right|_1^2 + \frac{1}{8\pi} \int (\mathbf{E}_{1_n} - \mathbf{E}_{2_n}) (\mathbf{E}_1 + \mathbf{E}_2) \, df.$$

Thus the electric traction on the interface of transition may be represented by a pull towards each side, along the direction of the resultant force $\mathbf{E}$; this pull is, on the side 2, of intensity

$$\mathbf{P}_{2_n} \mathbf{E}_2 + \frac{1}{8\pi} (\mathbf{E}_{1_n} - \mathbf{E}_{2_n}) \mathbf{E}_2,$$

or what is the same thing

$$\mathbf{P}_{2_n} \mathbf{E}_2 - \tfrac{1}{2} (\mathbf{P}_{2_n} - \mathbf{P}_{1_n} - \sigma) \mathbf{E}_2 = \tfrac{1}{2} (\sigma + \mathbf{P}_{1_n} + \mathbf{P}_{2_n}) \mathbf{E}_2,$$

in the direction of $\mathbf{E}_2$; on the face 1 the pull is

$$\tfrac{1}{2} (\sigma - \mathbf{P}_{1_n} - \mathbf{P}_{2_n}) \mathbf{E}_1,$$

now in the direction of $\mathbf{E}_1$. As the tangential component of the electric force $\mathbf{E}$ is under all circumstances continuous across the interface the total traction on both sides is along the normal and equivalent to

$$\tfrac{1}{2} (\mathbf{P}_{1_n} + \mathbf{P}_{2_n}) (\mathbf{E}_{2_n} - \mathbf{E}_{1_n}),$$

together with tractions $\tfrac{1}{2}\mathbf{E}_2\sigma$, $\tfrac{1}{2}\mathbf{E}_1\sigma$ acting on the true charge σ, all the quantities being now measured positive in any the same direction. In the case where there is no surface charge this simply reduces to a normal pull towards the side 2 of amount $-2\pi\mathbf{P}_{2_n}{}^2 + 2\pi\mathbf{P}_{1_n}{}^2$.

When the interface is between a dielectric 1 and a conductor 2, the traction is only towards the side 1 and is equal to $\tfrac{1}{2} (\mathbf{P}_{1_n} + \sigma) \mathbf{E}_1$, or $\tfrac{1}{2}\mathbf{D}_{1_n}\mathbf{E}_1$, per unit area, along the normal, which is now the direction of the resultant force.

231. All this is quite independent of the law of the connection between the polarisation and the electric force in the material medium. Thus under the most general circumstances as regards electric field, the forcive on the material due to its electric excitation consists of the interfacial tractions thus specified together with a force $\mathbf{F}$ and a torque $\mathbf{G}$ per unit volume given by the above formula, viz.

$$\mathbf{F} = (\mathbf{P}\nabla) \mathbf{E},$$
$$\mathbf{G} = [\mathbf{P}\mathbf{E}].$$

The assumptions underlying this analysis that the transitions are gradual, will be sufficiently satisfied even if the intermediate layer is only one or two molecules in thickness, for as these molecules are arranged slightly in and out, and not in exact rows along the interface, their polarity can still be averaged into a continuous density, as above. The aggregate tractions over a thin layer of transition thus do not depend sensibly on the nature of the transition, but only the circumstances on the two sides of the layer.

In the case of a fluid medium, the bodily part of the forcive produces and is compensated by a fluid pressure

$$\int (\mathbf{P}\,d\mathbf{E}),$$

where **P**, being polarisation induced by the electric force **E**, is for a fluid in the same direction as **E** and a function of its magnitude. This pressure will be transmitted statically in the fluid medium to the interfaces (i.e. a reacting pressure $\int (\mathbf{P}d\mathbf{E})$ exerted by the interface will keep the medium in internal equilibrium); combining it there with the surface tractions proper, it appears that the material equilibrium of fluid media is secured as regards forces of electric origin, if extraneous force is provided to compensate a total normal traction towards each side of each interface, of intensity

$$-\,2\pi\mathbf{P}_n{}^2 - \int (\mathbf{P}d\mathbf{E}).$$

In the case usually treated, in which a linear law of induction is assumed, so that the relation between **P** and **E** is

$$\mathbf{P} = \frac{\epsilon - 1}{4\pi}\,\mathbf{E},$$

the mechanical result of the electric excitation of the fluid medium is easily shown to be the same as if each interface were pulled towards each side by a Faraday-Maxwell stress made up of a pull $\frac{\epsilon\mathbf{E}^2}{8\pi}$ along the lines of force and an equal pressure in all directions at right angles. But this imposed geometrical self-equilibrating stress system would not be an adequate representation of the mechanical forcive in a solid medium; for then the bodily forcive, instead of being wholly transmitted is in part balanced on the spot by reactions depending on the solidity of the material. The forcive acting on isotropic material may however in every case, whether the induction follows a linear law or not, be expressible as an extraneous or imposed system, made up of a bodily hydrostatic pressure $\int (\mathbf{P}d\mathbf{E})$ (which in the case of a fluid only relieves the ordinary fluid pressure and so diminishes the compression)

together with normal tractions on the interfaces between dielectric media, of intensity $-2\pi \mathbf{P}_n^2 - \int \mathbf{P} d\mathbf{E}$ acting towards each side, and tractions

$$\tfrac{1}{2}\mathbf{D}_n\mathbf{E} - \int (\mathbf{P}\,d\mathbf{E})$$

on the surfaces of conductors acting towards the dielectric.

232. As a single example of these general principles let us consider the energy and ponderomotive forces of a homogeneous dielectric ellipsoid in an otherwise uniform field.

We have already solved this problem as far as obtaining the field is concerned. We then showed that the field inside the ellipsoid has, with the usual notation, a potential

$$\phi = -\frac{E_x x}{1 + A(\epsilon - 1)} - \frac{E_y y}{1 + B(\epsilon - 1)} - \frac{E_z z}{1 + C(\epsilon - 1)},$$

and thus the force intensity at any point inside the ellipsoid has components

$$\left(\frac{E_x}{1 + A(\epsilon - 1)}, \frac{E_y}{1 + B(\epsilon - 1)}, \frac{E_z}{1 + C(\epsilon - 1)}\right),$$

and the components of the polarisation intensity are therefore

$$\frac{1}{4\pi}\left(\frac{E_x}{A + \frac{1}{\epsilon - 1}}, \frac{E_y}{B + \frac{1}{\epsilon - 1}}, \frac{E_z}{C + \frac{1}{\epsilon - 1}}\right).$$

We thus conclude that the portion of the total energy of the field which is associated with the polarisation of the material of the ellipsoid and which serves as an energy function of the bodily forces on that ellipsoid is merely

$$\tfrac{1}{2}\int (PE)\,dv = \frac{\epsilon - 1}{8\pi}\int E^2 dv,$$

the integral being taken throughout the volume of the ellipsoid, E denoting the force intensity of the total field.

This is equal to

$$\frac{\epsilon - 1}{8\pi}\left[\Sigma \frac{E_x^2}{(1 + A(\epsilon - 1))^2}\right] \cdot \frac{4\pi abc}{3}$$

in the present case when the ellipsoid is homogeneous.

The mechanical force can either be obtained from the energy function or by the elementary methods. Adopting the former we see that it is a wrench of intensity $\mathbf{F}$ and in which the couple is $\mathbf{G}$ where

$$\mathbf{F} = \text{grad} \int \frac{\epsilon' E^2}{2}\,dv, \qquad \epsilon' = \frac{\epsilon - 1}{4\pi},$$

$$\mathbf{G} = \frac{\partial}{\partial \theta}\int \frac{\epsilon' E^2}{2}\,dv,$$

θ denoting a generalised angular component.

We thus see that if the field changes very slightly in the space occupied by the ellipsoid or if $\frac{\partial E^2}{\partial x}$ is uniform throughout the ellipsoid

$$\mathbf{F} = \tfrac{4}{3}\pi abc\epsilon' \text{ grad } E^2,$$

so that it will be attracted into places of stronger force (not necessarily higher potential).

To obtain some idea of the nature of the couple let us specialise our assumptions slightly. If the ellipsoid is fixed to rotate about its c-axis and the field is in a direction perpendicular to this axis, then

$$E_x = E\cos\theta, \qquad E_y = E\sin\theta, \qquad E_z = 0,$$

θ determining the angle between the plane of the c-axis and E and the plane of the c- and a-axes, then

$$W = \frac{E^2(\epsilon - 1)}{8\pi}\left[\frac{\cos^2\theta}{\{1 + A(\epsilon - 1)\}^2} + \frac{\sin^2\theta}{\{1 + B(\epsilon - 1)\}^2}\right],$$

so that

$$G_\theta = +\frac{\partial W}{\partial \theta}$$

$$= \left[\frac{1}{\{1 + B(\epsilon - 1)\}^2} - \frac{1}{\{1 + A(\epsilon - 1)\}^2}\right]\left(\frac{\epsilon - 1}{4\pi}\right) E^2 \sin 2\theta.$$

If $a > b$ then $A < B < 1$ and the quantity in the square bracket is negative, thus G_θ is negative when $\theta < \frac{\pi}{2}$. We thus see that the ellipsoid always tends to turn its longest axis into the direction of the field.

233. On electric displacement*. Since the vectors of the present theory satisfy exactly the same conditions as those of the original Maxwell-Faraday theory they must ultimately represent the same quantities. But the theory just developed is based on elementary physical notions regarding the behaviour of the dielectric medium when introduced into an electric field. The present ideas must therefore help us to explain the former theory and by means of it we should be able to obtain some insight into the nature of 'electric displacement.' This is best accomplished by considering a particular problem. viz. that of a parallel plate condenser with large surfaces with equal positive and negative charges; a plane slab of some dielectric substance (constant ϵ) is inserted parallel to the plates. We treat the air spaces as a vacuum because its density is so slight. The solution of this problem is easy and if the surface densities of charge on the plates of the condenser are $\pm\sigma$ the electric force is $4\pi\sigma$ all across the air spaces and is $\frac{4\pi\sigma}{\epsilon}$ in the dielectric. (The lines of force are straight across by symmetry and at the surface of the dielectric $\epsilon\mathbf{E}_2 = \mathbf{E}_1$.)

* Cf. Larmor, *Aether and Matter*; Lorentz, *Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern.*

Now on the present theory of the matter the dielectric substance is polarised: the molecules have a positive and negative pole and owing to the presence of the field the axes have a convergence towards a definite direction, viz. straight across between the plates, so that their moments no longer cancel. The intensity of polarisation of the medium is thus at each point directed straight across between the plates.

If now we consider a small rectangular volume element of the substance parallel to the lines of force straight across we see that the polarisation of all the molecules in it is equivalent to a small polar distribution in the volume, which is just the same as if it had a positive charge of density $+\sigma_t$ on one end and a negative charge of density $-\sigma_t$ on the other. All the little molecular moments can be summed up into a uniform polarisation: the irregular molecular distribution is smoothed out into a uniform average. At least this is an effective representation of the matter. It does not mean that we assert that there is an actual charge on each end of the little element but that the aggregate of the polarisation in the element can be replaced by these charges when investigating its action at external points. The essential thing for this purpose is the electric moment of the element and any distribution giving the right moment is an effectively correct one.

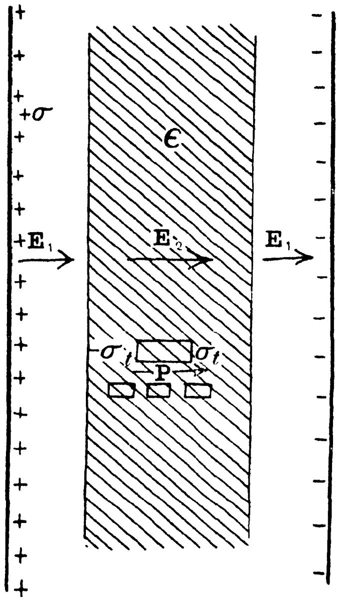


Fig. 49

Now by combining all these small rectangular elements so polarised into the finite piece of dielectric we see that there will be an uncompensated part* of the surface density (which is not necessarily the same for each element) where one rectangular block abuts on the next one and at an end at the boundary of the dielectric itself there remains the complete surface polarity. This amounts to what we have called the ideal electric distribution of Poisson; the outstanding parts throughout the medium correspond to the volume density and the complete polarity remaining at the surface of the medium corresponds to the ideal surface density. Regarded in this way it is obvious that this theoretical distribution and the actual one will not give the same field in the immediate neighbourhood of an element of the substance. The ideal distribution has been smoothed out from the other and it is only at a distance that it is effectively equivalent to the actual polarity.

* In the particular case examined this uncompensated charge is zero.

In the example under immediate consideration the field in the dielectric is uniform and so the intensity of polarisation will also be uniform throughout the medium: thus the charges on the ends of adjacent small elements will be the same and thus when put together there will be no uncompensated polarity: we shall merely have a surface density of ideal electric charge $-\sigma'$ on one face of the dielectric and $+\sigma'$ on the other. In the old-fashioned way of describing these things σ (the charge density on the plates of the condenser) might be called the *free* charge and σ' the *bound* charge (as it cannot be moved); σ' is only the end aspect of the polarisation in the medium which has a counterpart at the other side of the medium and they cancel across.

On our theory σ' is equal to the normal component of the polarisation at the surface and this is

$$\frac{\epsilon - 1}{4\pi} \mathbf{E}_2,$$

but since $\mathbf{E}_2 = \dfrac{\mathbf{E}_1}{\epsilon} = \dfrac{4\pi\sigma}{\epsilon}$ we have

$$\sigma' = \left(1 - \frac{1}{\epsilon}\right) \sigma.$$

So that
$$\sigma - \sigma' = \frac{\sigma}{\epsilon}.$$

234. Let us now examine another point. The polarisation of the element can be expressed by saying that an electric displacement in the element from one end to the other has taken place. Initially the positive and negative charges effectively coincide and cancel but on the application of an electric field they are separated and the electric moment can be considered to arise from the electric displacement of one charge relative to the other. We can at least theoretically imagine it to be like this. There is thus an actual movement of electric charge. Essentially the movement consists in the molecules being really strained round a bit, but when we aggregate these up for the small rectangular volume element as before, the effect is the same as if the positive charge were moved from one face of the volume element to the opposite one. If this is the case how can we measure the displacement? The proper measure is the product of each charge element by the distance through which it is moved and the total sum of the quantities so obtained because if we moved the same charge in each case through half the length it ought to give half the measure of the displacement. Thus the total electric displacement in our small rectangular volume element of end area $\delta\epsilon$ and length δl is

$$\sigma'\delta\epsilon \times \delta l = \sigma' d\tau,$$

where $\delta\tau$ is the volume of the element; but this is the moment produced in the element. Thus an effective measure of the displacement in the volume element is the intensity of polarisation multiplied into the volume.

Thus for the slab of dielectric in the example considered the result of the total electric displacements in the medium is merely to displace a surface charge σ' from one side of the slab to the other straight across. There is a true displacement of electricity equal to a displacement right across. The flux of electricity measured in this way is a true electric displacement.

But on the Faraday-Maxwell theory of electric action the electric displacement in any small volume dv is taken to be $\frac{\epsilon \mathbf{E}}{4\pi}$ which is equal to

$$\frac{(1 + 4\pi\epsilon')\mathbf{E}}{4\pi} dv = \frac{\mathbf{E}}{4\pi} dv + \mathbf{P}dv,$$

so that in addition to the true electric displacement represented by the term $\mathbf{P}dv$ as in the present theory there is something else which is quite a new thing altogether. If we call the true electric flux or induction a displacement the term $\frac{\mathbf{E}}{4\pi} dv$ represents quite an extraneous thing altogether. This part still exists in empty space when there are no dielectrics present so that it cannot possibly be ascribed to an electric displacement. Maxwell's flux vector is therefore not all electric displacement as part of it remains when there is no electricity present at that point. The part $\mathbf{P}dv$ is as we have already seen a *true* electric displacement and the other part $\frac{\mathbf{E}}{4\pi} dv$ we call the *aethereal* displacement. This latter part has the same properties as the former.

Thus if we want to retain the analogy between the simple displacement theory of Maxwell and the polarisation theory just developed we must introduce this new type of displacement so that the total electric displacement of Maxwell includes the true electric displacement of the present theory and the aethereal displacement.

235. The real significance of the matter is however best exhibited in another manner. Consider again the example of the condenser with the dielectric slab; what happens when it is charged? As far as we are at present concerned the condenser may be charged by transferring a positive charge $+Q$ round a wire connecting the two plates thereby leaving, in defect, a charge $-Q$ on the one plate and creating an excess of charge $+Q$ on the other plate. While this is being accomplished a displacement is taking place in the dielectric (the polarisation is being gradually set up) and a charge Q' is displaced across the medium from one side to the other. This is all the electric motion that takes place; an actual

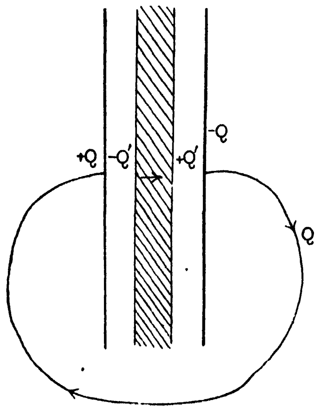


Fig. 50

charge Q moves round the wire and a change of polarisation in the dielectric corresponds to a motion of a charge Q' across the dielectric.

But since $Q = A\sigma$ and $Q' = A\sigma'$, where A is the equal area of the parallel faces of condenser plates and dielectric (the charge is assumed uniformly distributed) we have that

$$Q' = \left(1 - \frac{1}{\epsilon}\right) Q,$$

so that the displacement of charge across the dielectric is not equal to that round the wire: and there is nothing at all in the free spaces between.

If we stop here we shall have a very complicated state of affairs. It was merely in order to avoid all these complications in electrodynamic theory that Maxwell made the assumption that the displacement which takes place during any electric change is always circuital; that is always takes place in complete circuits. If we, with Maxwell, make the postulate that electricity always flows in complete cycles we shall find electrodynamic theory much easier to handle. This will appear later.

In the example above, Maxwell would therefore postulate a hypothetical total displacement equal to Q in the air and $\frac{Q}{\epsilon}$ in the dielectric; this being all that is required to complete the flow of the quantity Q all round. Estimated per unit volume this would mean adding a displacement $\frac{\mathbf{E}}{4\pi}$ at each point of space between the condenser plates. (It is assumed that $\mathbf{E} = 0$ everywhere except between the plates.)

236. This is easily seen to be the general result. If $\mathbf{E}$ is the force intensity at any point of an electric field Maxwell's theory adds a displacement equal to $\frac{\mathbf{E}}{4\pi}$ at that point to any true electric displacement that may occur there. If we do this then the flux of displacement is always in closed cycles. This additional displacement is not true electric displacement at all, as it exists at points in a vacuum; it is an aethereal displacement possessing all the electrodynamic properties of true electric flux.

A dynamical theory of electromagnetic actions should give a reason for this action in the aether, for the existence of this aethereal displacement which has the same properties as a flow of electricity but is not itself a flow of electricity. The hypothesis is however experimentally correct and it simplifies the theory immensely and there we shall leave it for the present.

On this view of the matter the aether is to be regarded as the seat of part of the energy associated with any electrostatic field, viz. that part associated with the production of the aethereal displacement. On a previous

analogy this part may be taken as distributed throughout the field with a density at any point equal to

$$\tfrac{1}{2}\left(\mathbf{E}\cdot\frac{\mathbf{E}}{4\pi}\right)=\frac{1}{8\pi}\mathbf{E}^2,$$

a result which is verified by the fact that all the energy in the field is located in the aether if no dielectric medium is present.

237. The relation of inductive capacity to density. One of the most successful ways of testing a constitutional theory of the present type is to formulate on the same basis the connection between the constitutional relations involved in it and the physical or chemical constitution of the medium. In the present case the whole constitutive character of the theory is involved in the one constant introduced in it, viz. the specific inductive capacity ϵ. If therefore we can formulate a connection between this constant and the constitution of the medium we shall have a definite means of testing the general validity of our theory. It is quite easy to obtain a relation between the constant ϵ and the density of the medium in certain simple cases and we shall find that it agrees very well with our experimental knowledge on the same question.

Let the dielectric medium contain n molecules per unit volume, these molecules being presumed to be merely concentrated when a change of density of the medium occurs. Each of the molecules becomes polarised to a moment $\mathbf{p}$ by the field of the electric force; this field is made up of the extraneous exciting field and that of the polarised molecules themselves; the latter again consists of a part arising from the polarised medium as a whole and a part involving only the immediate surroundings of the point considered; to obtain an estimate of these various parts let us consider again the method of their separation.

238. The total electric force acting on a single molecule is derived from the aggregate potential

$$\phi=\Sigma\,(\mathbf{p}\nabla)\,\frac{1}{r}.$$

This potential, when the point considered is inside the polarised medium, involves the actual distribution of the surrounding molecules; and thus the force derived from it changes rapidly, at any instant of time, in the interstices between the molecules. But when the point considered is outside the polarised medium, or inside a cavity formed in it (whose dimensions are large compared with molecular distances) the summation in the expression for ϕ may be replaced by continuous integration; so that $\mathbf{P}$ denoting the intensity of polarisation in the molecules of the dielectric medium

$$\phi=\int(\mathbf{P}\nabla)\,\frac{dv}{r},$$

and the force thus derived is perfectly regular and continuous. This expression may be integrated by parts since, the origin being now outside the region of the integral, no infinities of the function to be integrated occur in that region. Thus

$$\phi = \int \frac{\mathbf{P}_n}{r}\, df - \int \frac{\operatorname{div} \mathbf{P}}{r}\, dv$$

that is, the potential at points in free aether is due to Poisson's ideal volume density $\rho = -\operatorname{div} \mathbf{P}$ and a surface density $\sigma = \mathbf{P}_n$. When the point considered is in an interior cavity, this surface density is extended over the surface of the cavity as well as over the outer boundary. Now when it is borne in mind that, at any rate in a fluid, the polar molecules are in rapid movement and not in fixed positions which would imply a sort of crystalline structure, it follows that the electric force on a molecule in the interior of the material medium, with which we are concerned, is an average force involving the average distribution of these molecules, and is therefore properly due to an ideal continuous density like Poisson's, even as regards the elements of volume which are very close up to the point considered. To compute the average force which causes the polarisation of a given molecule we have thus to consider that molecule as situated in the centre of a spherical cavity whose radius is of the order of molecular dimensions; and we have to take account of the effect of a Poisson averaged continuous local polarisation surrounding the molecule, whose intensity increases from nothing at a certain distance from the centre up to the full amount **P** at the limit of the molecular range, this intensity being practically uniform in direction and a function of the distance only.

We therefore assume spherical stratification in the distribution of the Poisson ideal volume density near the point under investigation. To estimate the effect of an elementary shell in this stratification, the charge in it can be reckoned as a surface density on it of intensity

$$\delta\mathbf{P} \cos\theta,$$

$\delta\mathbf{P}$ denoting the small increment of **P** as we pass through the shell, and θ the polar angle between the direction of **P** and the normal at the point of the shell. This shell thus contributes a force at the centre in the direction of **P** equal to

$$\iint \frac{r^2 \delta\mathbf{P}}{r^2} \cos\theta \sin\theta\, d\theta\, d\phi$$

$$= \frac{4\pi}{3}\, \delta\mathbf{P}.$$

Thus on the whole the local part of the forcive is

$$\frac{4\pi}{3} \int_0^P \delta\mathbf{P} = -\frac{4\pi}{3}\mathbf{P}.$$

The force polarising the molecules is therefore

$$\mathbf{E} + \frac{4\pi}{3}\mathbf{P},$$

$\mathbf{E}$ denoting the total electric force. Now if the polarisation produced be presumed to be proportional to the polarising force

$$\mathbf{p} = \epsilon'\left(\mathbf{E} + \frac{4\pi}{3}\mathbf{P}\right),$$

and thus since $$\mathbf{P} = \Sigma\mathbf{p} = n\mathbf{p},$$

we have $$\mathbf{P} = n\epsilon'\left(\mathbf{E} + \frac{4\pi}{3}\mathbf{P}\right),$$

and by the definition of ϵ we have

$$\mathbf{P} = \frac{\epsilon - 1}{4\pi}\mathbf{E}.$$

Thus $$\frac{\epsilon - 1}{4\pi} = n\epsilon'\left(1 + \frac{\epsilon - 1}{3}\right) = n\epsilon'\left(\frac{\epsilon + 2}{3}\right).$$

Thus $$\frac{\epsilon - 1}{\epsilon + 2} = \frac{4\pi}{3}n\epsilon',$$

from which we see that the function

$$\frac{\epsilon - 1}{\epsilon + 2}$$

must be proportional to the density of the medium. This is the usual Lorentz formula* which has been satisfactorily verified in numerous cases.

239. On the mechanism of dielectric polarisation†. We have thus seen that the only consistent view of the action of a dielectric in transmitting the actions in electrostatic fields is the one based on the idea of the electric polarisation of the individual molecules of the matter in the field, each of which is thus presumed to contain as an essential part of its constitution certain electric charges more or less tightly bound in its interior, but which can be separated, the positive from the negative, when an electric field is applied from without. We have also mentioned the fact that in all probability these charges are constituted, the negative of a certain number of more or less identical atomic charges, the electrons, of extremely small mass and the positive in an as yet uncertain manner. Although the explanations developed above have in reality no reference to this very definite constitution of the involved charges and the mechanism of its binding in the atoms, it seems certainly of theoretical interest, if not of practical importance, to formulate the connection between

* This formula was determined for the optical case by R. Lorentz, *Ann. Phys. Chem.* (3), 11 (1880), p. 77; 20 (1883), p. 19; and independently by H. A. Lorentz, *Ann. Phys. Chem.* (3), 9 (1880), p. 642. The mode of deduction here given is due to Larmor, *Phil. Trans.* A, 190 (1897), p. 232.

† Lorentz, *Arch. Néerland* (1892), p. 363. Cf. also Livens, *Phil. Mag.* XXIV. p. 285 (1912).

the general theory and the special form of it treated as a branch of the general electron theory of these things. Of course any such procedure naturally brings us into close contact with the vexed question of the constitution of the molecule which is occupying so much attention at present. We can however obtain sufficient information in a tentative sort of way without making any very special hypothesis. Of course the state of affairs exemplified cannot in the least be said to depict the thing as it really exists. All we do is to imagine some simple system that will coordinate and explain the majority of the phenomena known to us.

240. We imagine therefore a body composed of innumerable molecules or atoms, of particles, as Lorentz calls them, each molecule containing a certain number of electrons and the necessary positive charge. Since there is as yet no definite evidence as to the distribution of this positive charge, it is best to adopt an hypothesis which renders most easy the task of deducing the properties of the atom from its structure. It will be supposed therefore that the positive electrification is distributed rigidly throughout the volume of the atom and this condition is probably secured in any case to a sufficient approximation on account of the comparatively large mass of the positive constituents. The model atom as pictured by Kelvin and worked with by J. J. Thomson*, was assumed to be spherical in form and the distribution of the positive charge uniform, but we need not make these special assumptions for our present theory, especially as it would seem to be inconsistent with the recent work on the question.

Now on the modern view of these things all electrical phenomena are concerned with the motions of this complicated electrical system, called an atom. The main part of the motion is that of the negative electrons and it is possible to discuss the subject in terms of these electrons. But of course any method of procedure in which the motion of the individual electrons is the object of our investigation is wholly useless when the distribution of the atoms is highly irregular. We have thus as usual to express ourselves in terms of statistical or averaged sums over all the electrons in the element of volume. Statistically the effect of an electric force on a body is to polarise the molecules, that is, to twist them round or alter them in some way so that they have a definite polarity. We can express this polarity as an averaged sum over all the electrons per unit volume.

241. The x-component of the moment of a simple bi-polar element is

$$ex' - ex,$$

(x', y', z') being the coordinates of the positive pole, and (x, y, z) those of the equal negative pole of the doublet referred to fixed rectangular axes.

* *Phil. Mag.* (5), XLIV. (1897), pp. 310. See also his books *Electricity and Matter* and *Corpuscular Theory of Matter.*

If the element of volume is small enough we have simply to add up vectorially the moments of the simple doublets contained in it to get the resultant moment of the element, i.e. its polarisation.

The x-component of the intensity of polarisation is thus

$$\mathbf{P}_x = \Sigma e\,(x' - x),$$

where Σ is taken per unit volume over all the doublets. This is simply Σex, taken for all the electrons and the elements of positive charge per unit volume, due regard being paid to sign. Thus

$$\mathbf{P}_x = \Sigma ex$$

is the x-component of the polarisation. Now let $(\bar{x}, \bar{y}, \bar{z})$ be the mean centre of the positive charge distribution in the element of volume at any instant, then, since the atom was originally neutral, this is also the mean position of the negative charges before they are displaced relatively to the positive charges. Thus Σex for the positive charges is

$$\bar{x}\Sigma e,$$

and for the negative charges it is

$$\bar{x}\Sigma e + \Sigma ex,$$

where (x, y, z) is now the displacement of the negative electron e from its neutral position of equilibrium.

But Σe for the positive charges is equal but opposite in sign to Σe for the negative electrons, and thus $\quad \mathbf{P}_x = \Sigma ex,$

Σ now being taken only for the negative electrons, (x, y, z) denoting the component displacements of that electron from its neutral position of equilibrium.

We have thus expressed the polarisation in terms of the negative electrons alone and it is in this form that we shall use it.

242. The position of an electron or better its displacement (x, y, z), and therefore the polarisation, depends on the forces acting on the electron. These forces are in the present instance of two types only.

Firstly there is the electric force due to the electric field in the aether. At first sight it might be thought that this is simply $e\mathbf{E}$, where $\mathbf{E}$ is the electric force intensity, and this is what is taken by most authors. On closer investigation it is however obvious that just as in the last paragraph a term must be added on account of the polarisation in the medium. This local force may generally be taken to be of the simple form*

$$a\mathbf{P},$$

* In isotropic media only. In the case where the medium possesses natural or induced aeolotropic characteristics the local force would be different in different directions and not necessarily parallel to the polarisation. Cf. Larmor, *Phil. Trans.* A, 190 (1897), p. 236.

and its direction is that of $\mathbf{P}$; a is some constant depending on the local distribution of atoms or molecules and which therefore would constantly vary with the temperature and pressure, but which in most cases is very nearly equal to $\frac{4\pi}{3}$.

Secondly there is the force of unknown origin and amount holding the electron inside the atom. Before the external field is applied the internal forces of this type will hold the electron in a certain position of stable equilibrium, and if the field is not too strong the displacement from this equilibrium position when it is applied will only be small*, so that the force holding it back may be taken to be proportional to the displacement. We can easily imagine in a general way that this quasi-elastic force has its origin in the mutual electrical actions between the charges in the atoms. Thus denoting by k a certain positive constant which depends on the properties of the atom, and which may be different for different electrons in the atom, we may write for the components of this force

$$-k(x, y, z)$$

(x, y, z) being interpreted as before stated, for each electron as the components in three definite directions of its displacement from the equilibrium position it occupied when the atom was undisturbed by any external actions. An average isotropy is assumed for the intra-molecular forces†.

Now in a condition of equilibrium these two forces, the internal and external forces, must balance so that for each electron we have three equations of the type

$$-kx + e(\mathbf{E}_x + a\mathbf{P}_x) = 0,$$

so that the displacement of the electron has an x-component equal to

$$\frac{k}{e}(\mathbf{E}_x + a\mathbf{P}_x),$$

and therefore

$$\mathbf{P}_x = \Sigma ex = \Sigma \frac{e^2}{k}(\mathbf{E}_x + a\mathbf{P}_x),$$

the sum Σ being taken per unit volume over all the contained electrons, each with their proper value of k. Thus

$$\mathbf{P}_x = \frac{\Sigma \frac{e^2}{k}}{1 - \Sigma \frac{ae^2}{k}} \mathbf{E}_x.$$

* The fields which can be created in actual practice are very feeble compared with those that probably exist inside the atoms or molecules.

† The question of aeolotropic intra-molecular forces is considered by Voigt (*Magneto u. Electro-optik*) as also is the case when the forces are not proportional to the displacement. The necessity for this extension does not however appear to exist at present, since all the phenomena which are explained by them can be more suitably accounted for on the assumption of aeolotropy in the local forcive, which is entirely neglected by Voigt.

Thus on the present form of theory the dielectric constant of the medium is ϵ where

$$\epsilon = 1 + \frac{4\pi\Sigma \frac{e^2}{k}}{1 - \Sigma \frac{ae^2}{k}},$$

and it is thus interpreted completely in terms of the electronic constitution of the medium. This formula was first obtained by Lorentz but owing to the uncertainty in the value of k it is of very little practical use in the present instance.

It might be thought that this view of the affair is necessarily restricted in that it leads essentially to a linear relation between **P** and **E**, which although apparently satisfactory from the practical standpoint, is nevertheless merely an approximation to the real state of affairs; but it must be remembered that our deduction is also essentially based on the linear relation between the quasi-elastic force resisting displacement and the displacement of the electron itself, and it is only so far as this assumption is justified that the above result is valid.

243. Pyro- and piezo-electricity*. We have generally assumed in the preceding discussions that the elements of all dielectric media are always permanently neutral as regards their electrical effect on external systems so long as they are not under the influence of an external polarising field. This would imply that anything in the nature of permanent polarity, which is such an important feature in the correlative subject of magnetism, is non-existent, or at least negligible, in the electrical case. Whether any such presumption is really justifiable it is difficult to say but there are certain phenomena which seem to suggest at least the possibility that it is not valid in every case.

Several substances like quartz and tourmaline which crystallise in asymmetric forms always appear to be polarised immediately after their temperature is changed, and in opposite directions according as it is raised or lowered. The polarisation exhibits itself mainly as an apparent separation of charge on the outer surface of the piece of the substance under investigation, one part of which appears positively charged and the opposite negatively charged. This is the phenomenon of *pyro-electricity*. If the substance is maintained for any period at the new temperature the polarisation gradually disappears and soon ceases to be observable at all.

244. Lord Kelvin explains† this phenomenon by assuming that the elements of the crystal substance are permanently polarised to an extent,

* A complete account of these phenomena with all the associated experimental and theoretical details can be found in Voigt, *Lehrbuch der Kristallphysik* (Leipzig, 1910). Cf. also the same author's *Kompendium der theoretischen Physik* (Leipzig, 1896), Bd. II, Teil 4, §§ 11–15, 20.

† *Nicol's Cyclopedia of Physical Science*, 1860. *Math. and Phys. Papers*, I. p. 315.

however, depending on the temperature, and that they are arranged with their electric axes in regular order in the crystalline media with which the phenomenon is associated. If the material of the crystal and in particular its outer surface are not perfectly non-conducting, the polarisation will ultimately give rise to a surface distribution of charge which neutralises the effect of its electric field at all external parts of the field. If the polarisation of the medium is altered by changing its temperature and the establishment of the neutralising charge takes place slowly, it should be possible to detect the polarisation before its external field is again neutralised.

This explanation of the phenomenon appears to be perfectly consistent with all the characteristic properties of the effect and, in addition, with the results of numerous experiments—based mainly on the independent variability of the polarisation and its neutralising charge—which have been performed with a view to testing it. It would thus appear to be highly probable that the underlying assumption of permanent molecular polarity is largely justified.

245. There is an inverse effect associated with the phenomenon of pyro-electricity, the existence of which was predicted by Lord Kelvin* but which was not observed until quite recently. If there is a relation of dependence between the polarisation of a medium and its temperature there must be a path of transformation open between the kinetic energy of thermal agitation of the molecules and the organised electric energy of their polarisations, and if the transformation can be carried out in either direction (i.e. if the effect is a reversible one) an alteration of the electric energy should produce a corresponding change in the thermal energy. The electrical energy of the polarisations may be altered by moving the body about in an electric field and thus we conclude that any such movement will give rise to a temperature variation in the substance. This electrocaloric effect has been observed by Straubel† and Lange‡, who find that the quantitative relation established for the phenomenon by Lord Kelvin by thermodynamic reasoning is satisfactorily verified.

246. Another effect of an analogous nature and of even more widespread character than the purely thermal effects just described has been observed in a large number of substances§. In these cases the observed polarisation of the substance is produced not by changing the temperature but by the application of pressure on opposite sides of the substance. This pressure gives rise to an additional strain in the material the main effect of which is that the constituents of the permanent polar elements take up new positions in the substance and the old neutralising surface charge is no longer effective in balancing their field at external points.

* *Math. and Phys. Papers*, I. p. 316, 1877.

† *Göttinger Nachr.* (1902), Heft. 2.

‡ Dissertation, Jena, 1905.

§ J. and P. Curie, *Paris C. R.* 91 (1880), pp. 294, 383.

Associated with this so-called *piezo-electric effect* there is an inverse phenomenon* in which an alteration in the pure elastic strain in the medium is effected simply by changing the energy of the electric polarisations by moving the substance about in an electric field of variable intensity.

247. Although these phenomena of pyro- and piezo-electricity seem to require for their explanation the assumption of permanent polar elements in the substance it is possible that the occurrence of such elements may be a result merely of the mutual interaction of the molecules, for it is only observed in crystalline substances in which the molecular structure is perfectly regular, and in which therefore the local interaction between any molecule and its neighbours would always be related in a definite manner to the crystal structure. Such a view is supported by the fact that the polarity, which in such cases is essentially a phenomenon of molecular grouping, depends on the physical conditions as to temperature and strain in the medium, which are just the conditions which are circumscribed by the mutual interaction of the molecules.

248. Associated with these reversible phenomena of pyro- and piezo-electricity, which depend essentially on the presence of permanent polar elements in the medium, there are irreversible phenomena arising from the induced polarity †, when the extent of the induction in a given field is a function of the temperature and strain conditions of the medium. These two new effects can of course only be exhibited in their inverse aspects and appear as a temperature and strain condition variation resulting from the polarisation of the medium induced by an external field. The former of these effects has never yet been detected and the latter is usually inseparably mixed up with the strain produced by the mechanical forces proper on the medium resulting from its polarisation (the effect of *electrostriction*), although arrangements can be devised by which it can be observed ‡.

* Lippmann, *Ann. chim. phys.* (5), 24 (1881), p. 164; *Jour. de phys.* (1), 10 (1881), p. 391. Cf. also Riecke, *Gött. Nachr.* (1893), pp. 3–13; Voigt, *Gött. Nachr.* (1894), Heft. 4.

† The thermal one was predicted by Lippmann, *Ann. chim. phys.* (5), 24 (1881), p. 171, and the mechanical one by Larmor, *Phil. Trans.* 190 A (1897), § 83. The magnetic aspect of these phenomena is more important. Cf. below, p. 270.

‡ Bidwell, *Phil. Trans.* A (1888), p. 228; *Proc. R. S.* 1894.

CHAPTER VI

MAGNETO-STATICS

249. Introduction. We now turn to the consideration of another static affair which occurs in electromagnetic theory: this is magnetism. The elementary ideas of the theory are derived from the action of certain bodies, called magnets, which if suspended so as to turn freely about a vertical axis at any part of the earth's surface except the magnetic poles, will in general tend to set themselves in a certain azimuth, and if disturbed from this position will oscillate about it. Such bodies are the iron ore called lode-stone, and pieces of steel which have been subjected to certain treatment.

It is found that the force which acts on the body tends to cause a certain line in the body to become parallel to a certain direction in space. This line we call the axis of the magnet.

Let us now suppose the axes of several magnets have been determined and that the end of each which points north is marked. Then if one of these magnets be freely suspended and another brought near to it, it is found that two marked ends repel each other as do also two unmarked ends, but a marked and an unmarked end attract. If the magnets are in the form of long rods or wires uniformly magnetised along their length, it is found that the greatest manifestations of force occur when the end of one magnet is held near the other, and that the phenomena can be accounted for by supposing that like ends of the magnets repel each other, that unlike ends attract each other and the intermediate parts of the magnets have no sensible effect.

The ends of a long thin magnet are commonly called poles. In the case of an indefinitely thin magnet uniformly magnetised in its length the extremities act as centres of force and the rest of the magnet appears devoid of magnetic action. In all actual magnets however the magnetisation deviates from uniformity so that no single points can be taken as poles. Coulomb however by using long thin rods magnetised with care succeeded in establishing the law of force between two poles.

The force between two magnetic poles is in the straight line joining them and numerically equal to the product of the strength of the poles divided by the square of the distance between them*.

* Coulomb, *Mem. de l'Acad.* 1785, p. 603. Cf. also Biot, *Traité de physique*, t. iii.

This law of course assumes that the strength of each pole is measured in terms of a certain unit, the magnitude of which may be deduced from the terms of the law. The unit pole is such that when placed at unit distance in air from a similar pole, it repels it with a unit force.

The quantity called the strength of a pole may also be called a quantity of magnetism (positive or negative), provided we attribute no properties to it, other than those observed in magnets. The quantity of magnetism at one pole of a magnet is always equal and opposite to that at the other so that the total quantity in any magnet is zero.

Since the expression of the law of force between given quantities of magnetism has exactly the same mathematical form as the law of force in electrostatic theory much of the mathematical treatment of magnetism must be similar to that of electricity. We shall in fact transfer the results of our previous analysis directly to the subject now before us.

250. If the middle of a long thin magnet be examined, it is found to possess no magnetic properties, but if the magnet be broken at that point, each of the pieces is found to have a magnetic pole at the place of fracture, and this new pole is exactly equal and opposite to the other pole belonging to the piece. It is impossible to procure, by any means, a magnet whose poles are unequal.

If we break the long thin magnet into a number of short pieces we shall obtain a series of short magnets, each of which has poles of nearly the same strength as those of the original long magnet.

Let us now put all the pieces of the magnet together as at first. At each point of junction there will be two poles exactly equal and of opposite kinds, placed in contact, so that their action on any other pole will be null. The magnet, thus rebuilt, has the same form as at first.

Since in this case we know the long magnet to be made up of little short magnets, and since the phenomena are the same as in the unbroken magnet, we may regard the magnet, even before being broken, as made up of small particles, each of which has two equal and opposite poles. The same idea is also found to be generally true for magnets of any shape. We are thus induced to transfer to this case the analysis and methods of the previous chapter on polarised media. We regard each particle of the body (molecule or molecular group) as a little magnet possessing two magnetic poles of equal strengths at a small distance apart. The magnetic field of force of such a particle is deduced in a manner exactly analogous to that already employed. The potential for instance at any point distant r from its centre in a direction making an angle θ with its axis is

$$\phi = \frac{m \cos \theta}{r^2},$$

m being the moment of the doublet. This may also be written in the form

$$\phi = (\mathbf{m}\nabla)\frac{1}{r}$$

when we take cognisance of the vector sense of the moment m.

The magnetic force intensity is then deduced as before by differentiation of the potential.

251. The field of a finite magnet is now obtained by regarding it as composed of a large number of small magnetic particles of the above kind.

The condition of magnetisation at any point of the finite magnet is completely specified by a vector **I**, called the intensity of magnetisation, which is such that if δv is any small element of volume of the substance at any point then $\mathbf{I}\delta v$ is the resultant effective moment of all the little bi-polar elements (magnets) in it. If the axes of these elementary magnets are distributed anyhow in all different directions then $\mathbf{I} = 0$, but if there is any degree of convergence to a definite direction **I** has a finite value.

The discussion of the potential and force in the field of this finite magnet then follows exactly the same lines as in the previous chapter.

The potential of the magnet at points external to the distribution of polarisation is obtained by the addition of the potentials of each of its constituent elements and is therefore

$$\psi = \int (\mathbf{I}\nabla)\frac{dv_1}{r_1},$$

and the force is obtained from this potential in the ordinary way, i.e. as its negative gradient. The application of these expressions at internal points however fails firstly on account of the uncertainty as to the law of action of a doublet very close up to it and secondly, as regards the expression for the force, on account of the non-absolute convergence of the integral. We are then led to the introduction of Poisson's ideal *magnetic matter** consisting of a volume density

$$\rho = -\operatorname{div}\mathbf{I}$$

at any part of the solid magnet together with a surface distribution of density

$$\sigma = \mathbf{I}_n$$

on the surface of the magnet. This distribution effectively replaces the distribution of bi-poles as far as the determination of the magnetic field outside the magnetic matter is concerned, a determination which is valid up to within a physically small differential distance from the matter. If however the point at which we wish to investigate the field is inside the magnet we must as usual put a physically small cavity round it and define the field there

* *Mem. de l'Acad.* 5 (1826), pp. 247, 488; 6 (1827), p. 441. Cf. also Maxwell, *Treatise*, II. § 385.

as the field inside this cavity due to the distribution beyond it which is effectively represented as above by the distribution of ideal magnetic matter and which therefore includes a part due to the distribution $\sigma = \mathbf{I}_n$ of magnetic matter on the surface of the cavity : together with the field due to the matter inside the cavity. This latter part of the field with the distribution on the walls of the cavity are the parts, and the only parts, of the field at the internal point which depend appreciably on the local molecular configuration, and which therefore in any real case are quite unknown : we have however separated them from the definite part of the field due to the magnetic body as a whole which is specified as due to the distribution of magnetic matter of density ρ throughout the magnet and σ on its outer boundary, the field of these latter involving no appreciable local part. We then follow the usual course in physical theories and ignore the local part of the field as being ineffective as regards the conditions of the matter in bulk.

We thus define the magnetic field inside the body as that field when the effect of the local parts is rejected. It is therefore completely defined as due to the distribution of ideal magnetic matter throughout the volume of the magnet and on its surface. The force and potential of the field due to such a distribution are expressible by definitely convergent integrals at internal as well as external points. Moreover on this definition the magnetic force is the vector which is represented by the gradient of the potential function.

252. If the direction of magnetisation I at each point of the body is the same, say parallel to the x-axis, then its potential at the point (ξ, η, ζ) is

$$\psi = \int I \frac{\partial}{\partial x}\left(\frac{1}{r}\right) d\tau = -\int I \frac{\partial}{\partial \xi}\left(\frac{1}{r}\right) d\tau,$$

and since I is not a function of (ξ, η, ζ), but only of (x, y, z) this is

$$\psi = -\frac{\partial}{\partial \xi}\int \frac{I}{r}\, d\tau.$$

Thus if we know the gravitational potential at the point (ξ, η, ζ) of the same body with a distribution of density $\rho = I$ throughout its volume we can at once deduce the potential of the field of the body magnetised at each place parallel to the axis of x to intensity I.

For example the gravitational potential of a sphere uniformly charged to density $\rho = I$ throughout its volume has a potential at external points distant r from its centre

$$\phi = \frac{4}{3}\frac{\pi a^3 I}{r},$$

and thus if we consider the same sphere uniformly magnetised parallel to the axis of x at each point to intensity I the potential of the magnetic field is

$$\psi = -\frac{4\pi a^3 I}{3} \cdot \frac{x}{r^3}.$$

253. The mathematical relations of the magnetic field. We have now a consistent scheme to which to apply our analysis and we proceed in this exactly as before: we denote the magnetic force by $\mathbf{H}$ and the potential by ψ and thus

$$\mathbf{H} = -\operatorname{grad}\psi,$$

and the magnetic potential ψ satisfies the conditions that at each point of space

$$\nabla^2\psi = -4\pi\rho,$$

and at a surface of discontinuity in the medium ψ is continuous but

$$\frac{\partial\psi_1}{\partial n} - \frac{\partial\psi_2}{\partial n} = 4\pi\sigma,$$

ρ and σ as before, in this chapter, denoting the volume and surface densities of Poisson's ideal magnetic matter, so that

$$\rho = -\operatorname{div}\mathbf{I},$$

$$\sigma = \mathbf{I}_n,$$

whence the conditions for ψ can be interpreted in the form

$$\nabla^2\psi = -\operatorname{div}\mathbf{H} = 4\pi\operatorname{div}\mathbf{I},$$

and $$\mathbf{H}_{1n} - \mathbf{H}_{2n} = 4\pi\mathbf{I}_n.$$

Thus $$\operatorname{div}(\mathbf{H} + 4\pi\mathbf{I}) = 0.$$

The vector $$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I},$$

is everywhere solenoidal: it satisfies the usual hydrodynamical equation of continuity and is usually called the magnetic flux or induction vector: it is of immense importance in the theory. The fact that it is solenoidal simply means that the surface integral of the normal induction over any closed surface whatever is zero. For if we take the integral

$$\int (\operatorname{div}\mathbf{B})\, dv$$

throughout any region bounded by the surface f it must vanish for at each point $\operatorname{div}\mathbf{B} = 0$; but by Green's theorem it consists of

$$\int_f \mathbf{B}_n\, df,$$

together with the surface integrals arising from discontinuities when we pass into the magnetic matter: these are the integrals of

$$\mathbf{B}_{1n} - \mathbf{B}_{2n}$$

over the surfaces concerned or of

$$\mathbf{H}_{n_1} - \mathbf{H}_{n_2} - 4\pi\mathbf{I}_n,$$

which is zero: this establishes the result.

254. Thus if we plot the magnetic field in terms of the lines of magnetic induction and form them into tubes, we see that the tubes of magnetic induction are always closed and the product of the strength into the cross-

section is constant along each tube so that the normal induction over any closed surface is zero. It follows that the value of the surface integral over any part of a surface depends solely on the boundary curve and it must be able to be expressed in terms of the position of the boundary. This is accomplished by means of Stokes' conversion of a surface integral into a line integral and requires us to find a vector which is such that its components ($\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{A}_z$) satisfy

$$\mathbf{B}_x = \frac{\partial \mathbf{A}_z}{\partial y} - \frac{\partial \mathbf{A}_y}{\partial z}$$

and two similar equations: this is expressed in the form that

$$\mathbf{B} = \text{curl}\,\mathbf{A},$$

and then for any surface f and the boundary curve s

$$\int_f \mathbf{B}_n\, df = \int (\mathbf{A}\, d\mathbf{s}),$$

the former integral being taken over f and the latter round its boundary curve.

The vector $\mathbf{A}$ is called the *magnetic vector potential** and its significance will subsequently appear.

255. We know that the potential of the particle placed along the axis of z at the origin is

$$\psi = \frac{mz}{r^3},$$

so that by definition of the vector potential $\mathbf{A}$

$$\begin{aligned}
\frac{\partial \mathbf{A}_z}{\partial y} - \frac{\partial \mathbf{A}_y}{\partial z} &= \mathbf{B}_x = -\frac{\partial \psi}{\partial x} = \frac{3mxz}{r^5},\\
\frac{\partial \mathbf{A}_x}{\partial z} - \frac{\partial \mathbf{A}_z}{\partial x} &= \mathbf{B}_y = \qquad\quad\; = \frac{3myz}{r^5},\\
\frac{\partial \mathbf{A}_y}{\partial x} - \frac{\partial \mathbf{A}_x}{\partial y} &= \mathbf{B}_z = \qquad\quad\; = -\frac{m}{r^3} + \frac{3mz^2}{r^5}.
\end{aligned}$$

These equations are solved by

$$\mathbf{A}_x = -\frac{my}{r^3}, \quad \mathbf{A}_y = \frac{mx}{r^3}, \quad \mathbf{A}_z = 0.$$

The result is that the vector potential in the case of a magnetic particle is at right angles to the axis of the magnet and to the radius to the point and is of magnitude $\frac{m \sin\theta}{r^2}$, where θ is the angle between the axis of the magnet and the radius to the point and its sense is that of positive rotation round the axis of the magnet. The expression $\frac{m \sin\theta}{r^2} = \frac{mr \sin\theta}{r^3}$ shows that we

* Maxwell, *Treatise*, II. § 405.

may break it up into components for $mr \sin\theta$ may be interpreted as twice the area of the triangle formed by m and r and the vector is at right angles to the area and its components are the projections. If we now use generally

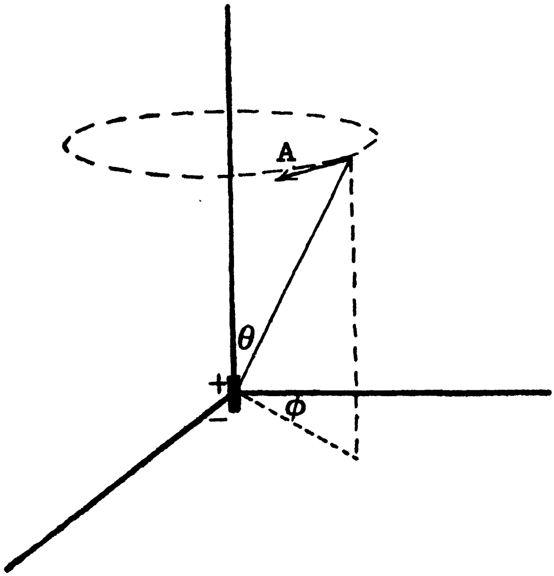


Fig. 51

$\mathbf{r}$ to denote the radius vector from the centre of the magnet to any external point in the field, $\mathbf{m}$ the vectorial moment of the particle and $\mathbf{r}_1$ the unit vector along $\mathbf{r}$, then we see that

$$\mathbf{A} = \frac{[\mathbf{m}\mathbf{r}_1]}{r^2} = [\mathbf{m}\nabla]\frac{1}{r}$$

is the general expression for the vector potential at the point in the field.

256. The above formulae for the vector potential of a magnetic particle may now be used to enable us to write down a vector potential for any finite magnet. We have merely to replace $\mathbf{m}$ by $\mathbf{I}dv$ and then integrate over the whole body. We get at once

$$\mathbf{A} = \int [\mathbf{I}\nabla]\frac{dv}{r}^*,$$

a formula which certainly applies at points outside the magnetism. At points inside the magnetism r can vanish in an element of the integral but the integral is nevertheless quite convergent and $\mathbf{A}$ is thus representable quantitatively, in a physical theory which neglects purely local actions, by this integral at every point of the field. But as regards its differential coefficients, by means of which $\mathbf{B}$ is derived from it, it is dependent on the

* Maxwell, *Treatise*, II. § 405.

unknown distribution of the local polarity. This is of course quite in keeping with the fact that the magnetic field inside the magnet, considered as due to the aggregate of the molecular magnetic elements, by means of which the vector potential is defined through the relation curl $\mathbf{A} = \mathbf{B}$, itself involves this local contribution. But in the previous theory we were able to discard the local part of the magnetic force, depending on the molecular character of the distribution at the point, from which alone indefiniteness arises. It may be surmised that we should in like manner discard from the vector potential the purely local contribution which is the source of its discontinuity. This may be effected as usual by the aid of integration by parts. At a point inside the material medium the field may be then separated into two parts; the first due to the medium beyond a physically small closed surface surrounding it and for this part the function $\mathbf{A}$ and its first gradients can be represented analytically by integrals of the above type which are entirely convergent and determinate since r cannot be less than a finite lower limit; the second part is due to the elements inside the surface thus drawn. On integration by parts the contribution of the former to the expression for $\mathbf{A}$ may be put in the form

$$\mathbf{A} = \int \frac{\text{curl}\,\mathbf{I}}{r}\,dv + \left| \int_f [\mathbf{n}_1 \mathbf{I}] \frac{df}{r} \right|_1^2,$$

and is thus expressible as a volume integral together with an integral over interfaces of transition of the magnetism, and also an integral over the surface of the cavity: the volume integral is convergent and does not depend on the form of the cavity, while the integral over the surface of the cavity is finite and with the part due to the distribution inside the small surface drawn is the sole representative of the influence of the local molecular configuration; in our present procedure it depends on the form of the cavity; in actual practice it depends on the local molecular configuration. By the general principle, the mechanically effective functions are the analytical integrals obtained by excluding this undetermined local part. This leads to an expression for the total vector potential of the medium treated as continuous

$$\mathbf{A} = \int \frac{\text{curl}\,\mathbf{I}}{r}\,dv + \int \frac{[\mathbf{n}_1 \mathbf{I}]}{r}\,df^*,$$

the latter surface integral being taken over the transition boundary of the magnetism.

257. As a final result we may quote the case of a uniform magnetic shell of strength τ on the surface f. This is obtained from the form for the particle by replacing $\mathbf{m}$ by $\mathbf{n}_1 \tau df$ and integrating over the surface of the shell: this gives

$$\mathbf{A} = \tau \int_f [\mathbf{n}_1 \nabla] \frac{df}{r},$$

* Larmor, *Aether and Matter*, p. 260.

which on reconversion by Stokes's Theorem gives

$$\mathbf{A} = \tau \int \frac{d\mathbf{s}}{r},$$

this integral being taken round the boundary of the shell.

258. So far we have been dealing with generalities and have considered the actual distribution of magnetisation in a magnet as given explicitly among the data of the problem. We have made no assumption as to whether this magnetisation is permanent or temporary, except in those parts of our reasoning in which we have supposed the magnet broken up into small portions, or small portions removed from the magnet in such a way as not to alter the magnetisation of any part.

We must now consider, as in the analogous dielectric problem, the magnetisation of bodies with respect to the mode in which it may be produced and changed. A bar of iron held parallel to the direction of the earth's magnetic field is found to become magnetic, with its poles turned the opposite way from those of the earth, or the same way as those of a compass needle in stable equilibrium.

Any piece of soft iron placed in a magnetic field is found to exhibit magnetic properties. If the iron is removed from the field, its magnetic properties are greatly weakened or disappear altogether. On the other hand a piece of hard iron or steel retains its magnetic properties acquired when placed in a magnetic field.

259. If a magnet could be constructed so that the distribution of its magnetisation is not altered by any magnetic force brought to act on it, it might be called a permanently or rigidly magnetised body. There is no known magnetic substance which perfectly fulfils this condition, but it is nevertheless convenient for scientific purposes to make a distinction between the permanent and temporary magnetisation, defining the permanent magnetisation as that which exists independently of the magnetic forces, and the temporary magnetisation as that which depends on those forces. This distinction is however not founded on a knowledge of the intimate nature of the magnetisable substances: it is only the expression of a convenient hypothesis*.

* Various forms of the mathematical theory of magnetism have recently been constructed on the assumption of the existence of a distribution of permanent magnetic matter. In this case the magnetic induction vector is no longer solenoidal, its divergence determining the density of the magnetic matter. This procedure is adopted in order to secure a closer analogy with the electric case and to remove certain discrepancies supposed to exist in the more usual form of the theory: in reality it confuses the point at issue and only complicates a perfectly valid theory. Cf. E. Cohn, *Das electromagnetische Feld*, p. 510; R. Gans, *Ann. der Physik*. 13 (1904), p. 634 and *Encyclopädie der math. Wissensch.* Bd. v. Art. 15.

Our theory thus divides itself into two distinct parts which although they are involved and dependent on one another are conveniently treated separately, we shall therefore begin by a short discussion of the relations between rigidly magnetised substances and the field they create and follow it by the more general case of induced magnetisation.

260. Permanent magnetism. The above analysis of course applies to this case. If the magnetic field is due to the permanent magnets alone the field of their polarised elements is all that there is. The potential ψ of this field is due to the distributions of ideal magnetic matter consisting of a volume density

$$\rho_0 = -\operatorname{div} \mathbf{I}$$

throughout the volume of the magnets with

$$\sigma_0 = \mathbf{I}_n$$

over their surfaces. **I** is of course the vector defining the intensity of the permanent magnetisation at each point of the body. The magnetic force **H** is defined as the force due to the above distributions so that everywhere

$$\mathbf{H} = -\operatorname{grad} \psi,$$

and also $$\nabla^2\psi = -\operatorname{div} \mathbf{H} = -4\pi\rho_0 = 4\pi \operatorname{div} \mathbf{I},$$

so that if the induction is $$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I},$$

we have $$\operatorname{div} \mathbf{B} = 0,$$

with the condition at the surface of the magnet that

$$\mathbf{H}_{1_n} - \mathbf{H}_{2_n} = 4\pi\mathbf{I}_n,$$

or interpreted in terms of the induction

$$\mathbf{B}_{1_n} - \mathbf{B}_{2_n} = 0,$$

so that **B** is a solenoidal vector as in the general case. The vector potential of the permanent magnetism **A** is thus derived so that

$$\mathbf{B} = \operatorname{curl} \mathbf{A}.$$

These general results of the analysis of the field are useful only in as far as they enable us to determine the really essential and accessible quantities of such fields, viz. those defining the observable mechanical relations of the magnets giving rise to it. Let us consider a few applications in this respect.

261. What is the work done in introducing a permanent magnet into a magnetic field of specified intensity? In other words, what is the mechanical potential of the magnet when existing in a given position in any given field?

In displacing a small elementary doublet of moment **m** through a small distance ds the work done is, as before,

$$-\left(\mathbf{m} \cdot \frac{\partial \mathbf{H}}{\partial s}\right) ds,$$

H denoting the strength of the field at the position of the doublet.

If now we consider all the little magnets in the element dv of the magnet in any position we can sum them up into a polarisation per unit volume, the intensity $\mathbf{I}_1$ being such that

$$\mathbf{I}_1 dv = \Sigma \mathbf{m};$$

and each of these elementary magnets receives the same small displacement ds at a place where the total magnetic force is $\mathbf{H}$ (due partly to the original field and partly to the magnet itself) and is practically the same for them all. Thus the work done in the displacement of the small volume element of the magnet is

$$-\left(\mathbf{I}_1 \cdot \frac{\partial \mathbf{H}}{\partial s}\right) dv\, ds,$$

or for the whole finite magnet the work done during the displacement ds of each element is

$$\delta W = -\, ds \int \left(\mathbf{I}_1 \frac{\partial \mathbf{H}}{\partial s}\right) dv.$$

If we make the usual assumptions as to the reversibility of the actions we can conclude in the usual way that the potential energy of the magnet in the field is

$$W = \int (\mathbf{I}_1 \mathbf{H})\, dv,$$

since $\mathbf{I}_1$ is constant for any displacement of the magnet which does not alter its shape or form. The linear component of the force in any direction acting on the magnet in this position is

$$\mathbf{F} = -\frac{\partial W}{\partial s} = \int \left(\mathbf{I}_1 \frac{\partial \mathbf{H}}{\partial s}\right) dv$$

$$= \frac{\partial}{\partial s} \int (\mathbf{I}_1 \mathbf{H})\, dv.$$

262. But the magnetic field consists of two parts, one due to the original field and the superposed part due to the magnet itself; denoting the intensities in these separate parts by $\mathbf{H}_2$ and $\mathbf{H}_1$ respectively, so that

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2,$$

we have
$$W = \int (\mathbf{I}_1 \mathbf{H}_1)\, dv + \int (\mathbf{I}_1 \mathbf{H}_2)\, dv,$$

and the first integral on the right is independent of the position of the magnet in the field; it is a constant for any displacement of the magnet which does not alter its form or magnetisation. We may thus simply regard

$$W_{12} = \int (\mathbf{I}_1 \mathbf{H}_2)\, dv$$

as the potential of the given magnet in the external field $\mathbf{H}_2$.

263. Now suppose this external field $\mathbf{H}_2$ is the field due to a second permanent magnet, the work of bringing the first magnet up from a great distance to its relative position is, as we have seen, equal to

$$W_{12} = \int (\mathbf{I}_1\mathbf{H}_2)\, dv.$$

If however the second magnet had been placed in position first and the first magnet brought up so as to obtain the same relative configuration, the same work would obviously have been done, but this time it is expressed by

$$W_{21} = \int (\mathbf{I}_2 \,.\, \mathbf{H}_1)\, dv,$$

so that we must have

$$W_{12} = W_{21} = \tfrac{1}{2}\int [(\mathbf{I}_2\mathbf{H}_1) + (\mathbf{I}_1\mathbf{H}_2)]\, dv,$$

where the combined integral is taken over the whole of the space including the two magnets. Since however

$$\int (\mathbf{I}_1\mathbf{H}_1)\, dv \quad \text{and} \quad \int (\mathbf{I}_2\mathbf{H}_2)\, dv$$

are both constant for displacements of the two magnets we may take

$$W = \tfrac{1}{2}\int (\mathbf{I}_1 + \mathbf{I}_2,\ \mathbf{H}_1 + \mathbf{H}_2)\, dv + \text{constant}$$

for the potential energy of these magnets relative to one another. If we use now

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2$$

to denote generally the polarisation in any element of the field ($\mathbf{I} = \mathbf{I}_1$ in the first magnet and $\mathbf{I}_2$ in the second and is zero everywhere else), we have then

$$W = \tfrac{1}{2}\int (\mathbf{IH})\, dv + \text{const.},$$

the integral being extended throughout the whole field.

A convenient interpretation of this quantity in terms of the coordinates of the respective magnetic masses would enable us to determine the whole force system exerted by one magnet on the other.

264. The general properties of the fields of permanent magnets and their mechanical reactions are illustrated by application to the simplest case when the magnets are very small doublets; we have first to examine more closely the field of a single doublet and then we can easily determine the mutual potential and reaction forces of two such doublets.

The field of a magnetic particle. The potential due to the small magnetic particle AB of moment μ at a point P distant r from its centre is

$$\psi = \frac{\mu \cos\theta}{r^2}.$$

The components of force in the magnetic field at P are

(i) along OP

$$-\frac{\partial\psi}{\partial r}=\frac{2\mu\cos\theta}{r^3},$$

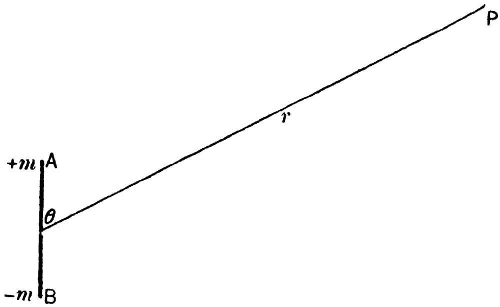


Fig. 52

(ii) at right angles to OP

$$-\frac{1}{r}\frac{\partial\psi}{\partial\theta}=\frac{\mu\sin\theta}{r^3}.$$

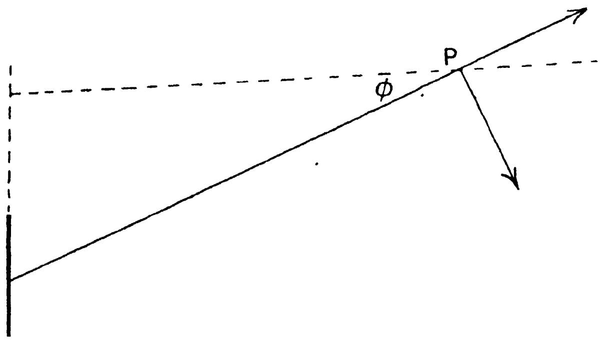


Fig. 53

The resultant force makes with the radius vector r an angle

$$\phi=\tan^{-1}\left(\tfrac{1}{2}\tan\theta\right),$$

and the tangent to the line of force at P meets the axis of the magnet produced at a distance equal to

$$\frac{r\sin\phi}{\sin(\phi+\theta)}=\frac{r}{\cos\theta+2\cos\theta}=\frac{r}{3\cos\theta}.$$

265. *The mutual potential of two small magnetic particles**. The mutual potential of the two particles is the potential of one in the field of the other.

* Tait, *Quarterly Jour. of Math.* Jan. 1860. Cf. his book on *Quaternions*, §§ 442, 443.

The two particles are μ at AB and μ' at AB'. We calculate the potential of the second magnet in the field of the first as the sum of the potentials due to its constituent poles m' at A' and $-m'$ at B'. It is therefore the limit of

$$\mu\left\{\frac{m'\cos AOA'}{OA'^2}-\frac{m'\cos AOB'}{OB'^2}\right\}$$

$$=\text{Lt}\,\mu\left[\frac{m'}{OA'^3}\left(r\cos\theta+\frac{ds'}{2}\cos\epsilon\right)-\frac{m'}{OB'^3}\left(r\cos\theta-\frac{ds'}{2}\cos\epsilon\right)\right]$$

$$=\text{Lt}\,\mu r\cos\theta\left[m'\left(r+\frac{ds'}{2}\cos\theta'\right)^{-3}-m'\left(r-\frac{ds'}{2}\cos\theta'\right)^{-3}\right]+\frac{\mu\mu'\cos\epsilon}{r^3}$$

$$=\frac{\mu\mu'}{r^3}(\cos\epsilon-3\cos\theta\cos\theta'),$$

where ϵ is the angle between the positive directions of the axes of the particles and θ, θ' the angles these axes make with the radius joining the centres from one definite magnet to the other.

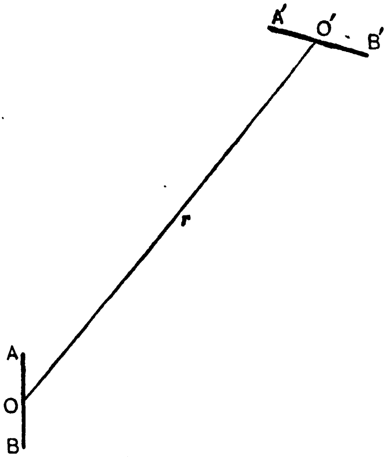


Fig. 54

If now we use (l_1, m_1, n_1), (l_2, m_2, n_2) as the direction cosines of the axes of these magnets and (λ, μ, ν) as the direction cosines of the radius vector the mutual potential can be written in the form

$$\frac{\mu\mu'}{r^3}[(l_1l_2+m_1m_2+n_1n_2)-3(l_1\lambda+m_1\mu+n_1\nu)(l_2\lambda+m_2\mu+n_2\nu)],$$

which we might evidently have obtained otherwise as

$$\mu\mu'\left(l_1\frac{\partial}{\partial x}+m_1\frac{\partial}{\partial y}+n_1\frac{\partial}{\partial z}\right)\left(l_2\frac{\partial}{\partial x}+m_2\frac{\partial}{\partial y}+n_2\frac{\partial}{\partial z}\right)\frac{1}{r},$$

where $$r^2\equiv x^2+y^2+z^2.$$

This may be written in the form

$$\frac{\mu\mu'}{r^3}(l_1l_2 + m_1m_2 + n_1n_2) - \frac{3\mu\mu'}{r^5}(l_1x + m_1y + n_1z)(l_2x + m_2y + n_2z).$$

266. *Determine the wrench exerted by one magnet on the other.* This can be determined from the above value of the mutual potential, for if this is W the force component parallel to the x-axis is $-\dfrac{\partial W}{\partial x}$ or

$$\begin{aligned}\frac{3\mu\mu'}{r^5}\,&x\,(l_1l_2 + m_1m_2 + n_1n_2)\\ &- \frac{15\mu\mu'}{r^7}x\,(l_1x + m_1y + n_1z)(l_2x + m_2y + n_2z)\\ &+ \frac{3\mu\mu'}{r^5}\{l_1(l_2x + m_2y + n_2z) + l_2(l_1x + m_1y + n_1z)\}.\end{aligned}$$

A simplification is however introduced if we resolve the resultant force into three directions parallel to the radius and the axes of the two magnets respectively. If these components are I, II, III then the above expression is

$$\mathrm{I}l_1 + \mathrm{II}l_2 + \mathrm{III}\,\frac{x}{r},$$

and by comparison with the above we see that

$$\begin{aligned}\mathrm{I} &= \frac{3\mu\mu'}{r^4}\left(\frac{l_2x}{r} + \frac{m_2y}{r} + \frac{n_2z}{r}\right),\\ \mathrm{II} &= \frac{3\mu\mu'}{r^4}\left(\frac{l_1x}{r} + \frac{m_1y}{r} + \frac{n_1z}{r}\right),\\ \mathrm{III} &= \frac{3\mu\mu'}{r^4}(l_1l_2 + \ldots) - \frac{15\mu\mu'}{r^7}x\,(l_1x + \ldots)(l_2x + \ldots),\end{aligned}$$

or returning to the original notation,

$$\begin{aligned}\mathrm{I} &= \frac{3\mu\mu'\cos\theta'}{r^4},\\ \mathrm{II} &= \frac{3\mu\mu'\cos\theta}{r^4},\\ \mathrm{III} &= \frac{3\mu\mu'\cos\epsilon}{r^4} - \frac{15\mu\mu'}{r^4}\cos\theta\cos\theta'.\end{aligned}$$

267. We must now determine the couple on the second magnet and to do this we reduce the wrench on it to a force at its centre and the couple. The force is as given above: to determine the couple we proceed in a slightly different manner although it is possible to deduce it from the potential energy. The couple acting on any small magnet in any field when the centre is the base point is at once written down when the force at the centre due to the field is known. If H be this force there is a pair of equal and opposite parallel forces mH acting on the poles and the moment of the couple

is $\mu H \sin\phi$ and it is in the plane of the force and the axis tending to set the axis along the force; ϕ is the angle between the axes and the direction of the force.

[The potential energy of the small magnet in the field is

$$\mu\left(l\frac{\partial\psi}{\partial x}+m\frac{\partial\psi}{\partial y}+n\frac{\partial\psi}{\partial z}\right)=-\mu H\cos\phi.]$$

We have already seen that the force at the centre of the second magnet can be resolved into two components, one $\frac{3\mu\cos\theta}{r^3}$ outwards along the radius

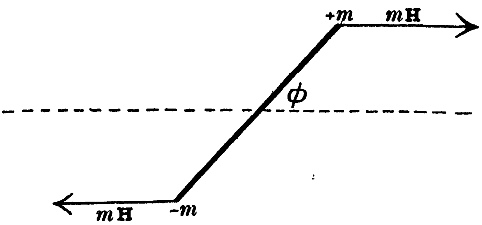


Fig. 55

and the other $\frac{\mu}{r^3}$ parallel to the axis of the first magnet. It follows that the couple on the second magnet in the plane of its axis and the radius vector is towards the radius vector and of amount

$$\frac{3\mu\mu'\cos\theta\sin\theta'}{r^3},$$

and the couple in the plane of the axis and a line parallel to the axis of the first magnet through its centre is

$$\frac{\mu\mu'\sin\epsilon}{r^3}.$$

268. There are two particular cases of these general results which are of historical interest as they provided the means by which Gauss proved that the law of magnetic attractions was that of the inverse square.

Fig. 56

The positions of the magnets are clearly indicated in the diagrams, the axes being in each case in one plane.

In the first case the force at the centre of the second magnet is $\frac{\mu}{r^3}$ and so the couple on it is $\frac{\mu\mu'}{r^3}$; in the second case the couple is $\frac{2\mu\mu'}{r^3}$: in the one case the couple is double what it is in the other and this is Gauss' result.

269. Induced magnetisation. We shall now investigate temporary magnetisation on the assumption that the magnetisation of any particle of the substance may be dependent on the magnetic force acting on that particle. This magnetic force may arise partly from external causes and also partly from the magnetisation of the neighbouring particles.

A body whose magnetisation is altered in virtue of the action of a magnetic force is said to be magnetised by induction, and the alteration in magnetisation is said to be induced by the magnetising force.

The problem before us is really the determination of the alteration of the magnetic field due to the introduction of a piece of a magnetisable substance (which may for generality also have permanent magnetism in it). The theory states that this alteration is due to the fact that the substance becomes magnetised, i.e. each little bit of it becomes a little magnet under the influence of the field. If this is so then a knowledge of the intensity of magnetisation induced would be sufficient to enable us to determine the whole of the circumstances. But this is what we do not know. We can however make a theory and see how it actually agrees with the facts. This is the procedure adopted for dielectrics in the previous chapter and we shall follow the same course of reasoning as there set out.

The presumption is that the magnetisation is conditioned by the magnetic force and thus if there is to be any law about the matter at all we must have the polarisation intensity $\mathbf{I}$ at any point in the medium as a function of the total magnetic force $\mathbf{H}$ at that point

$$\mathbf{I} = f(\mathbf{H}).$$

If the theory is to be at all workable this relation must be a linear one, which in isotropic media assumes the form

$$\mathbf{I} = \mu'\mathbf{H} + \mathbf{I}_0,$$

where $\mathbf{I}_0 = 0$ if there is no permanent magnetisation.

This simple law is of course right if the field is small or when the substance is but slightly magnetic. In other cases it is of no use but we can do no better.

270. Now let us examine the magnetic field in which such a relation holds. The mathematical formulation involves three vectors: (i) $\mathbf{I}$, the intensity of magnetisation, (ii) $\mathbf{H}$, the magnetic force which is the gradient of a potential ψ and (iii) $\mathbf{B}$, the magnetic flux which is a stream vector; and then

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I},$$

so that only two of the variables are really independent in the general theory. Now we have

$$\mathbf{I} = \mathbf{I}_0 + \mu'\mathbf{H},$$

so that

$$\mathbf{B} = \mathbf{H}(1 + 4\pi\mu') + 4\pi\mathbf{I}_0,$$

where $\mathbf{I}_0$ represents the distribution of permanent magnetism.

The coefficient $(1 + 4\pi\mu')$ is called the *coefficient of induction* of the medium and is usually denoted by μ, so that

$$\mathbf{B} = \mu\mathbf{H} + 4\pi\mathbf{I}_0,$$

μ is Kelvin's *permeability**: if μ is big the substance is very permeable to magnetic force.

We can express everything in terms of the magnetic potential ψ: in the most general case

$$\operatorname{div}\mathbf{B} = 0,$$

and

$$\mathbf{H} = -\operatorname{grad}\psi,$$

so that

$$\operatorname{div}(\mu\mathbf{H} + 4\pi\mathbf{I}_0) = 0,$$

or in other words

$$\operatorname{div}(\mu\operatorname{grad}\psi) = 4\pi\operatorname{div}\mathbf{I}_0,$$

which is the characteristic equation for the magnetic potential in the theory. To obtain a solution for the problem of the disturbance of the magnetic field by the introduction of such magnetic media we have to find suitable integrals of this equation in the various regions involved and fit them up at the boundaries.

271. A knowledge of the conditions which hold at a surface of discontinuity in the medium is therefore essential to the theory. These are easily obtained in the usual manner. In crossing any interface in the medium the normal component of the induction is always continuous or

$$\mathbf{B}_{1_n} - \mathbf{B}_{2_n} = 0$$

suffices 1 and 2 denoting the different media on the two sides of the surface. But

$$\mathbf{B}_1 = \mathbf{H}_1 + 4\pi(\mu_1'\mathbf{H}_1 + \mathbf{I}_{0_1}),$$

and

$$\mathbf{B}_2 = \mathbf{H}_2 + 4\pi(\mu_2'\mathbf{H}_2 + \mathbf{I}_{0_2}),$$

so that

$$\mu_1\mathbf{H}_1 + 4\pi\mathbf{I}_{0_1} = \mu_2\mathbf{H}_2 + 4\pi\mathbf{I}_{0_2},$$

or in terms of the potential

$$\mu_1\frac{\partial\psi_1}{\partial n} - \mu_2\frac{\partial\psi_2}{\partial n} = 4\pi(\mathbf{I}_{0_1} - \mathbf{I}_{0_2}).$$

If as is often the case one of the media is air then we can put $\mu_2 = 1, \mathbf{I}_{0_2} = 0$ so that the condition is

$$\mu_1\frac{\partial\psi_1}{\partial n} - \frac{\partial\psi_2}{\partial n} = 4\pi\mathbf{I}_{0_1},$$

* "Theory of induced magnetism," *Reprint of Articles on Electricity, etc.* p. 484.

or if there is no permanent magnetism at all

$$\mu_1 \frac{\partial \psi_1}{\partial n} = \frac{\partial \psi_2}{\partial n}.$$

In addition it is obvious that the potential ψ must be continuous across any such interface and therefore also the tangential component of the magnetic force is also continuous.

272. To solve the problem of the disturbance produced by and in a given piece of magnetisable substance introduced into a magnetic field we have to determine ψ, a continuous function, to satisfy the following conditions:

1. In free space $\qquad \nabla^2 \psi = 0,$

and at a great distance from the field ψ and the force components are zero.

2. In the magnetisable substances

$$\frac{\partial}{\partial x}\left(\mu \frac{\partial \psi}{\partial x}\right) + \ldots = 0,$$

and at the boundary, if ψ_i and ψ_0 refer to internal and external potential

$$\mu \frac{\partial \psi_i}{\partial n} = \frac{\partial \psi_0}{\partial n},$$

where δn is the element of the outward normal.

3. In the rigid magnets if any

$$\nabla^2 \psi = 4\pi \operatorname{div} \mathbf{I},$$

and at their surface $\qquad \operatorname{grad}_n \psi_i - \operatorname{grad}_n \psi_0 = 4\pi \mathbf{I}_n.$

273. *A sphere of soft iron* in a uniform magnetic field.* Here ψ is regular except at infinity where it is like $-Hx$ and $\nabla^2 \psi = 0$ everywhere. Try

$$\psi_i = Ax \text{ inside},$$

$$\psi_0 = -Hx + \frac{Bx}{r^3} \text{ outside}.$$

These satisfy all but the surface conditions. Continuity of potential at the surface $r = a$ gives

$$A = -H + \frac{B}{a^3},$$

and continuity of induction gives

$$\mu A = -H - \frac{2B}{a^3},$$

so that $\qquad A = -\frac{3}{2+\mu} H, \qquad B = \frac{\mu - 1}{\mu + 2} Ha^3,$

* Poisson, *l.c.* p. 164. Cf. also Somigliana, *Rend. del r. Inst. Lomb.* 2 (36), (1903); Boggio, *ibid.* (2) (37), (1904), p. 123 and *Nuovo Cimento* (5), 11 (1906), p. 1.

and the inside potential is

$$\psi_i = -\frac{3Hx}{2+\mu},$$

and the outside one

$$\psi_0 = \left(\frac{\mu-1}{\mu+2}\frac{a^3}{r^3} - 1\right) Hx.$$

The strength of the field inside the iron is

$$\frac{3H}{2+\mu},$$

and is considerably smaller than H if μ is very big. The intensity of magnetisation on the other hand is

$$I = \frac{3\mu'}{\mu+2} H,$$

and since $\mu = 1 + 4\pi\mu'$ this is

$$I = \frac{3\mu'}{3+4\pi\mu'} H = \frac{3}{4\pi + \frac{3}{\mu'}} H,$$

and if μ' is large this is practically

$$\frac{3}{4\pi} H,$$

so that as the susceptibility of the substance increases the intensity of the magnetisation induced in it by a given field approaches a limiting value beyond which it cannot go.

274. The induction in the iron is

$$B = \mu H = \frac{3\mu}{2+\mu} H,$$

but outside at a distance from the sphere

$$B = H.$$

Now consider what this means. Take a tube of magnetic induction whose cross-section at a distance from the sphere is ds. Now we know that along this tube

$$Bds$$

is constant so that the cross-section of this tube in the sphere is ds_i where

$$Hds = \mu H ds_i,$$

$$ds_i = \frac{ds}{\mu},$$

so that if μ is very big the cross-section of the tube in the iron is very small compared with that outside at a distance. This illustrates very vividly how

the tubes of magnetic induction are gathered up into the iron so that the magnetic flux in the field converges into and is concentrated by the sphere. This is of course precisely the important practical property. The function of soft iron in dynamos is to collect the magnetic flux, but not the force. The force in the iron is in fact very small if μ is big.

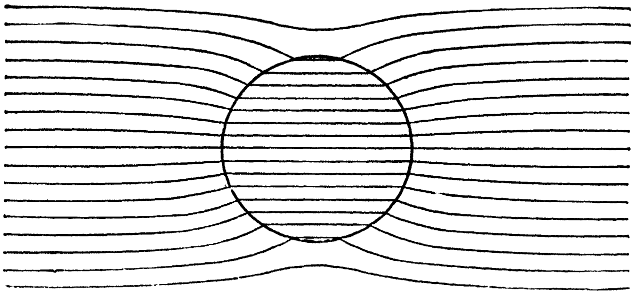


Fig. 57

Notice again that as μ (or μ') increases the magnetic induction also reaches the limit $3H$.

275. *The spherical shell of magnetic material in the uniform field*.* Let H be the strength of the field, then $-Hx$ is the part of the potential due to the given field and it is evident that the expressions for the potential in the three given regions must be of the form :

(i) inside the shell $\quad \psi_1 = Ax,$

(ii) in the material of shell

$$\psi_2 = Bx + \frac{Cx}{r^3},$$

(iii) outside $\quad \psi_3 = -Hx + \frac{Dx}{r^3}.$

Continuity of potential gives

$$Aa = Ba + \frac{C}{a^2} \quad \text{..........................(1)},$$

$$-Hb + \frac{D}{b^2} = Bb + \frac{C}{b^2} \quad \text{..........................(2)}.$$

Continuity of induction gives

$$\mu B - \frac{2\mu C}{a^3} = A \text{......................................(3)},$$

$$\mu B - \frac{2\mu C}{b^3} = -H - \frac{2D}{b^3} \text{..........................(4)},$$

from which we obtain

$$\frac{A}{3\mu} = \frac{B}{2\mu+1} = \frac{C}{a^3(\mu-1)} = \frac{D}{} = \frac{-3H}{(2+\mu)(2\mu+1) - \frac{2a^3}{b^3}(\mu-1)^2}.$$

* Poisson, *l.c.* p. 164.

The internal field has a potential

$$\frac{-9Hx}{(2+\mu)(2\mu+1)-\frac{2a^3}{b^3}(\mu-1)^2}.$$

This is the interesting result. The field inside the shell is reduced in the ratio

$$9:(2+\mu)(2\mu+1)-\frac{2a^3}{b^3}(\mu-1),$$

which is very small if μ is very big however thin the shell may be. This illustrates the problem of magnetic screening; in order to shield delicate instruments from influence by dynamos or other external actions they are enclosed in a shell of iron.

276. As a final example we may examine the induction in an ellipsoid* of magnetic matter placed in a uniform field of force with one of its axes parallel to the direction of the undisturbed field. The analysis is precisely similar to that given for the analogous case in dielectric media and need only be briefly indicated in this case.

The potential of the field in the undisturbed state is, except for a constant,

$$\psi=-Hx,$$

if the axes of coordinates are chosen along the principal axes of the ellipsoid with the origin at the centre.

We are then induced to try potentials

$$\psi_0=-Hx+Lx\int_\lambda^\infty\frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}}$$

in outside space and

$$\psi_i=-L'x$$

for the space inside the ellipsoid; and we must now try and find L, L' to satisfy the continuity conditions at the boundary.

Continuity of potential requires

$$-H=L'-L\int_0^\infty\frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

whilst continuity of induction requires

$$-H+L\int_0^\infty\frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}}-\frac{2L}{a^3bc}=\mu L'.$$

Whence using, as before

$$A=\tfrac{1}{2}a^3bc\int_0^\infty\frac{dt}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

* Neumann, *Jour. f. Math.* 37 (1848), p. 21; *Vorlesungen über die Theorie des Magnetismus* (Leipzig, 1881); Guilani, *Nuovo Cim.* (3), 11 (1882).

we find that
$$L = \frac{a^3bc}{2} \frac{(\mu - 1) H}{1 + A(\mu - 1)},$$
$$L' = \frac{-H}{1 + A(\mu - 1)},$$
so that the modified forms of the potential for the disturbed field are
$$\psi_0 = -Hx + \frac{a^3bc}{2} \frac{(\mu - 1) Hx}{1 + A(\mu - 1)} \int_\lambda^\infty \frac{dt}{(a^2 + t)\sqrt{(a^2 + t)(b^2 + t)(c^2 + t)}},$$
$$\psi_i = \frac{-Hx}{1 + A(\mu - 1)}.$$
The strength of the field inside the ellipsoid is
$$\frac{H}{1 + A(\mu - 1)},$$
and is as a rule considerably smaller than H, at least if the ellipsoid is made of iron for which μ is very big. The important new point is however that the strength of this field depends on A, and if A is very small, it is in fact identical with the applied field. This means that if the ellipsoid is very long in the direction of the field, as compared with its dimensions in the perpendicular directions, the field in its interior is practically the same as the applied field.

This fact is employed in the determination of the coefficient μ; for if we know the magnetisation induced in a piece of iron by a given field the ratio of the two determines the constant μ' if the internal field of the magnetisation is negligibly small, as would for instance be the case if the piece of metal is very extended in one direction. For this reason long iron wires placed parallel to the lines of the applied field are used in such determinations.

The question of the forces on the ellipsoid magnetised by induction is identical with the corresponding question for dielectric media and the results are analogous. We need not therefore stop to consider these questions further*.

277. *Neumann's theorem.* There is an important reciprocal relation established by Neumann which, although it really depends on principles to be subsequently established, is worth quoting at this stage on account of its importance in determining details of the polarisation induced in pieces of magnetic metal in any field. If in a given inducing magnetic field $\mathbf{H}$ a piece of iron has induced in it a polarity $\mathbf{I}$ and in another field $\mathbf{H}_0$ the polarity is $\mathbf{I}_0$ then, if the polarisation follows a linear law,
$$\int (\mathbf{H}_0\mathbf{I})\, dv = \int (\mathbf{H}\mathbf{I}_0)\, dv,$$
both integrals being taken throughout the mass of the iron.

* The problem of two spheres has been examined by C. Neumann, *Hydrodynamische Untersuchungen* (Leipzig, 1883), p. 282; R. A. Herman, *Quarterly Jour. for Math.* 22 (1887), p. 204; Boggio, *Rend. del. r. Inst. Lomb.* (2). 37 (1904), p. 405.

Suppose the total field when the iron is in position is in the first case $\mathbf{H} + \mathbf{H}'$ and in the second $\mathbf{H}_0 + \mathbf{H}_0'$.

Now the law of induction gives us that the components of $\mathbf{I}$ are conjugate linear functions of the components of $(\mathbf{H} + \mathbf{H}')$ however aeolotropic the medium may be. Similarly the components of $\mathbf{I}_0$ are linear functions of $(\mathbf{H}_0 + \mathbf{H}_0')$ with the same coefficients. It thus easily follows that

$$(\mathbf{I} \,.\, \mathbf{H}_0 + \mathbf{H}_0') = (\mathbf{I}_0 \,.\, \mathbf{H} + \mathbf{H}').$$

Thus the given result is true if

$$\int (\mathbf{I} \,.\, \mathbf{H}_0')\, dv = \int (\mathbf{I}_0 \,.\, \mathbf{H}')\, dv,$$

where of course $\mathbf{H}_0'$ is the field due to the distribution of magnetisation specified by $\mathbf{I}_0'$, and $\mathbf{H}'$ that due to the $\mathbf{I}'$: the left-hand side of this equation is therefore the work required to bring a body polarised to intensity $\mathbf{I}$ into its position in the field $\mathbf{H}_0'$, which is obviously equal to the work required to bring the body polarised to intensity $\mathbf{I}_0$ into the field $\mathbf{H}$. The result is therefore established.

An important application of this theorem is obtained when we know that one of the fields is uniform, $\mathbf{H}_0$ say, for then we have

$$\int (\mathbf{I}_0 \mathbf{H})\, dv = \left(\mathbf{H}_0 \,.\, \int \mathbf{I}\, dv\right).$$

If therefore we know the details of the distribution of polarisation when the body is in the uniform field (i.e. $\mathbf{I}_0$) then we can find

$$\int \mathbf{I}\, dv,$$

when the body is placed in any field.

We know for example the polarisation induced in an ellipsoid in a uniform field. We could therefore write down the total moment of the same ellipsoid in any field; it depends upon the volume integral of the inducing force.

This is a short sketch of the general theory of induced magnetism constructed from the view which regards it as a particular example of the general theory of polarised media. With the help of this theory we can investigate the fundamental questions concerning the energy and ponderomotive forces acting on such magnetic media. But before doing this a few words may be said as to the validity of the theory from the point of view of actual experience.

278. Paramagnetism, diamagnetism and ferromagnetism. The theory of induced magnetism given above is based on the linear relation between the magnetisation intensity $\mathbf{I}$ and the magnetic force $\mathbf{H}$ which in isotropic media is of the form

$$\mathbf{I} = \mu' \mathbf{H},$$

or in terms of the permeability μ

$$\mathbf{I} = \frac{\mu - 1}{4\pi} \mathbf{H}.$$

If $\mu > 1$ we see that the magnetisation has the same sign as the magnetising force but if $\mu < 1$ the signs are opposite. The important thing is now that there are bodies of both classes: there are in fact many quite ordinary substances for which $\mu < 1$ and in which therefore the direction of the magnetisation induced is in the opposite direction to the inducing field. Such substances are called *diamagnetic substances* to distinguish them from the more regular substances in which $\mu > 1$ and which are called *paramagnetic* substances.

In all diamagnetic substances the value of μ' is extraordinarily small: it is largest in bismuth where however it only amounts to $2{\cdot}5 \,.\, 10^{-6}$. In paramagnetic substances it is always equally small except in substances of the iron group. The above theory would therefore be completely effective for all substances except perhaps those of the latter group, which are called *ferromagnetic substances*. The typical examples of the ferromagnetic substances are iron, nickel and cobalt.

279. In the ferromagnetic substances, all of which are strongly magnetic, the simple relation between the magnetisation and magnetic force on which the above theory has been built is by no means valid except perhaps when the magnetic inducing force is very small in which case it is necessarily sufficient. In such cases it is found best to represent the relation by a graph which can be determined empirically for any particular mass of the substance under consideration. The type of curve which is always obtained is exhibited in the figure where the ordinates represent for a particular specimen of iron the values of **I**, the magnetisation induced, the abscissae the values of **H**, the magnetic force.

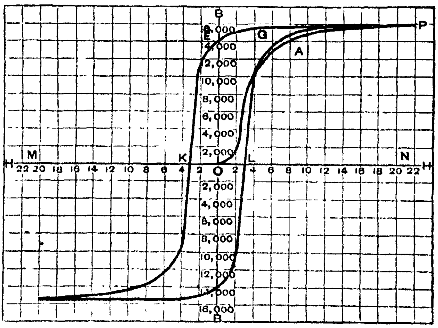


Fig. 58

For very small values of **H** the curve is straight indicating that the permeability, which is still defined by the relation

$$\mu = 1 + \frac{4\pi \mathbf{I}}{\mathbf{H}},$$

is independent of the magnetic force. When however the magnetic force increases beyond about ·02 absolute units the curve begins to rise rapidly and the value of μ is greater than it was before for small magnetic forces. The curve rises rapidly for some time when the force is about five absolute units : it then begins to get flatter and there are indications that for very great values of the magnetic force the curve becomes a straight line parallel to the axis of the magnetic force. This means that the intensity of magnetisation does not further increase as the magnetic force increases. When this is the case the iron or other magnetisable substance is said to be *saturated.* For a specimen of soft iron examined by Ewing* saturation was practically obtained when the magnetic force was about 2000 absolute units. For steel the magnetic force required for saturation is very much greater than for soft iron.

280. When a piece of iron or steel is magnetised in a strong magnetic field it will retain a considerable proportion of its magnetisation even after the applied field has been removed and the iron is no longer under the influence of any applied magnetic force. This power of remaining magnetic after the magnetic force has been removed, is called magnetic *retentiveness*; permanent magnets are a familiar instance of this property. This effect of the previous magnetic history of a substance on its behaviour when exposed to given magnetic conditions has been studied in great detail by Prof. Ewing, who has given to this property the name of *hysteresis*†. To illustrate this property let us consider the curve above, Fig. 58, which represents the relation for a sample of soft iron between the intensity of magnetisation (ordinates) and the magnetising force (the abscissa) when the magnetic force increases from zero up to ON, then diminishes from ON through zero to $-OM$, and then increases again to its original value. When the force is first applied we have the state represented by the portion OP of the curve, which begins by being straight, then increases more rapidly, bends round and finally reaches P, the point corresponding to the greatest magnetic force applied to the iron. If now the force is diminished it will be found that the magnetisation for a given force is greater than it was when the magnet was initially under the action of the same force, i.e. the magnet has retained some of its previous magnetisation, thus the curve PE, when the force is diminishing, will not correspond to the curve OP but will be above it. OE is the magnetisation retained by the magnet when free from magnetic force; in some cases this amounts to more than 90 per cent. of the greatest magnetisation attained by the magnet. When the magnetising force is reversed the magnet rapidly loses its magnetisation and the negative force represented by OK is sufficient to deprive it of all the magnetisation. When the negative magnetic force is increased beyond

* His results are given in detail in his book *Magnetic Induction in Iron and other metals.*

† The first real hysteretic effect was examined by Kelvin, *Phil. Trans.* 170 (1879), p. 68. Cf. also Warburg, *Wied. Ann.* 13 (1880), p. 141; Ewing, *Proc. R. S.* 34 (1882), p. 29.

this value, the magnetisation is negative. After the magnetic force is again reversed it requires a positive force equal to OL to deprive the iron of its negative magnetisation. When the force is again increased to its original value the relation between the force and induction is represented by the portion LGP of the curve. If after attaining this value the force is again diminished to $-ON$ and back again the corresponding curve is the curve PEK.

281. Various more or less successful attempts have been made to account for this special complexity in the magnetic properties of iron and steel. The matter is of the utmost importance from the technical point of view and has consequently taken a prominent place in theoretical discussions but it is only during the last few years that any substantial progress has been made on the theoretical side. It has long been known that paramagnetisation is mainly a constitutive phenomenon of the medium which exhibits it, and it must therefore be very considerably affected by any accidental character in the constitution of such medium. Now the structure of ordinary iron is known to be very complex and extremely irregular, so that its magnetic behaviour must be expected to be irregular to a corresponding extent. It thus appears to be necessary to examine some other substance with a more definite and controllable constitution and which exhibits the same paramagnetic phenomena, before any attempt at a theoretical discussion can be effectively made. The most suitable substances for this purpose are such minerals as pyrrhotite, hematite or magnetite, which can easily be obtained in regular crystalline form.

The crystals of pyrrhotite, which is a sulphide of iron, possess three mutually perpendicular planes of magnetic symmetry and are much more easily magnetised in a direction perpendicular to one of these planes than in any other direction. Moreover the induction phenomena for this direction are extraordinarily simple*. If the crystal is originally unmagnetised it remains so until the magnetic force H reaches a definite critical value H_s, when the intensity of magnetisation suddenly assumes its saturation value I_s, which it retains perfectly constant until the magnetic force reaches the value $-H_c$. At this point the magnetisation is immediately reversed to the value $-I_s$, a value which is retained constant until the magnetic force again passes through the value $+H_s$ when it is reversed. The hysteresis curve showing the relation between I and H is thus a simple rectangle of area $4H_sI_s$ and the phenomenon is irreversible.

282. An obvious explanation of these phenomena at once suggests itself. If we assume that the elements of the crystal are permanently magnetised to a definite intensity, it would appear that the two positions of an element in which its magnetic axis is parallel to the main magnetic axes of the crystal

* They were discovered by Weiss and are fully described in various papers in *Jour. de Phys.* for 1905.

are the only ones which are stable under the action of the internal constitutive forces; and further that either of these positions ceases to be a position of stable equilibrium as soon as the opposing magnetic force reaches its critical value. The equilibrium in either case is largely maintained by kinetic action, so that any configuration would be immediately destroyed as soon as it ceases to be a stable one.

This explanation is still further supported by the fact that the magnetic behaviour of the substance in any direction other than along its principal axis is perfectly continuous and in complete accord with the theoretical consequences of such a view. The fact that the magnetisation parallel to the principal axis is much greater than that in any other direction suggests that the forces holding the elements in position are extremely large.

Broadly speaking the other ferromagnetic crystalline minerals exhibit the same general features as pyrrhotite; there are important differences in detail but they all possess different magnetic properties along the different axes of symmetry and usually one axis of conspicuously easy magnetisation.

283. Whatever the ultimate explanation of these properties of the ferromagnetic crystals may be it is probable that they furnish an indication of the direction in which we must look for an explanation of the behaviour of the ferromagnetic metals. We can in fact fully account in a general way for the more complex behaviour of such substances as iron on the assumption that it is constituted of an irregular conglomeration of small crystals of the simpler type. It is well known that an ordinary piece of iron is a complex aggregate of minute crystals and when impurities are present, as is generally the case, the crystals may vary considerably in size and composition, so that no unnatural assumption is thereby involved. Moreover when the constitution of the iron is known to be regular, as for instance when it is deposited electrolytically under the action of a strong magnetic field, its magnetic behaviour is just like that for pyrrhotite, the hysteresis curve being rectangular.

284. The fact that the ferromagnetic quantity is essentially a constitutional one is best illustrated by the behaviour of iron in particular, which is the more important practical substance. At ordinary temperatures the simple relation between polarisation and polarising force assumed in the analytical developments is by no means true for the case of iron in a magnetic field. At least the factor μ' in the relation

$$I = \mu' H$$

is a function of H. There are also in addition very considerable hysteretic losses of energy, as our analysis would imply in any failing case. The reason of all this is that the polarisation induced in the iron at ordinary temperatures is a property of molecular groups and not of the single molecules. The experimental test of this fact is obtained by breaking up the groups by heating

the metal. It has been long known* that iron at high temperatures is only slightly magnetisable; but the important discovery was made by Hopkinson†. There is another property of iron which goes by the name of 'recalescence.' If we heat a piece of iron to a bright heat and then let it cool gradually, it first gradually darkens by loss of heat but then at a definite temperature (about 800° C.) it suddenly brightens up again. This means that at this temperature there is a sudden release of internal energy in the form of heat. Hopkinson showed that the temperature at which this occurs is just the temperature at which the magnetic property changes and the metal ceases to be ferromagnetic. The release of internal energy must mean a change in the state of aggregation of the molecules and this is also indicated, on the present theory, by the change in the magnetic properties. A striking and probably just analogy is drawn by Curie between the simple law of magnetisation of a substance like iron at high temperatures and this sudden change which it undergoes when the temperature is lowered beyond the recalescence point and the simple law of expansion of a gaseous substance at high temperature and the sudden change which it undergoes on lowering the temperature beyond a critical point so that the mutual attractions of the molecules come into play and produce the liquid state.

There is another but less marked critical temperature (1280° C.) on both sides of which the substance behaves as an ordinary paramagnetic body, but in the passage through this temperature the susceptibility suddenly changes. This of course indicates that a further alteration of the molecular configuration takes place at this temperature.

The occurrence of these properties is by no means confined to iron. Similar features are presented by all the other known ferromagnetic substances and also by some diamagnetic substances. In the case of tin a number of transition points have been observed and the element is sometimes diamagnetic and sometimes paramagnetic according to its temperature.

285. The energy and mechanical relations of induced magnetism. We can now turn to an examination of the fundamental mechanical relations of induced magnetism. The analysis is however to a certain extent analogous to the similar problem connected with polarised dielectric media discussed in the previous chapter. We need therefore only to give a short resumé of the results as far as they are concerned with the present case, interpreting them of course in terms of the vectors of the present theory.

The energy required to establish the polarisation $\mathbf{I}$ in the element dv regarded as the mathematical equivalent of the work done in separating the poles of each small doublet is

$$(\mathbf{IH})\, dv,$$

* Barrett, *Phil. Mag.* 44 (1873), p. 472.

† *Phil. Trans.* A, 180 (1890), p. 443.

and this represents the total work required by the element as a whole on account of the polarisation in it. It consists essentially of two distinct parts of fundamentally different origins and we try and separate them as in the previous chapter.

Before doing this however we must recapitulate some remarks made in the last chapter respecting the energy of polarised media in general. It was found that the general equation of virtual work for a polarised medium existing in any field of force (now magnetic) would be of the general form

$$\delta T_i = \delta W_m + \delta W - \delta W_i - \delta W_i',$$

where δT_i represents the variation in the thermal energy of the irregular heat motion of the molecules; δW_m is the virtual work of the magnetic forces of attraction due to the field; δW is the virtual work of the applied mechanical forces; δW_i is the increase of the store of ordinary elastic energy and $\delta W_i'$ that of the intramolecular energy of quasi-elastic strain and motion. However in the most important cases which come under review in the theory of magnetism it is not possible to draw a close distinction between thermal and magnetic processes so that this equation cannot now be simplified as in the previous application in the last chapter. We can however write it in the convenient form

$$-\delta W_m = \delta W - \delta E_i,$$

wherein δE_i represents the variation of the total internal energy of the matter, both elastic and motional, molecular and intramolecular.

286. Pass* the magnetic body through a cycle by moving it around a path in a permanent magnetic field **H**. An infinitesimal displacement of the volume δv from a place where the field is **H**, to one where it is $(\mathbf{H} + \delta\mathbf{H})$ does mechanical work, arising from the magnetic attractions, of amount

$$(\mathbf{I}\delta\mathbf{H})\,\delta v.$$

The integral of this throughout the whole connected system gives the virtual work for that displacement, from which the forces assisting it are derived as usual. Confining attention to the element δv the work supplied by it from the field, to outside mechanical systems which it drives, in traversing any path is thus

$$\delta v\,(\mathbf{I}\delta\mathbf{H}),$$

the integral being taken along the path. If **I** is a function of **H**, that is if the magnetism is in part thoroughly permanent and in part induced without hysteresis, so that the operation is reversible, this work must vanish for a complete cycle; otherwise energy would inevitably be created either in the direct path or else in the reversed one of the complete system of which δv is a part. Thus the negation of perpetual motion in that case demands that

$$(\mathbf{I}\delta\mathbf{H}) = d\phi,$$

* The treatment here followed is given by Larmor implicitly in *Phil. Trans.* A, 190 (1897) and *in extenso* in *Proc. R. S.* 71 (1903), pp. 236–239.

$d\phi$ being the complete differential of a function ϕ of $(\mathbf{H}_x, \mathbf{H}_y, \mathbf{H}_z)$ involving only even powers. If the polarisation follows a linear law, as it must do if the field is small, ϕ is quadratic and in the simple case of isotropic media is simply

$$\phi = \tfrac{1}{2}\mu'\mathbf{H}^2$$

so that

$$\mathbf{I} = \mu'\mathbf{H},$$

as we assumed before.

287. But if there is hysteresis, so that the cycle is not reversible

$$-\delta v \int (\mathbf{I}\delta\mathbf{H})$$

represents negative mechanical work done, or energy degraded in the cycle But this is

$$-\delta v \int (\mathbf{I}\delta\mathbf{H}) = -\Big|(\mathbf{HI})\Big|\delta v + \delta v \int (\mathbf{H}\delta\mathbf{I}),$$

and in the average of a large number of cycles $|(\mathbf{IH})|\,\delta v = 0$ so that the energy lost is also measured by

$$\delta v \int (\mathbf{H}\delta\mathbf{I}).$$

It is also equal to

$$-\frac{\delta v}{4\pi}\int (\mathbf{B} - \mathbf{H}\,.\,\delta\mathbf{H}) = -\frac{\delta v}{4\pi}\int (\mathbf{B}\delta\mathbf{H}) + \frac{\delta v}{4\pi}\int (\mathbf{H}\delta\mathbf{H}),$$

and the latter part vanishes in a cycle so that the loss of energy is equal to

$$-\frac{\delta v}{4\pi}\int (\mathbf{B}\delta\mathbf{H}),$$

taken throughout the cycle. This is the practical method of obtaining the hysteresis loss of energy, the expression obtained being that of Warburg† and Ewing‡ for the magneto-hysteretic waste of mechanical energy in driving electric machines.

288. In addition to this energy concerned with the attractions, the external field expends energy in polarising or orientating the individual molecules against the internal forces of the medium, of aggregate amount in the whole cycle

$$\delta v \int (\mathbf{H}\delta\mathbf{I}).$$

This part has nothing to do with the mechanical forces. It is stored up as internal energy of a purely elastic or thermal character. If the polarisation

* It is important to notice that this expression represents the area of the hysteresis loop on the magnetisation curve given in § 278.

† *Ann. Phys. Chem.* (3), 13 (1881), p. 140. Cf. also *Rapports Congrès internat. de phys.* 2 (Paris, 1900), p. 509, and *Phys. Zeitschr.* 2 (1901), p. 367.

‡ *Proc. R. S.* 48 (1890), p. 342.

gradually breaks away some or all of this is lost in heat and the phenomenon of hysteresis results.

In any case, whatever the hysteresis, the sum of this second part and the first reversed is integrable independently of the path, giving

$$\delta v \,|\, (\mathbf{HI}) \,|,$$

namely, the change in the total energy in the element, thus vanishing for a complete cycle which restores things to their original state, as it ought to do. The former part of this total represents the average waste of direct mechanical energy in moving the magnetic substance through the cycle and accounts for the heat thus evolved which is represented by the second part.

When the relation between **I** and **H** is linear and isotropic the two parts of the total energy, the mechanical and molecular parts are both equal

$$\delta v \int (\mathbf{H}\delta\mathbf{I}) = \delta v \int \mu' \frac{\delta\,(\mathbf{H}^2)}{2} = \delta v \int \delta \left(\frac{\mu'\mathbf{H}^2}{2}\right)$$

$$= \delta v \int (\mathbf{I}\delta\mathbf{H}),$$

and this is the usual theoretical result. If there is no hysteretic loss of any kind we may take

$$\int \frac{\mu'\mathbf{H}^2}{2}\,dv,$$

where the integral is extended throughout the substance, as the potential energy of the mechanical forces acting on the medium.

289. Now let us apply these results in a particular case. Suppose the magnetic field arises from a distribution of rigid magnetic polarity of density $\mathbf{I}_0$ at any point in the field. The total energy in the field can then be calculated as the mechanical work done in building the rigid magnetism up gradually in the presence of the magnetisable substances, the induced magnetism simultaneously takes the appropriate value at any stage of this process. Suppose that at any instant the force intensity in the field is **H**, then the work done in bringing up an additional small increment of polarity $\delta\mathbf{I}_0$ to each place in the field is clearly equal to

$$-\int (\mathbf{H}\,.\,\delta\mathbf{I}_0)\,dv$$

integrated throughout the field.

290. If now we assume that there are no sudden discontinuities in the magnetic distribution in the field (and any such might be treated as a continuous rapid transition) then we can prove that the integral

$$\int (\mathbf{BH})\,dv,$$

when extended throughout the whole field is zero. If ψ is the magnetic potential it is in fact equal to

$$-\int (\mathbf{B}\nabla)\,\psi dv,$$

and by Green's lemma this transforms to

$$\int \psi \operatorname{div} \mathbf{B}\, dv - \int \psi \mathbf{B}_n df,$$

the latter integral being taken over an indefinitely extended surface bounding the field. When the field is regular at infinity this latter integral vanishes and, since

$$\operatorname{div} \mathbf{B} = 0$$

everywhere, the former does also. Thus we have

$$\int (\mathbf{HB})\, dv = 0.$$

In a similar manner it is verified in particular that

$$\int (\mathbf{B}\delta\mathbf{H})\, dv = \int (\mathbf{H}\delta\mathbf{B})\, dv = 0,$$

if $\delta\mathbf{B}$ denotes the small variation in the value of $\mathbf{B}$ at any place consequent on the above defined small variation on the distribution of magnetic polarity, since

$$\operatorname{div} \delta\mathbf{B} = \operatorname{div} (\mathbf{B} + \delta\mathbf{B}) - \operatorname{div} \mathbf{B} = 0.$$

291. Now

$$\mathbf{B} = \mathbf{H} + 4\pi (\mathbf{I}_0 + \mathbf{I}),$$

where $\mathbf{I}$ denotes the intensity of the induced magnetic polarity corresponding to the field of $\mathbf{I}_0$; thus

$$\delta\mathbf{B} = \delta\mathbf{H} + 4\pi (\delta\mathbf{I}_0 + \delta\mathbf{I}),$$

so that

$$-(\mathbf{H}\delta\mathbf{I}_0) = (\mathbf{H}\delta\mathbf{I}) + \frac{1}{4\pi}\{(\mathbf{H}\delta\mathbf{H}) - (\mathbf{H}\delta\mathbf{B})\}$$

$$= (\mathbf{H}\delta\mathbf{I}) + \frac{1}{4\pi}(\mathbf{B}\delta\mathbf{H}) - \frac{1}{4\pi}(\mathbf{B}\delta\mathbf{B}) + 4\pi(\mathbf{I} + \mathbf{I}_0,\ \delta\mathbf{I} + \delta\mathbf{I}_0).$$

Thus the work done in increasing the rigid polarity by $\delta\mathbf{I}_0$ may be reckoned as

$$\delta W = -\frac{1}{8\pi}\int \delta\{\mathbf{B}^2 - 16\pi^2(\mathbf{I} + \mathbf{I}_0)^2\}\, dv + \int (\mathbf{H}\delta\mathbf{I})\, dv.$$

The total work done in establishing the magnetic field is therefore

$$\int_0 \delta W = -\frac{1}{8\pi}\int \{\mathbf{B}^2 - 16\pi^2(\mathbf{I} + \mathbf{I}_0)^2\}\, dv + \int (\mathbf{H}\delta\mathbf{I})\, dv.$$

The second part of this total energy represents the internal elastic energy stored up in the magnetic media on account of the magnetic polarity induced in them. The first part therefore represents the true magnetic potential

energy of the field and on a tentative theory we could regard it as distributed throughout the field with a density

$$-\frac{1}{8\pi}\{\mathbf{B}^2 - 16\pi^2(\mathbf{I}+\mathbf{I}_0)^2\}$$

at any place.

292. In the case of a linear law of induction and when the medium is isotropic we have

$$\begin{aligned}\mathbf{B} &= \mathbf{H}(1+4\pi\mu') + 4\pi\mathbf{I}_0\\ &= \mu\mathbf{H} + 4\pi\mathbf{I}_0,\end{aligned}$$

so that

$$\mathbf{H} = \frac{\mathbf{B} - 4\pi\mathbf{I}_0}{\mu},$$

and

$$\mathbf{I} = \mu'\mathbf{H} = \frac{\mu-1}{4\pi}\left\{\frac{\mathbf{B}-4\pi\mathbf{I}_0}{\mu}\right\}.$$

Thus the energy stored in the magnetic media on account of the polarisations induced in them appears as the integral

$$\int\frac{\mu-1}{8\pi\mu^2}\{\mathbf{B}-4\pi\mathbf{I}_0\}^2\,dv.$$

In this case also we have

$$\mathbf{I}+\mathbf{I}_0 = \frac{1}{4\pi\mu}\{\overline{\mu-1}\mathbf{B} + 4\pi\mathbf{I}_0\},$$

so that the total energy of the aethereal field is the integral of

$$\begin{aligned}&-\frac{1}{8\pi}\mathbf{B}^2 + \frac{1}{8\pi\mu^2}\{\overline{\mu-1}\mathbf{B}+4\pi\mathbf{I}_0\}^2\\ &= -\frac{\mathbf{B}^2}{8\pi\mu} - \frac{\mu-1}{8\pi\mu^2}\mathbf{B}^2 + \frac{\mu-1}{\mu^2}\mathbf{B}\mathbf{I}_0 + \frac{2\pi\mathbf{I}_0^2}{\mu^2}\\ &= -\frac{\mathbf{B}^2-16\pi^2\mathbf{I}_0^2}{8\pi\mu} - \frac{\mu-1}{8\pi\mu^2}\{\mathbf{B}-4\pi\mathbf{I}_0^2\}.\end{aligned}$$

The total energy associated with the magnetic field is therefore

$$-\frac{1}{8\pi}\int\frac{\mathbf{B}^2-16\pi^2\mathbf{I}_0^2}{\mu}\,dv.$$

The second or purely local part of this energy depending solely on $\mathbf{I}_0$ would in most cases remain constant; but in any case it would be foreign to a mechanical theory concerned with the system in bulk. Thus the mechanically effective part of the energy associated with the system is

$$-\frac{1}{8\pi}\int\frac{\mathbf{B}^2}{\mu}\,dv,$$

and it can therefore be regarded as distributed throughout the field with the density

$$\mathbf{B}^2$$

At points of the field where there is no rigid magnetism this is the same as

$$-\frac{\mu\mathbf{H}^2}{8\pi},$$

which is the expression usually given in the text books, but with the opposite sign. The discrepancy of sign is due to inconsistencies in the usual formulation of the subject, which gives a density for the static potential energy differing in the general case from the density deduced above by the amount

$$-\frac{1}{4\pi}(\mathbf{BH}),$$

giving a zero total on the whole. The formulation adopted above is the only one that appears to be consistent with the subsequent developments of the theory in its dynamical aspects.

293. Of the true magnetic potential energy in the field

$$-\frac{1}{8\pi}\int \{\mathbf{B}^2 - 16\pi^2(\mathbf{I}+\mathbf{I}_0)^2\}\, dv,$$

a part

$$-\int\int_0^H (\mathbf{I}\delta\mathbf{H})\, dv$$

corresponds to the polarisation induced in the magnetic media: it is concerned mainly with the magnetic attractions, or their equivalent mechanical forces, exerted by the field on the polarised media as a whole, of which it is the potential function. The remainder

$$\frac{1}{8\pi}\int \{\mathbf{B}^2 - 16\pi^2(\mathbf{I}+\mathbf{I}_0)^2\}\, dv + \int dv \int_0^H (\mathbf{I}\delta\mathbf{H})$$

corresponds to the rigid polarisations $\mathbf{I}_0$ and is a potential function of the mechanical reactions on the permanent magnets giving rise to the field. Since

$$\mathbf{B} = \mathbf{H} + 4\pi(\mathbf{I}+\mathbf{I}_0)$$

this part may be written in the form

$$\frac{1}{4\pi}\int dv \int_0^B (\mathbf{H}\delta\mathbf{B}) - \int dv \int_0^H (\mathbf{I}_0\delta\mathbf{H}),$$

which, since in every stage of the process

$$\int (\mathbf{H}\delta\mathbf{B})\, dv = 0,$$

reduces to

$$-\int dv \int_0^H (\mathbf{I}_0\delta\mathbf{H}),$$

in agreement with a former result when there are no magnetic substances about.

294. The forces acting on a magnetically polarised medium can now be obtained either from the energy expressions or by an analysis similar to that given in the previous chapter. Excluding as there the part arising wholly

from the interaction of neighbouring molecules, which is not transmitted by material stress, but is compensated on the spot by molecular action due to change of physical state induced by it, the magnetic force proper is made up of a bodily force F and torque G where

$$\mathbf{F} = (\mathbf{I}\nabla)\,\mathbf{H},$$

and*
$$\mathbf{G} = [\mathbf{IH}].$$

Under the usual circumstances these expressions are identical with the ones given by Maxwell in his treatise. The remarkable property is there established, and is the direct analogue of our result for polarised media, that independently of the form of the relation between the magnetic induction and magnetic force in the medium and whether there is permanent magnetism or not, this bodily forcive can be formally represented in explicit terms as equivalent to an imposed stress: viz. it is the same as would arise from (i) a hydrostatic pressure $\frac{H^2}{8\pi}$, (ii) a tension along the bisector of the angle θ between H and B, equal to $HB\cos^2\theta/4\pi$, (iii) a pressure along the bisector of the supplementary angle, equal to $BH\sin^2\theta/4\pi$, together with an outstanding bodily torque turning from $\mathbf{B}$ towards $\mathbf{H}$ equal to $HB\sin 2\theta/4\pi$. When $\mathbf{B}$ and $\mathbf{H}$ are in the same direction, the torque vanishes, and a pure stress remains in the form of a tension $(BH - \frac{1}{2}H^2)/4\pi$ along the lines of force and a pressure $H^2/8\pi$ in all directions at right angles to them. There is of course no warrant for taking this stress to be other than a mere geometrical representation of the bodily forcive. It is however a convenient one for some purposes. Thus the traction acting on the layer of transition between two media, in which $\mathbf{H}$ changes very rapidly, which might be directly deduced in the same manner as the electric traction above may also be expressed directly as the resultant of these Maxwellian tractions towards the two sides of the interface. As there cannot be free magnetic surface density the traction on the interface is represented, under the most general circumstances, whatever extraneous magnetic field may there exist, by purely normal pull of intensity $2\pi\mathbf{I}_n{}^2$ towards each side.

295. On the thermal relations of the energy of magnetisation. We have briefly explained in a previous paragraph the difference between paramagnetic and diamagnetic substances, but we did not stop to think what the essential difference between these two kinds of magnetisation might be. How is it that both para- and diamagnetism exist in nature? The answer must

* A general form of the theory of magnetic stress based on the method of energy and analogous to Helmholtz's theory for dielectrics has been given by Cohn (*Das electromagnetische Feld*, p. 510) and further elaborated by Gans (*Ann. der Phys.* 13 (1904), p. 634, and *Encyclopädie der math. Wissensch.* v. 15), Koláček (*Ann der Phys.* 13 (1904), p. 1). Sano (*Phys. Zeits.* 3 (1902), p. 401). This theory is however open to the same criticism as levelled by Larmor at Helmholtz's procedure.

obviously be that there is an essential difference in the origin of the two kinds and this assertion is fully confirmed by experiment. As a result of an extensive investigation of the magnetic properties of matter the law has recently been formulated by Curie* that in all feebly paramagnetic substances, including gases, the coefficient of magnetisation varies inversely as the absolute temperature, with a degree of accuracy which tends to perfection at high temperatures: that in strongly magnetic substances such as iron and nickel the same law is ultimately reached when the temperature is high: while in diamagnetic substances the coefficient is usually nearly independent of temperature and also of changes in the chemical state of the material. The inference is made by Curie that this points to diamagnetism being an affair of the internal constitution of the molecule, having only slight relation to the bodily motions of the molecules on which temperature depends, in accordance with the first views of polarisation of a medium. On the other hand, paramagnetism is an affair of orientation of the molecules in space without change of internal conformation, so that alteration of the mean state of translational motion is involved in it and we should expect a temperature effect: this is the idea underlying the Weberian theory of magnetisation (the second alternative hypothesis mentioned to explain the cause of polarisation). This relation between paramagnetisation and temperature proves to be so simple that it must be the expression of a theoretical principle. The following considerations in fact derive it from Carnot's principle: the argument is precise so long as the induced magnetisation is so slight that the exciting magnetic force on the separate molecules is practically that of the inducing field, but it loses exactness as soon as, owing to the diminution of energy of agitation with falling temperature, the molecules begin to exercise sensible magnetic control over each other, and thus introduce the phenomena of grouping and consequent hysteresis that are associated with the ferromagnetic state.

296. Consider† a mass of paramagnetic material, moved up from a place where the intensity of the field is $\mathbf{H}$ to a place where it is $\mathbf{H} + \delta\mathbf{H}$. The aggregate per unit volume of the total magnetic energies of its molecules is thereby altered from $-(\mathbf{IH})$ to $-(\mathbf{I} + \delta\mathbf{I}, \mathbf{H} + \delta\mathbf{H})$. The mechanical work done by the mass in virtue of its attraction by the field is $(\mathbf{I}\delta\mathbf{H})$. Thus there remains a loss in the total magnetic energy of the molecules, equal to $(\mathbf{H}\delta\mathbf{I})$; this can only have passed into heat in the material; for we can work on the hypothesis that the field of force $\mathbf{H}$ is due to an absolutely permanent

* *Ann. de Chimie* (1895). A considerable number of exceptions to the law have been found particularly at low temperatures but these are attributed to changes in the molecular configuration of the material. Cf. Kammerlingh Onnes and Perrier, *Konink. Akad. Wetensch. Amsterdam Proc.* XIV. p. 115 (1911).

† The argument here is due originally to Larmor, *Phil. Trans.* A, 190 (1897), §§ 71 and 72. Cf. also *Proc. R. S.* 71 (1903), p. 235.

magnetic system, so that no energy is used up in producing magnetic dis placements in the inducing magnets. Now let us apply Carnot's principle to a reversible cycle in which the material is moved up in the field at temperature $\theta + \delta\theta$ and moved back at temperature θ, with adiabatic transition between these temperatures. Let $h + dh$ be the thermal energy per unit volume which it must receive from without at the higher temperature, and h that which it must return at the lower, in order to perform the amount of work δW, equal to

$$\frac{\partial \mathbf{I}}{\partial \theta} \delta \mathbf{H} \delta \theta$$

in the cycle; then by Carnot's principle

$$\frac{\delta W}{\delta \theta} = \frac{h}{\theta},$$

but
$$h = -(\mathbf{H} \delta \mathbf{I})$$

as above so that
$$\frac{\partial \mathbf{I}}{\partial \theta} \delta \mathbf{H} = -\frac{\mathbf{H} \delta \mathbf{I}}{\theta},$$

or
$$\frac{\partial \mathbf{I}}{\partial \theta} = -\frac{\mathbf{H}}{\theta} \frac{\partial \mathbf{I}}{\partial \mathbf{H}},$$

so that
$$\mathbf{I} = f\left(\frac{\mathbf{H}}{\theta}\right),$$

f being some arbitrary function. When the magnetising force $\mathbf{H}$ is very small or the temperature θ very large this relation is approximately equivalent to

$$\mathbf{I} = \frac{\kappa}{\theta} \mathbf{H},$$

so that
$$\mu' = \frac{\kappa}{\theta},$$

which is Curie's law.

297. Conversely, assuming Curie's law we can deduce that in paramagnetic bodies magnetisation consists in orientation of the molecules without sensible change in their internal energies. In an analytical form the argument is as follows:

$$dh = (\mathbf{M} d\mathbf{I}) + N d\theta,$$

and
$$dE = dh + \frac{1}{\mu'} (\mathbf{I} d\mathbf{I}),$$

whence by the thermodynamic formula

$$\frac{\mathbf{M}}{\theta} = -\frac{d}{d\theta} (\mu'^{-1} \mathbf{I}),$$

so that
$$\frac{\mathbf{M}}{\mathbf{I}} = -\theta \frac{d}{d\theta} \left(\frac{1}{\mu'}\right) = -\frac{1}{\mu'}$$

by Curie's law; hence
$$dh = -(\mathbf{H} d\mathbf{I}) + N d\theta,$$

so that at constant temperature

$$-h = \int(\mathbf{H}\delta\mathbf{I}),$$

that is the heat that the material develops during magnetisation is the equivalent of the magnetic energy that is not used up in mechanical work. This is precisely what we should expect if the material is a gas: for there is then no internal work by which this energy could be used up, and the magnetisation arises from the effort of the magnetic field to orientate the molecules which are spinning about as the result of gaseous encounters. The law of Curie thus indicates that the same is sensibly true for all paramagnetic media at high temperatures: at lower temperatures they generally pass into the ferromagnetic condition. It is the magnetisation so to speak of an ideal perfect ferromagnet in which the controlling force that resists the orientating action of the field is practically wholly derived from the magnetic interaction of the neighbouring molecules, which for this purpose form elastic systems. In ordinary paramagnetic substances this mutual magnetic control of the molecules is insensible compared with the control due to other molecular causes, and our conclusion is that these causes are such that the magnetic energy expended in working against them is transformed into heat energy, not into internal energy of any regular elastic type.

298. The argument for the simple case of a gaseous medium exhibiting paramagnetic properties has been further extended by Langevin, who has succeeded in calculating, on the basis of the kinetic theory, the actual form of the functional relation between the intensity of magnetisation and magnetising force for this ideal case. The problem considered is that in which a simple gas, such as oxygen, the molecules of which are regarded as rigidly magnetised to moment m, exists in a uniform field of magnetic force of intensity H. In this case the organising potential energy of the molecules, which is of a purely magnetic nature, the mutual interaction of the molecules themselves being neglected, is balanced entirely by the disorganising effect of the kinetic energy of thermal agitation. Now the potential energy of a magnetic doublet m in a field of magnetic force of intensity H is

$$-mH\cos\chi,$$

if χ is the angle between the direction of the force H and the axis of the doublet. It follows then by the usual considerations of the kinetic theory that the number dn of molecules per unit volume whose magnetic axes make with the direction of the field an angle lying between χ and $\chi + d\chi$ is

$$dn = 2\pi A e^{\frac{mH\cos\chi}{R\theta}} \sin\chi\, d\chi,$$

wherein R is the usual constant of the kinetic theory, θ is the absolute

temperature and A is a constant determined by the fact that the total number of molecules per unit volume is

$$n = 2\pi A \int_0^\pi e^{\frac{mH\cos\chi}{R\theta}} \sin\chi\, d\chi$$

$$= \frac{4\pi AR\theta}{mH} \sinh\left(\frac{mH}{R\theta}\right).$$

Thus $$A = \frac{mH}{4\pi R\theta} \operatorname{cosech}\left(\frac{mH}{R\theta}\right).$$

In this case the resultant intensity of magnetisation I is in the direction of H, by symmetry, and is actually given by

$$I = \int m\cos\chi\, dn$$

$$= 2\pi Am \int_0^\pi e^{\frac{mH\cos\chi}{R\theta}} \cos\chi \sin\chi\, d\chi$$

$$= 4\pi Am \left[\frac{\cosh\left(\frac{mH}{R\theta}\right)}{\frac{mH}{R\theta}} - \frac{\sinh\left(\frac{mH}{R\theta}\right)}{\left(\frac{mH}{R\theta}\right)^2}\right].$$

Substituting the value of A from above we find that

$$I = nm\left[\coth\left(\frac{mH}{R\theta}\right) - \frac{R\theta}{mH}\right].$$

Since m is determined solely by the structure of the molecules we see that for a given density of the gas the intensity I of the magnetisation induced is a function of H/θ, in accordance with the conclusion drawn from the thermodynamic reasoning Moreover for small values of the inducing force the functional relation becomes a mere proportionality or

$$I = \frac{nm^2 H}{2R\theta},$$

so that the susceptibility is

$$\mu' = \frac{nm^2}{2R\theta},$$

and varies inversely as the absolute temperature in accordance with Curie's law.

299. This theory can be applied to the calculation of the moment m of the molecular magnet in the case of those substances easily obtainable in the simple gaseous state. It has been extended by Weiss so as to apply to a substance in dilute solution in a feebly magnetic liquid. In this way determinations of m can be made for a large number of substances, and the results obtained prove to be related to one another in a most remarkable manner. It is in fact found that the average magnetic moment of any

molecule is always equal, within the very narrow limits of experimental error, to a certain small *integral* multiple of a certain universal constant. The constant integers for the large number of substances examined range from 4 to 56 and extremely few cases are found which cannot be so expressed with great accuracy.

To explain this regularity Weiss was led to the very natural hypothesis that the paramagnetic properties of substances arise from the presence of an ultimate unit, the so called *magneton*, in the atom of the substance. It is apparently necessary that these elements should be capable of annihilating each other temporarily, as the same substance may contain different numbers of magnetons at different temperatures.

It has not yet been found possible to justify this assumption of an ultimate magnetic unit, and we shall not therefore make any essential use of it in our future discussions. The importance of the idea however seems to warrant the brief reference just given to it.

300. The theory of ferromagnetism. It has not yet been found possible to test directly the general relation deduced by Langevin connecting the magnetisation induced in a gas and the strength of the inducing field. The theoretical formula has however been applied with great boldness by Weiss* to build up a theory of the behaviour of ferromagnetic substances in general, and the success which has attended his investigations removes a good deal of doubt as to the justification of his ideas.

According to Weiss the circumstances in ferromagnetic substances such as the crystals described above differ from those in the simple case to which we have just applied the kinetic theory, only in so far as the effects considered are modified by the mutual magnetic influence of the molecules of the substance. In the case of a gas the mutual interaction of the neighbouring molecules can always be neglected, but when the molecules are packed so close together as they are in the case of a solid body, and when the intensity of magnetisation becomes so large as is the case with iron, this local interaction may greatly preponderate. It is of course quite impossible for us to know very much about the local magnetic field† of force surrounding any molecule, even in such ideal substances as the simplest crystals appear to be; but we do know that under the most favourable circumstances it will vary rapidly in distances quite inaccessible to our perceptions. A good deal of suggestive information can however be obtained by taking, after Weiss, a physically average view of the matter and assuming this local field at an average estimate smoothed out from the highly irregular field that actually exists. This

* *Jour. de Phys.* VI. p. 661 (1907).

† It is interesting to notice that neither the theory nor the facts imply that this local field is magnetic in nature, although it may be convenient so to regard it. Cf. Eucken. *Die Theorie der Strahlung und der Quanten* (Halle, 1914), p. 321 (Bericht Langevin).

average local field arises entirely in the magnetisation induced in the medium and its intensity will, on the simplest hypothesis, be proportional to the intensity of this magnetisation, the constant of proportionality being however related to direction in the medium, if this latter possesses aeolotropic characteristics. We shall consider the case of isotropic media and denote the local field strength by aI.

301. Weiss now formulates the theory of ferromagnetism exactly on the lines suggested by Langevin but with the force acting on the molecular magnets now equal to

$$H + aI.$$

The value of I for a given value of H is then to be obtained from the formula

$$I = mn\left[\coth\left\{\frac{m}{R\theta}(H + aI)\right\} - \frac{R\theta}{m(H + aI)}\right].$$

The maximum value of the magnetisation or its saturation value is

$$I_s = nm.$$

If we introduce this value in the above relation between I and H it becomes

$$I = I_s\left[\coth\left\{\frac{I_s(H + aI)}{nR\theta}\right\} - \frac{nR\theta}{I_s(H + aI)}\right].$$

This is the relation on which Weiss builds his theory.

In the first case the phenomenon of permanent magnetisation is fully accounted for. In this case $H = 0$ so that the magnetisation is given by

$$I = I_s\left[\coth\left(\frac{aII_s}{nR\theta}\right) - \frac{nR\theta}{aII_s}\right].$$

Under ideal circumstances m and a are both independent of I and θ and there is then, in general, a solution of this equation for I which is different from zero. Moreover this non-zero solution when it exists represents the stable condition of the medium, because the magnetic potential energy in it is a minimum. The zero solution corresponds to a maximum value of the potential energy and is therefore in the general case unstable.

As the temperature rises the permanent magnetisation intensity gradually decreases and at a certain temperature θ_s it vanishes altogether. Beyond this temperature the only possible real solution of the equation for I is

$$I = 0,$$

so that the substance is then incapable of permanent magnetisation. The temperature θ_s may therefore be interpreted as the temperature at which the ferromagnetic quality disappears, i.e. the so-called critical temperature. In the neighbourhood of this temperature the magnetisation is always small so that using

$$x = \frac{aII_s}{nR\theta},$$

we have approximately

$$\frac{1}{I_s}\frac{dI}{dx} = \frac{d}{dx}\left[\coth x - \frac{1}{x}\right] = 1 - \operatorname{cosech}^2 x + \frac{1}{x^2} = \tfrac{1}{3}.$$

But

$$\frac{dI}{dx} = \frac{nR\theta}{aI_s},$$

so that at the actual critical temperature

$$\frac{I_s}{3} = \frac{nR\theta_s}{aI_s},$$

or

$$\theta_s = \frac{aI_s^2}{3nR}.$$

302. If now we introduce this temperature into the relation determining the intensity of permanent magnetisation it assumes the form

$$I = I_s\left[\coth\left(\frac{3I\theta_s}{\theta I_s}\right) - \frac{\theta I_s}{3I\theta_s}\right],$$

so that the ratio I/I_s is a function of θ/θ_s and the function is the same for all substances.

Thus if we express the intensity of permanent magnetisation in terms of the maximum possible intensity I_s and the absolute temperature θ in terms of the absolute critical temperature, we obtain a characteristic equation for the intensity of permanent magnetisation at a given temperature, which is identical for all ferromagnetic substances. This relation has been tested by Weiss* in the case of magnetite and he finds that it is satisfied with great accuracy except at very low temperatures (− 79° C.) and in the neighbourhood of the critical temperature, where however the deviations are not large. When it is observed that there are no disposable constants in the formula this agreement between the observed facts and what at first sight appears to be merely a provisory theory can only be regarded as remarkable.

303. The theory proves equally successful in explaining the observed facts of the phenomenon of induced magnetisation. Of course, as we have already noticed, any simple theory of the present type is hardly likely to be directly applicable to the ferromagnetic metals, the magnetic behaviour of which is complicated by various secondary causes. It is however found that the simple phenomena accompanying the magnetisation of the various crystalline ferromagnetic minerals when placed in a field parallel to their principal magnetic axis fit in admirably with the theoretical conclusions to be drawn from the theory.

The general relation between I and H is expressed by the equation

$$I = I_s\left[\coth\frac{I_s(H + aI)}{nR\theta} - \frac{Rn\theta}{I_s(H + aI)}\right].$$

* *Jour. de Phys.* VI. p. 665 (1907).

This requires that in general I should be a continuous function of H when θ is maintained constant, which is just the opposite to what was actually found to be the case, the magnetisation being practically independent of the magnetic field and equal to its saturation value at the given temperature. This could be the case only when the internal local field of intensity aI far exceeds in actual magnitude any field that we can experimentally produce. Now this actually appears to be the case, as we shall soon see; so that under all circumstances except when I is very small, that is in the neighbourhood of the critical temperature, the intensity of magnetisation is determined by

$$I = I_s \left[\coth\left(\frac{aII_s}{nR\theta}\right) - \frac{nR\theta}{aII_s}\right],$$

and is equal to the saturation intensity in all fields.

304. In the neighbourhood of the critical temperature, when I is small, the previous argument is invalid. But then the general relation may be simplified by approximation, and for all practical purposes it is equivalent to

$$\frac{I}{I_s} = \frac{I_s(H + aI)}{3nR\theta},$$

or introducing the critical temperature

$$(\theta - \theta_s) I = \frac{H\theta_s}{a},$$

in which form it has been satisfactorily verified in numerous cases, even with iron. This latter agreement is not surprising because in the neighbourhood of the critical temperature the constitutional irregularities are becoming very unstable.

In all cases where the last relation is experimentally verified, a determination of the constant a may be made, and hence an estimate of the strength of the local magnetic fields. From measurements of this kind Weiss finds that the intensity of the local field aI in iron is $6{\cdot}56 \,.\, 10^6$ absolute units and it is of a similar order of magnitude in other substances. This is very much greater than the strongest magnetic field ($5 \,.\, 10^4$ units) which can be produced in the laboratory, so that the peculiar result that the intensity of induced magnetisation I_s does not, under the simplest circumstances, appreciably alter with the external field is satisfactorily accounted for.

It is not possible in the scope of the present work to include any further account of many interesting developments of Weiss's theory; but it is hoped that the above discussion of its more elementary aspects will at least indicate the great advance which this theory represents in the theoretical examination of the behaviour of ferromagnetic substances.

CHAPTER VII

ELECTRIC CURRENTS IN METALLIC CONDUCTORS

305. Introduction. When a conductor is introduced into an electric field a separation of electricity takes place until the field in its interior is compensated. Until this state is attained there is a flow of electricity, or a current, as we say. To illustrate the matter more fully let us consider a conductor somewhat in the form of that shown in the figure. The end a is charged with a positive charge $+q$ and the end b with a negative charge $-q$. There is then an electric field partly inside and partly outside the

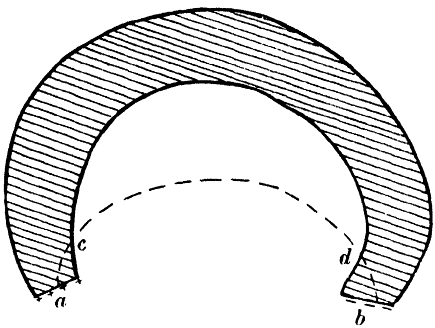


Fig. 59

conductor, the lines of force in which may cross the surface of the conductor. Such a state of affairs if initially established is however not a possible equilibrium one so that there follows immediately a separation of the charges at each point of the conductor the total result of which is the final annulling of the charges at a and b. We shall now suppose that we can continually renew the charges at a and b in such a way as will maintain a constant potential difference between the two ends of the conductor. The manner in which this is accomplished will be hereinafter discussed.

306. Now the force driving the charge is the electric force of the field and so the initial charge flux must follow the lines of force. There will thus be initially a displacement of electricity along a line of force such as that shown in the figure by the dotted line *acdb*. But this displacement can only proceed as far as c where the surface of the conductor is reached. There will thus be

an initial accumulation of positive charge on the surface of the conductor and this excites a field in the interior of the conductor, whose normal component at the surface is opposed to the normal component of the original force which was directed along the line *abcd*. This charge at *c* accumulates until this normal component is actually compensated, i.e. until the original line of force is altered into one running from *a* to *b* inside, and nearly parallel to the surface of the conductor.

Thus the first part of the electric flow is concerned merely with charging the surface of the conductor so that all the lines of force starting inside it from *a* remain inside it till *b* is reached.

In external space the normal component of the force originally along the line (*a*, *b*, *c*, *d*) is not compensated by the field of the surface charge, since they are both in the same direction. There is in fact in the external space a complicated electrostatic field composed of the original field superposed on that due to the surface charge.

It must however be said that the charges and force intensity in the field thus brought into existence are both extremely small. The electric elements in a conductor are so extremely mobile that it requires only a very small electrostatic force to produce an appreciable current.

307. The current can now flow undisturbed from *a* to *b* in the interior of the conductor along the new line of force and if the charges at *a* and *b* are continually supplied so as to maintain the constant potential difference a condition of stationary streaming is attained. The field in the whole space then remains constant. Moreover the amount of electricity crossing any section of the conductor per unit time must be the same as otherwise there would be an accumulation of charge in the conductor and a slight accumulation would create a back electromotive force which would tend to stop the current. A very slight accumulation would produce a sufficient back electromotive force to stop the current.

The process of starting the current thus requires a very slight accumulation of charge on the conductor which is just enough to make the flow steady or the current uniform and stationary. In this steady state the stream lines of the electric flow are also the lines of force of the electric field inside the conductor.

This idea of a current as a flow of electricity did not exist even for a considerable time after the discovery of batteries. It was Ohm (1827) who started the notion*.

308. Definition of an electric current. We must now define the electric flow in such a manner as to render it susceptible of calculation. If we adopt

* *Die galvanische Kette, mathematisch bearbeitet* (Berlin, 1827). Translated in Taylor's *Scientific Memoirs.*

the general method we should specify the flux of electricity in any direction at a point in the conductor by the amount Cds which crosses per unit time a small surface ds placed perpendicular to the direction with its mean centre at the point. We can easily show that this defines a vector quantity if we choose rectangular axes with their origin at the point under consideration and consider the flow in and out of the small tetrahedral volume $OABC$ at the origin of coordinates with edges $\delta x \delta y \delta z$ along the axes. The area ABC has projections on the axial planes equal to

$$\tfrac{1}{2}(\delta y \delta z,\ \delta z \delta x,\ \delta x \delta y) \quad \text{or} \quad (\mathbf{n}_{1_x},\ \mathbf{n}_{1_y},\ \mathbf{n}_{1_z})\,\delta s$$

where δs is the area ABC and $\mathbf{n}_1$ its direction vector.

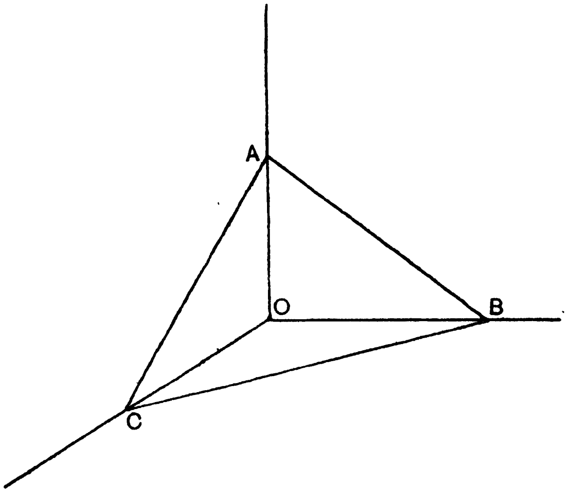


Fig. 60

The equation of continuity of flow expresses that the aggregate flux out of this volume is equal to *minus* the rate at which the total charge inside is increasing. If ρ is the density of charge inside

$$\delta v \cdot \frac{\partial \rho}{\partial t} = -\ (\mathbf{C}_x \mathbf{n}_{1_x} + \mathbf{C}_y \mathbf{n}_{1_y} + \mathbf{C}_z \mathbf{n}_{1_z})\,\delta s + \mathbf{C}_n \delta s,$$

where $\mathbf{C}_n$ is the flux component normal to δs and $\mathbf{C}_x$, $\mathbf{C}_y$, $\mathbf{C}_z$ those normal to the axial planes. The volume δv $(ABCO)$ is infinitely small of the third order and the surface δs is infinitely small of the second order and thus ultimately when the volume is very minute the left-hand side of this equation is zero and thus

$$\mathbf{C}_n = \mathbf{C}_x \mathbf{n}_{1_x} + \mathbf{C}_y \mathbf{n}_{1_y} + \mathbf{C}_z \mathbf{n}_{1_z},$$

which proves that $\mathbf{C}$ is a vector with components $(\mathbf{C}_x,\ \mathbf{C}_y,\ \mathbf{C}_z)$.

309. This is the general method of definition. In order however to obtain a closer insight into the true nature of the flux involved we must proceed in a slightly different manner. Consider again the small surface δs and construct on it a small cylinder of length δl with its axis parallel to the resultant motional velocity v_1 of the electric charge at the point (the direction of the lines of force at the point), this direction making an angle a with the normal at δs. If the density of the positive electricity at the point is ρ_1 the quantity of positive electricity in this cylinder is $\rho_1 \delta l \delta s \cos a$ and during a time

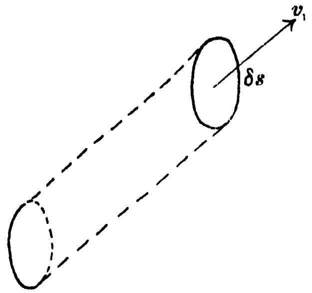


Fig. 61

$\delta t = \dfrac{\delta l}{v_1}$ all of this electricity flows out across δs. Thus for this surface

$$\mathbf{C}_1 \delta s = \frac{\rho_1 \delta l \delta s \cos a}{\delta t} = v_1 \rho_1 \cos a \delta s,$$

or

$$\mathbf{C}_1 = \rho_1 v_1 \cos a$$

measures the current of positive electricity in the direction normal to ds at the point.

If there is at the same time a flux of negative electricity of density $-\rho_2$ we know that it takes place in the opposite direction to that of the positive although perhaps with a different velocity v_2. The current across ds normally in the same direction due to the negative charge is therefore

$$\mathbf{C}_2 = -\rho_2 (-v_2) \cos a = \rho_2 v_2 \cos a$$

and the total current in this direction is

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 = (\rho_1 v_1 + \rho_2 v_2) \cos a,$$

from which again we see that $\mathbf{C}$ is the component of a vector $(\rho_1 v_1 + \rho_2 v_2)$; which we call the *current density* of the electric flow at the point.

310. Ohm's Law*. The force driving the charge and imparting to it the motional velocity is the electric force: the positive elements of charge are moving in the positive direction of the lines of force and the negative ones in the opposite direction and the two motions having the same effect constitute the current. This being the case there must be some relation between the electric force and the current. Ohm tried to reason the relation out by considering the phenomena of currents as analogous to the conduction of heat down a temperature gradient. There are two kinds of streaming motion recognised in physics; that of steady diffusion in which the *velocity* is proportional to the driving force; and that of free motion in which the change

* This law was anticipated by Cavendish in 1781. Cf. his *Electrical Researches*.

of velocity is proportional to the force. In diffusion the motion is so modified by impeding frictional forces that a state of steady motion is attained in which the velocity is proportional to the force. The conduction of heat is the typical example of a process of steady diffusion.

If now we assume with Ohm that the electric flow in the conductor is a process of steady diffusion under the action of the electric force we must assume with him that both v_1 and v_2 are *proportional* to the driving electric force $\mathbf{E}$. Thus if we now use $\mathbf{C}$ for the resultant electric current we have in the vector sense

$$\mathbf{C} = \kappa \mathbf{E},$$

where κ is a physical constant, which is usually called the *conductivity* of the substance at the point.

311. Now consider the case of a steady current flowing in a wire of finite thickness. The surfaces of the wire form a tube of force for the internal field and also a tube of flow for the current. The ends a, b of the wire are

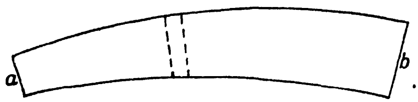


Fig. 62

presumed to cut the lines of force of the internal field everywhere normally and the same assumption tacitly underlies the choice of any other cross-section subsequently made. These sections will then be equi-potentials of the internal field.

Now consider any cross-section of the wire of total area f and suppose it resolved into small elements of area df. If $\mathbf{C}$ is the current density at a point in the wire the total quantity of electricity crossing the section per unit time is

$$J = \int_f \mathbf{C}_n \, df;$$

J is called the strength of the current in the wire, or simply the current.

Again since the normal to df is in the direction of the line of force in the field at the place

$$\mathbf{C}_n = \mathbf{C} = \kappa \mathbf{E},$$

so that

$$J = \int_f \kappa \mathbf{E} \, df.$$

Now at any infinitely small distance from the section f draw another equi-potential section f' and let δs be the distance between corresponding points of the sections, then

$$J = \int_f \frac{k\mathbf{E}\,\delta s}{\delta s} \, df.$$

But $\mathbf{E}\delta s = \delta\phi$ is a constant over the whole surface f, viz. the constant difference of potential between the two surfaces f and f'. Thus

$$J = \delta\phi \int \frac{\kappa df}{\delta s}.$$

The quantity

$$\delta k = \frac{1}{\int_f \frac{\kappa df}{\delta s}}$$

is called the 'resistance' of the portion of the conductor between the cross-sections f and f': thus at this point of the wire

$$J = \frac{\delta\phi}{\delta k}.$$

But J is a constant all along the wire and thus if ϕ_a and ϕ_b are the potentials at the ends a, b of the wire

$$\phi_a - \phi_b = Jk,$$

where $k = \int_a^b \delta k$ is the resistance of the wire between the two ends.

This is Ohm's law in its original form. The procedure adopted by Ohm was however rather different from that sketched above. He tried to extend the mathematics just previously developed by Fourier for the conduction of heat down a temperature gradient. In doing this he had of course to assume something analogous to temperature and it did not require much to convince him that the potential was the required quantity. The current in a wire is proportional to the fall in potential from one end to another

$$J = \frac{\phi_1 - \phi_2}{k}*.$$

This idea that currents go by diffusion was at first merely an hypothesis, but on the modern theory of electrons it appears as the actual state of affairs.

312. There is an important hydrostatic analogy which enables us to picture the process more clearly. If liquid is forced through a tube blocked by a number of small obstacles so that no eddies can be formed and if the motion is a steady pushing through with the hydrostatic pressure as the driving force the amount of the flow is

$$= \frac{\text{difference of pressures}}{\text{resistance of channel}}.$$

This is the more direct analogy with the electrical case. The term electric resistance is coined on this basis.

The analogy goes even farther and enables us to talk of the driving force in the electrical case as an electrostatic pressure. The modern theory of the flow of electricity basing the current on the flow of electrons is in fact a direct application of this analogy. A current consists largely of free electrons being pushed through among the obstacles presented by the molecules of the matter.

* These considerations are of course confined to a steady system: it is only when a steady state of flow has been attained that a potential exists (see Ch. IX).

The notion that electric pressure is the same as potential dates back to Volta's time. He knew that it was electrostatic pressure that pushed the current and he made a condensing electroscope sufficiently sensitive enough to show this. The distinction between free motion and diffusion was however due to Ohm.

313. The Volta potential difference. We have so far assumed that we are able to maintain a constant potential difference between two points on the surface of a conductor. We can now discuss how this is attained. The foundation of the method is Volta's discovery that when two conductors or generally any two different substances, are in contact, there is a definite potential difference between them*.

If we place two different conductors in contact, then an adjustment of charge will take place so that the one conductor will acquire a definite negative charge and the other a definite positive charge. The quantities of electricity involved in the rearrangement depends on the different conditions such as the form, size and relative position of the bodies, but with the same two substances the potential difference thus set up between them has a definite value as long as we always work at the same temperature.

If the substances are conductors as is usually the case then in the equilibrium condition the potential of the electrostatic field which arises has a constant value at all points inside either conductor except very near the surface where it is in contact with the other. The change from the potential of the one conductor to that of the other thus takes place in the infinitely thin contact surface so that we can speak of a sudden jump of the potential. It therefore also follows that the origin of the action is situated in the immediate neighbourhood of the surface of contact. Between the two substances at the adjacent faces there is a certain stress of chemical affinity which results in an inequality in the forces exerted by each metal on the elements of charge in the other. We could for example explain the potential difference between zinc and copper by imagining that there is a greater attraction on the positive charges exerted by the zinc than the copper, and on the negative charges exerted by the copper than the zinc. This electrochemical stress results in an electric displacement either by polarisation

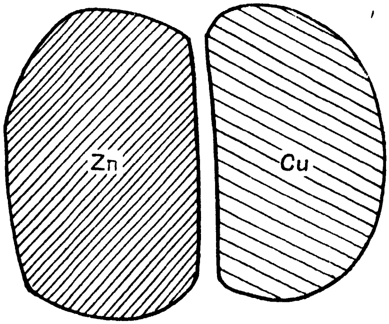


Fig. 63

* *Ann. de Chim.* 40, p. 225. Cf. also *Gilbert's Annalen*, 9, p. 380 (1801); 10, p. 425 (1802); 12, p. 498 (1803).

of the molecules (in non-conductors) or by actual finite separation of the charge between them. The resultant of the electric displacement is that one of the two contact surfaces appears positively charged and the other negatively, the double sheet thus created accounting for the jump of potential.

314. When the substances are conductors the charges on each will distribute themselves over the surfaces of the separate conductors. The charges on the conductors must be equal and opposite (their total must be zero) and so will be practically all concentrated on the adjacent surfaces at the surface of contact. The potentials of the conductors being ϕ_1 and ϕ_2, the difference $\phi_1 - \phi_2$ is always the same for the same conductors under the given conditions; and is usually called their *volta difference* of potential.

When equilibrium has been established, which usually requires only an extremely short time, there is an electrostatic field surrounding the conductors which obeys all the laws of electrostatics. In the interior of the surface of contact between the adjacent surface charges the electric field is however compensated by the contact forces of chemical affinity. Thus for example in the Zn-Cu case mentioned the zinc attracts the positive charge from the copper and the copper the negative charge from the zinc, the result being that the zinc becomes positively charged and the copper negatively. But each addition to the positive charge on the zinc repels the remaining positive charge on the copper and so lessens the total attraction of the zinc on it. The separation thus goes on till the attraction of the zinc for any further positive charge becomes balanced by the repulsion of that charge by the positive electricity on the zinc.

The electrostatic field of the conductors is practically that due to the double sheet on the surface of separation, the small remaining charges on the further parts of the surface having no effect.

315. Now consider several such conductors joined in a ring. There would be a potential difference at each junction of two different conducting surfaces caused by the creation of a double sheet as indicated above. There would thus be a potential gradient at each junction in the circuit and thus the system is ready for an electric current to flow. But no current can flow or else the principle of the conservation of energy would be violated. The non-existence of the current may be explained by the fact that the currents arising from the single impressed electromotive forces at the separate junctions so flow as to cancel one another out.

The energy principle is no longer violated and a current can result if only we could supply energy from some external source at one of the junctions. We shall presume the possibility of this supply, postponing the discussion

of the exact method in which it is applied. We should then have a current in the circuit, its density at any point being determined by

$$\mathbf{C} = \kappa \mathbf{E},$$

$\mathbf{E}$ being the electric force intensity of the field in the interior of the conductor; but $\mathbf{E}$ is composed of parts $\mathbf{E}_1$, $\mathbf{E}_2$, ... so that

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots,$$

where $\mathbf{E}_1$, $\mathbf{E}_2$, ... are the components in the direction of the resultant $\mathbf{E}$ of the several force intensities in the separate fields arising from the double sheet distributions at the various junctions.

316. Adopting the notation of the previous paragraph and considering for the present the current of strength dJ flowing through an elementary tube of flow (or tube of force) in the circuit of cross-section df at any place we have

$$J = \int_f \frac{dJ}{df}\, df,$$

where $$dJ = \mathbf{C}df = \frac{\kappa\,(\mathbf{E}_{1_s}\delta s + \mathbf{E}_{2_s}\delta s + \ldots)\, df}{\delta s},$$

or by integration round the whole circuit, i.e. with respect to s

$$dJ \,.\int \frac{ds}{\kappa df} = \int_s (\mathbf{E}_{1_s} ds + \mathbf{E}_{2_s} ds + \ldots).$$

But $$\int_s \mathbf{E}_{1_s} ds = \phi_1' - \phi_2 = \phi_{12}$$

is the volta potential difference of the two metals at the first junction of the circuit reckoned in a definite sense round the circuit. In this integral the element of the circuit between the double sheet concerned is of course not included, although the current crosses through this sheet; the argument for this may be stated as follows. If there were equilibrium with no current flowing the electric force intensity $\mathbf{E}$ in the contact sheet is exactly balanced by the contact forces (of chemical affinity), say $\mathbf{S}$, so that

$$\mathbf{S} - \mathbf{E} = 0.$$

If however the charges separated by the contact forces can break away and flow off as an electric current there is no longer an exact balance. But even in this case the outstanding difference between $\mathbf{S}$ and $\mathbf{E}$ is small compared with either of them, because sufficient charge accumulates in any case to establish the volta potential difference. Thus since $\mathbf{S}$ and $\mathbf{E}$ are both large of the first order the difference $\mathbf{S} - \mathbf{E}$ can be at most finite, say $\mathbf{E}'$, so that

$$\mathbf{S} - \mathbf{E} = \mathbf{E}'.$$

Since now the value of κ in the contact surfaces is certainly not large (it is at most finite) the values of $\mathbf{E}'d$ and $\kappa\mathbf{E}'d$ may be neglected in comparison

with finite quantities, d being the thickness of the sheet. Thus practically the only driving force for the current is that in the interior of the metals. We have thus

$$dJ\int\frac{ds}{\kappa df}=\phi_{12}+\phi_{23}+\ldots,$$

or

$$J=\frac{1}{k}(\phi_{12}+\phi_{23}+\ldots),$$

where

$$k=\int_f\frac{df}{\int_s\frac{ds}{\kappa df}}$$

represents the total resistance of the circuit. This is of course Ohm's law for the circuit.

317. If no energy is supplied at any part of the circuit then $J=0$ and thus

$$\phi_{12}+\phi_{23}+\ldots\phi_{n1}=0,$$

which is *Volta's law*. This sum is however no longer zero when a current is flowing or when energy is being added from outside at one of the junctions.

A direct consequence of this result is that the volta difference of potential of two metals in contact is exactly the same as the potential difference of the same two metals connected through a whole series of other metals. No direct electrometer reading will therefore ever detect the potential difference here described, because the instrument merely records the difference between the pieces of metal forming its quadrants (which are of course of the same metal*).

318. If we integrate in the positive direction round only part of the circuit, say that between the two sections α and β we get

$$\int_\alpha^\beta Eds=\phi_\alpha-\phi_\beta+\Sigma\phi_{r,r+1},$$

where ϕ_α, ϕ_β are the potentials at the sections α, β and $\Sigma\phi_{r,r+1}$ refers to the volta potential difference for all the junctions occurring in the section of the circuit between α and β. Consequently we have also

$$J=\frac{\phi_\alpha-\phi_\beta+\Sigma\phi_{r,r+1}}{k_{\alpha\beta}},$$

where $k_{\alpha,\beta}$ is the resistance of the part of the circuit concerned.

If the sections α, β are very near but on opposite sides of a given junction the resistance $k_{\alpha\beta}$ is very small and so

$$\phi_\alpha-\phi_\beta+\phi_{r,r+1}=0,$$

or

$$\phi_\alpha-\phi_\beta=-\phi_{r,r+1},$$

* By constructing electrometers with opposite quadrants of different metals it is possible to determine the difference of potential between the metals (Kelvin).

so that the potential in the circuit jumps at each junction by the corresponding volta difference of potential for that junction. If we put the sections α, β close together but so that practically the whole circuit is included between them and then remove the small portion of the circuit not included $J = 0$ and thus

$$\phi_\alpha - \phi_\beta = - \Sigma\phi_{r,r+1},$$

the potential difference between the ends of an open circuit is equal to the electromotive force operative in the same circuit when closed.

319. The general practical method of obtaining an effective electromotive force of this kind in the circuit is obtained by the insertion in it of a voltaic element or cell. The general principle involved may be illustrated by the following particular example. If we put into a vessel which contains dilute sulphuric acid a plate of copper and a plate of zinc at a small distance apart, it will be found by connecting the ends of the metals projecting from the acid to the quadrants of an electrometer that the pair of quadrants connected with the copper is at a higher potential than the other. In this experiment we have a series of conductors; brass, copper, acid, zinc and brass. The potential in the interior of each conductor has the same value at each point but it jumps at each surface of contact and the observed potential difference is the algebraic sum of the jumps. If ϕ_B, ϕ_C, ϕ_S, ϕ_Z, ϕ_B denote the potentials of the metals in order, and if also we use ϕ_{BC} for $\phi_B - \phi_C$ then we have the observed difference equal to

$$\phi_{BC} + \phi_{CS} + \phi_{SZ} + \phi_{ZB}.$$

One or more of these terms may be negative and if the acid were replaced by a metal the sum would be zero. It is only because we have an acid (or fluid conductor) in the series that the expression has a definite positive value different from zero.

The apparatus here described and many others of a like nature, which are composed of rigid and fluid conductors and which possess the property of creating a potential difference between two pieces of the same metal is called a galvanic element or cell. They were first invented and used by Volta and are called after him*.

With such an apparatus it is possible to produce a permanent steady current by connecting the metal ends projecting from the liquid through a simple metallic circuit. The chemical affinities of the elements of metal and those of the acid produce an electric separation in the manner previously described and the double sheet so produced makes the sudden jump of potential in crossing from the metal to the liquid. Chemical attractions prevent the separate induced charges combining across the liquid and so they have to go round the metals closing the circuit. The double sheet thus

* Cf. the letter in *Phil. Trans.* (1800), p. 402. Also *Gilbert's Annalen*, 6, p. 340.

dissipated by these charges going round is then continually renewed by chemical action and this makes a permanent current. The energy supplied at the junctions here is the chemical energy of combination of the one substance with the other. The precise nature of the action involved will be discussed in the next chapter.

320. On impressed forces in mechanics : an analogy*. The mechanical distinction between the two cases here discussed, when the current can exist and when it cannot, is easily recognised. The closed circuit of physically and chemically homogeneous conductors in itself is a self-contained mechanical system so that no current can flow in it unless there are external forces of some kind acting. Thus if a current flows in such a circuit there must on the whole be an impressed electromotive force equal to Jk (as this is the electromotive force in the circuit) which arises from actions of a purely external nature.

This conception of electromotive force corresponds exactly to the conception of external or impressed forces in the mechanics of ponderable bodies. The idea is that of a force, not determined by the conditions which govern the system under consideration, but which nevertheless acts on it in an arbitrary manner and by means which are in no way in essential connection with the system. For example in a system of elastic bodies, the elastic stresses and internal forces are in perfect accord with one another and with the internal deformations and motions; but external forces may act on the system in any arbitrary manner. Of course, these forces influence the distribution of the internal stresses but only in the sense that they alter the conditions under which the system exists.

321. There are two reasons why we introduce the idea of external forces into ordinary mechanics. The first is that we can thereby limit our discussions to the consideration of a particular system by itself. If in the above example of elastic bodies the external forces are produced by weights, their action is determined by mechanical laws; such forces would of course be internal forces if gravity were also included in the mechanical system. Another important example will be given later, it involves the electromotive forces which can be produced by the oscillations of a magnetic field in the neighbourhood of a current circuit or to the motion of that circuit through a magnetic field. Such forces are external as long as we prefer to leave the magnetic field out of the calculation. They become internal forces, whose action is determined by the ordinary laws of electrodynamics, as soon as the magnetic field is considered part of the system.

The second reason for introducing external forces into mechanics is that they often represent actions, which cannot be sufficiently well explained on

* Cf. Abraham, *Theorie der Elektrizität*, I. p. 199.

a purely mechanical basis. Examples are provided in connection with the motions of magnets or electrically charged bodies. If mechanics attempted to explain all the motions of natural bodies, it would tacitly ignore all those motions which cannot be explained on a purely dynamical basis. In the cases mentioned, for example, a complete dynamical description of the motions is not possible until we have a mechanical explanation of the magnetic and electric actions. Mechanics treats only of one side of natural phenomena, it is useless when we are dealing with phenomena of a non-mechanical nature. A modified usefulness is however attained by admitting ignorance of their fundamental basis but representing the actions of these non-mechanical processes by means of impressed forces. The present type of impressed electromotive force in a circuit is of this nature. Such forces could not be treated from the standpoint of pure electric theory, because we are in reality involved in them in chemical and thermal phenomena, the laws in which are known only in a few special cases. Thus if we limit ourselves to the description of electrodynamic phenomena, we must in such cases resort to the idea of the impressed electromotive force to explain the action of such processes on those under direct review.

322. On the energy relations of an electric current*. We must now enquire into the amount of work expended in driving the current. Consider for this purpose any conductor in which a current J is flowing. Let ϕ_1 and ϕ_2 be the electrostatic potentials of the internal electrical field at two sections of this conductor between which the resistance is k_{12}. In a time δt an amount of electricity equal to $J\delta t$ is transferred from the one section to the other through the conductor (or at least this is the effective result of the electrical flow during this time). This means that an amount of electrical energy

$$(\phi_1 - \phi_2)\, J\delta t,$$

has been lost in this part of the conductor during the time δt. Per unit time this is

$$J\,(\phi_1 - \phi_2).$$

What has become of this energy? The driving of the current is an affair of diffusion, the electric force is pushing the electric charges along among a large number of obstacles. The electric atoms get up a velocity, but impart it by collision to the molecules of the matter, so that their own motion becomes irregular. This is the essence of frictional resistance; the motion of the electric elements becomes irregular through collision with the obstacles. The wasted energy thus appears again as irregular motion of the electric charges and also partly of irregular motion of the molecules of the matter, that is it appears as heat in the conductor. Thus the heat developed per unit time in the portion of the conductor considered is

$$J\,(\phi_1 - \phi_2).$$

* Kelvin, *Phil. Mag.* Dec. 1851.

If we introduce Ohm's principle that

$$J = \frac{\phi_1 - \phi_2}{k_{12}},$$

the heat developed appears as of amount

$$J^2 k_{12},$$

per unit time. Thus in the whole circuit the total heat developed is

$$J^2 k,$$

where k denotes the resistance of the circuit. This expression however not only gives the total amount of heat but also its location.

323. If a voltaic cell is supplying the current the energy which appears as heat in the circuit must also come from the cell. Moreover this energy is available energy for if we had conductors of small resistance we could turn it into work. (If the conductors are of big resistance the work is entirely wasted in them.) This work is introduced into the circuit in the battery at the places where the substances are decomposed (i.e. at the surfaces of metals in liquid). The supply of available energy comes in from the liquid and may be used to drive a machine somewhere if it does not waste. The location of the energy supply is different from that of its emergence. Thus an electric current is a means of transmitting power.

If a voltaic cell is the source of the current the total heat developed in the circuit appears as the work required to raise the quantity of electricity supplied by the current through any cross-section of the circuit through the various potential jumps at the contact surfaces in the circuit. Because $(\phi_1 - \phi_2)$ for a whole circuit is simply

$$\Sigma\phi_{r,r+1},$$

for that circuit. The work per unit time is thus

$$J \,.\, (\Sigma\phi_{r,r+1}),$$

or since $\Sigma\phi_{r,r+1}$ is the quantity measured as the electromotive force of the cell, it is measured by the product of the current by the electromotive force of the cell.

It is of course assumed that all the work done on the electric charges in driving them forward is spent in increasing their velocity and is thus dissipated by collision and appears as heat. This only applies when none of the energy is turned into mechanical work as is usually the case in circuits containing dynamos.

324. The above results, based on the idea of diffusion, were experimentally tested and verified by Joule 75 years ago*. From his results he formulated his law expressing that the total amount of heat developed as expressed above is correct and also that the distribution given is correct.

* *Phil. Mag.* 19 (1841), p. 260.

He was also able to formulate the principle for electric flow which in its generalised form states that in all cases of diffusion the flow distributes itself so as to give the least possible heat for a given current. In other words if a given current is introduced into a network of conductors it distributes itself among the conductors so that the energy wasted is least.

The generalised principle in the form that in any steady dynamical motion of a given material system, when the forces are only frictional, the motion is such as to make the waste of energy the least possible, was given and proved by Lord Kelvin. The particular case here quoted is however usually called *Joule's law of minimum dissipation.*

325. The proof of this law in the electrical case is easy. Let there be n electrodes joined by $\frac{n(n-1)}{2}$ wires and let there be given electromotive forces in the wires and given conditions of supply and withdrawal of current at the electrodes so that the currents in the wires are steady.

Let $\phi_1, \phi_2, \ldots \phi_n$ be the potentials of the n points; $Q_1, Q_2, \ldots Q_n$ be the amounts of electricity supplied per unit time at these points so that in a steady state

$$Q_1 + Q_2 + \ldots Q_n = 0,$$

and let E_{rs} be a possible internal electromotive force in the wire joining the rth and sth points so that

$$E_{rs} = -E_{sr};$$

and also let K_{rs} denote the reciprocal of the resistance in this same wire so that $K_{rs} = K_{sr}$; we also use other symbols $K_{11}, K_{22}, \ldots$ having no physical significance but which are such that

$$K_{11} + K_{12} + \ldots + K_{1n} = 0,$$
$$K_{21} + K_{22} + \ldots + K_{2n} = 0,$$
etc.

In applying Ohm's law to each conductor and examining the flow at each point we have

$$Q_1 = K_{12}(E_{12} + \phi_1 - \phi_2) + K_{13}(E_{13} + \phi_1 - \phi_3) + \ldots\ldots + K_{1n}(E_{1n} + \phi_1 - \phi_n),$$
$$Q_2 = K_{21}(E_{21} + \phi_2 - \phi_1) + \ldots\ldots + K_{2n}(E_{2n} + \phi_2 - \phi_n),$$
....

These equations, usually ascribed to Kirchhoff*, can be written in the form

$$K_{11}\phi_1 + K_{12}\phi_2 + \ldots K_{1n}\phi_n = -Q_1 + K_{12}E_{12} + \ldots K_{1n}E_{1n},$$
....

The sum of the left and right-hand sides of these equations is zero and they therefore reduce to $(n-1)$ independent ones.

* Kirchhoff. *Ann. Phys. Chem.* 64 (1845), p. 512; 72 (1847), p. 497; Wheatstone, *Phil. Trans.* 2 (1843), p. 323; *Ann. Phys. Chem.* 62 (1844), p. 535. Cf. also Maxwell, *Treatise*, vol. I.

These are the conditions of flow, if the currents obey Ohm's law and we have now to show that the heat developed in this case is the least possible.

326. We multiply the above equations in order by $\phi_1, \phi_2, \ldots \phi_n$ and then add the n equations together; this gives us

$$-K_{rs}(\phi_r - \phi_s)^2 = -\sum_{r=1}^{n} Q_r\phi_r + \Sigma K_{rs}(\phi_r - \phi_s) E_{rs},$$

whence also $\quad \Sigma Q_r\phi_r = \Sigma K_{rs}(\phi_r - \phi_s)(\phi_r - \phi_s + E_{rs}),$

but if J_{rs} is the current in the conductor joining the rth and sth points we have

$$K_{rs}(\phi_r - \phi_s + E_{rs}) = J_{rs},$$

and thus $\quad \Sigma Q_r\phi_r = \Sigma J_{rs}(\phi_r - \phi_s),$

or also if we use $k_{rs} = 1/K_{rs}$

$$\Sigma Q_r\phi_r = \Sigma k_{rs}J_{rs}^2 - \Sigma J_{rs}E_{rs},$$

either of which expresses the energy equation. The total heat lost in the wires is equal to the total energy supplied at the junctions plus that drawn from the cells in the circuits.

In these equations J_{rs} is the actual current in the typical conductor determined by Ohm's law. Suppose now that $J_{rs} + x_{rs}$ be a modification of this current which is compatible with the conditions of supply and output; so that

$$x_{12} + x_{13} + \ldots x_{1n} = 0,$$
$$x_{21} + x_{23} + \ldots x_{2n} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and then we have

$$\Sigma k_{rs}(J_{rs} + x_{rs})^2 - \Sigma k_{rs}J_{rs}^2 = \Sigma k_{rs}x_{rs}^2 + 2\Sigma x_{rs}k_{rs}J_{rs}$$
$$= \Sigma k_{rs}x_{rs}^2 + 2\Sigma x_{rs}(\phi_r - \phi_s),$$

if there are no internal electromotive forces in the circuits to complicate matters. But then the last term on the right is

$$2\Sigma(x_{rs}\phi_r + x_{sr}\phi_s),$$

and is zero in virtue of the linear relations among the x's and thus since

$$\Sigma k_{rs}(J_{rs} + x_{rs})^2 - \Sigma k_{rs}J_{rs}^2,$$

is essentially positive in this case the heat developed in the actual state, with no internal electromotive forces, is less than that in any other state. The more general theorem when internal electromotive forces are included will be discussed on a future occasion.

327. The general relations for a network of conductors. We have already obtained the general equations for the distribution of steady conduction currents in a network of $\frac{n(n-1)}{2}$ wires joining n electrodes. We can now indicate the general method in which they may be solved and also the chief characteristics of the solution.

It is required to find the currents in the wires when the following data are specified :

$$\phi_1, \phi_2, \ldots \phi_n \text{ the potentials at the } n \text{ points,}$$

$$Q_1, Q_2, \ldots Q_n \text{ the supplies per unit time,}$$

so that $$Q_1 + Q_2 + \ldots Q_n = 0.$$

E_{rs} the general type of internal potential in the wire joining the rth to the sth conductor so that

$$E_{rs} = -E_{sr},$$

and K_{rs} the inverse resistance in the same wire so that

$$K_{rs} = K_{sr}.$$

We then introduced other symbols of type K_{rr} so that

$$K_{r1} + K_{r2} + \ldots + K_{r(r-1)} + K_{rr} + \ldots + K_{rn} = 0,$$

and then found that the general equations could be written in the form

$$K_{11}\phi_1 + K_{12}\phi_2 + \ldots K_{1n}\phi_n = -Q_1 + K_{12}E_{12} + K_{13}E_{13} + \ldots + K_{1n}E_{1n},$$
$$K_{21}\phi_1 + K_{22}\phi_2 + \ldots K_{2n}\phi_n = -Q_2 + K_{21}E_{12} + \ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$K_{1r}\phi_1 + K_{2r}\phi_2 + \ldots K_{rn}\phi_n = -Q_r + K_{r1}E_{r1} + \ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The sum of the left and right-hand sides of these equations are zero and so they are not independent; they reduce in fact to $(n-1)$ equations which determine the differences of potential, which is all we are concerned with. On account of the relations between the K's these equations are equivalent to the $(n-1)$ equations

$$K_{r1}\phi_{1n} + K_{r2}\phi_{2n} + \ldots K_{r,n-1}\phi_{n-1,n} = -Q_r + K_{r1}E_{r1} + \ldots r = 1, 2, 3, n-1,$$

where we have used $$\phi_{rn} = \phi_r - \phi_n$$

for the potential difference between the rth and nth electrode.

328. These equations can be easily solved for the ϕ's and they lead for example to an expression for ϕ_{rn} of the form

$$\phi_{rn} = (-Q_1 + K_{12}E_{12} + \ldots K_{1n}E_{1n})\frac{\Delta_{1r}}{\Delta}$$
$$+ (-Q_2 + K_{21}E_{21} + \ldots K_{2n}E_{2n})\frac{\Delta_{2r}}{\Delta} + \ldots\ldots$$

where $$\Delta \equiv \begin{vmatrix} K_{11}, & K_{12}, & K_{13} \ldots & K_{1,n-1} \\ K_{21}, & K_{22}, \ldots & & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ K_{n-1,1}, & K_{n-1,2} & & K_{n-1,n-1} \end{vmatrix}$$

and Δ_{rs} the minor of K_{rs} in this determinant.

Having thus determined ϕ_{rn} we can easily determine the current in the particular conductor joining the rth and nth electrodes for by Ohm's law this is

$$K_{rn}(\phi_{rn} + E_{rn}).$$

Suppose for example that the whole system of currents in the network is produced by a current Q entering at the rth electrode and leaving at the sth, there being no batteries in the network. Then all the E's are zero and all the Q's as well except Q_r and Q_s, these being given by

$$Q_r = -Q_s = Q.$$

We thus get that
$$\phi_{r'n} = -Q_r\frac{\Delta_{rr'}}{\Delta} - Q_s\frac{\Delta_{sr'}}{\Delta}$$
$$= -\frac{Q}{\Delta}(\Delta_{rr'} - \Delta_{sr'}),$$

similarly
$$\phi_{s'n} = -\frac{Q}{\Delta}(\Delta_{rs'} - \Delta_{ss'}),$$

so that
$$\phi_{r's'} = \phi_{r'n} - \phi_{s'n} = -\frac{Q}{\Delta}(\Delta_{rr'} + \Delta_{ss'} - \Delta_{sr'} - \Delta_{s'r}),$$

from the symmetry of which we see that the potential fall between the r'th and s'th points in the network when unit current traverses the network from the rth to the sth points is the same as the potential fall between the rth and sth points when unit current traverses the network from the r'th to the s'th points.

Many other relations of a like nature can immediately be deduced.

329. All these results follow however equally well without troubling to solve the equations as above. In fact if the conditions in any second distribution of currents are denoted by the same notation as above but with dashed letters then we have a similar set of equations in these dashed letters. Now multiply the first set of equations by $\phi_1', \phi_2', \ldots$ and the second set by $\phi_1, \phi_2, \ldots$ and sum each set: we evidently get the same on the left-hand sides and therefore the right-hand sides must also be equal. This leads to the relation

$$\Sigma\phi_r'(-Q_r + k_{r1}E_{r1} + \ldots) = \Sigma\phi_r(-Q_r' + K_{r1}E_{r1}' + \ldots).$$

Let there now be no electromotive forces and let a current Q go in at the rth point and emerge at the sth point and in the second state let the current Q' go in at the r'th point and come out at the s'th point. Now in the first state
$$Q_r = Q, \qquad Q_s = -Q,$$
and in the second
$$Q_{r'} = Q', \qquad Q_{s'} = -Q',$$
and the above theorem reduces to

$$Q(\phi_r' - \phi_s') = Q'(\phi_{r'} - \phi_{s'}),$$

which is the same result as determined above.

330. An important application of this theory is in the bridge arrangement invented by Wheatstone* for comparing resistances.

The bridge is represented diagrammatically in the figure. The current enters it at the point A and leaves it at the point D; these points being connected by the lines ABD, ACD arranged in parallel. The line ABD is composed of two conductors (AB) and (BD) of resistances k_{12}, k_{24} and the line (ACD) is composed of two conductors of resistances k_{13}, k_{14}. The points (B, D) are connected by a wire of resistance k_{23}. Thus with the notation as above, if ϕ_1, ϕ_2, ϕ_3, ϕ_4 are the respective potentials of the four points A, B, C and D and if the rate of supply of current is Q units per second then

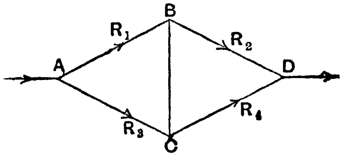


Fig. 64

$$\begin{aligned} K_{11}\phi_1 + K_{12}\phi_2 + K_{13}\phi_3 \qquad\qquad &= -Q, \\ K_{21}\phi_1 + K_{22}\phi_2 + K_{23}\phi_3 + K_{24}\phi_4 &= 0, \\ K_{31}\phi_1 + K_{32}\phi_2 + K_{33}\phi_3 + K_{34}\phi_4 &= 0, \\ K_{42}\phi_2 + K_{43}\phi_3 + K_{44}\phi_4 &= Q, \end{aligned}$$

where the coefficients K, the reciprocal resistances, are subject to the conditions

$$\begin{aligned} K_{11} + K_{12} + K_{13} &= 0, \\ K_{21} + K_{22} + K_{23} + K_{24} &= 0, \\ K_{31} + K_{32} + K_{33} + K_{34} &= 0, \\ K_{42} + K_{43} + K_{44} &= 0. \end{aligned}$$

These are the general equations from which the potentials ϕ_1, ϕ_2, ϕ_3 and ϕ_4 can be directly determined and thence also the currents in each of the wires.

The important case occurs when there is no current in the conductor (23). The condition for this is

$$\phi_2 = \phi_3 = \phi \text{ say,}$$

and then the second and third equation from each set give

$$\begin{aligned} K_{21}(\phi_1 - \phi) &= K_{24}(\phi - \phi_4), \\ K_{31}(\phi_1 - \phi) &= K_{34}(\phi - \phi_4), \end{aligned}$$

or we must have

$$\frac{K_{21}}{K_{31}} = \frac{K_{24}}{K_{34}},$$

or since $K_{rs} = \dfrac{1}{k_{rs}}$ this condition is equivalent to the condition that

$$\frac{k_{12}}{k_{13}} = \frac{k_{24}}{k_{34}}.$$

* *Phil. Trans.* 2 (1843), p. 309; *Pogg. Ann.* 62, p. 509.

Thus if we know the resistances k_{13}, k_{24}, k_{34} we can conclude from the attainment of the condition mentioned the resistance k_{12}. In the simplest form of the bridge, the line ACD is a single uniform wire and the position of the point C can be varied by moving a 'sliding contact' along the wire. The ratio of the resistances $k_{13} : k_{34}$ is in this case simply the ratio of the two lengths $AC : CD$ of the wire so that the ratio $k_{12} : k_{24}$ can be found by sliding the contact C along the wire ACD until there is observed to be no current in BC, and then reading the lengths AC and CD.

331. Stationary current streaming in three dimensions. As a further example of the general principles of the present chapter we may now examine the general problem of stationary currents in three dimensions.

The currents are presumed to be generated as a result of the application of external electromotive forces, which will be generally taken to be distributed throughout the field with an intensity at each point specified by $\mathbf{E}_e$. In the process of establishing the current flow a slight accumulation of charge will take place at certain parts of the conductor, these being necessary to secure the subsequent uniform streaming. When the steady state is attained the electrostatic field of these accumulated charges will be steady, and therefore possesses at each point a potential ϕ. The additional electromotive force is therefore

$$\mathbf{E} = -\operatorname{grad}\phi.$$

The total force driving the steady current is $\mathbf{E} + \mathbf{E}_e$ and thus if the specific conductivity of the medium is κ, the current density in the steady streaming is

$$\mathbf{C} = \kappa(\mathbf{E} + \mathbf{E}_e).$$

We have another condition which states that in the steady state there is no further accumulation at any point, and thus the integral

$$\int_f \mathbf{C}_n df,$$

taken over any closed surface in the field must vanish: this means that

$$\operatorname{div}\mathbf{C} = 0,$$

or the current is a stream vector.

The driving of the current is a process of diffusion and its energy is thus gradually being converted into heat energy; the amount of heat developed at any point in the conductor, is per unit volume

$$(\mathbf{C}\,.\,\mathbf{E} + \mathbf{E}_e) = \kappa(\mathbf{E} + \mathbf{E}_e)^2,$$

and is supplied by the impressed electromotive forces.

The condition of steady streaming implies that ϕ satisfies the characteristic equation

$$\operatorname{div}(\kappa \operatorname{grad}\phi) = \operatorname{div}\kappa\mathbf{E}_e,$$

isotropy of the medium being of course presumed. If κ is constant this reduces to

$$\nabla^2\phi = \operatorname{div}\mathbf{E}_e.$$

It is therefore only in the case when div $\mathbf{E}_e = 0$ that no current flow is possible. This case occurs in every closed circuit composed of conductors at the same temperature and which is such that no chemical changes whatever result from an actual flow of current produced in any way.

332. The general problem in this subject is to obtain the appropriate solutions of the general characteristic equation

$$\operatorname{div}(\kappa \operatorname{grad} \phi) = \operatorname{div} \mathbf{E}_e,$$

in each different part of the field and then fit them up across the boundaries between. This of course necessitates the specification of certain boundary conditions which are always satisfied. These are obviously:

(i) The normal component of the current must be continuous across any surface of discontinuity in the medium, otherwise there would be accumulation of charge on the surface

$$\mathbf{C}_{n_1} = \mathbf{C}_{n_2},$$

or

$$\kappa_1 (\mathbf{E}_{n_1} + \mathbf{E}_{e_{n_1}}) = \kappa_2 (\mathbf{E}_{n_2} + \mathbf{E}_{e_{n_2}}),$$

or

$$\kappa_1 \frac{\partial \phi}{\partial n_1} - \kappa_2 \frac{\partial \phi}{\partial n_2} = \kappa_1 \mathbf{E}_{e_{n_1}} - \kappa_2 \mathbf{E}_{e_{n_2}}.$$

(ii) The potential difference between near points one on each side of the surface of discontinuity is equal to the contact difference of potential for the two media in contact there $\phi_1 - \phi_2 = \phi_{12}$,

which, if not obvious, is easily deduced by an examination of the integral

$$\int \mathbf{E}_s ds = -\int \operatorname{grad}_s \phi \, ds,$$

taken along any path between two such points without going through the interface.

The important point in this result is that it implies continuity of the tangential electric force across the surface

$$\mathbf{E}_{t_1} = \mathbf{E}_{t_2},$$

or

$$\frac{\partial \phi}{\partial t_1} = \frac{\partial \phi}{\partial t_2}.$$

We can easily prove in the usual manner that these conditions determine ϕ to an additive constant, provided of course the distribution of impressed electromotive forces $\mathbf{E}_e$ is specified.

333. If the tangential components of the impressed electromotive forces vary steadily through a transition layer between two conducting metals in the field or if they are zero then continuity of tangential electric force also implies

$$\mathbf{E}_{t_1} + \mathbf{E}_{e_{t_1}} = \mathbf{E}_{t_2} + \mathbf{E}_{e_{t_2}},$$

and thus for the tangential components of the current intensities on either side of the surface we have a relation

$$\frac{\mathbf{C}_{t_1}}{\kappa_1} = \frac{\mathbf{C}_{t_2}}{\kappa_2},$$

and the first condition gave us

$$\mathbf{C}_{n_1} = \mathbf{C}_{n_2},$$

so that here we have a definite law for the refraction of the stream lines of the current flow through the surface. If we denote by α_1 and α_2 the angles $\left(< \frac{\pi}{2}\right)$ which $\mathbf{C}_1$ and $\mathbf{C}_2$ make with the normal to the interface at the point of incidence under consideration, these equations show that

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\kappa_1}{\kappa_2},$$

and the 'incident' and 'refracted' currents lie in a plane with this normal*.

334. In the non-conducting parts of the field $\mathbf{C} = 0$ everywhere and the above characteristic equations do not give us anything; but we know that in these parts div $\mathbf{E}$ is the density of the free charge at the points and if we exclude the existence of such we have

$$\nabla^2\phi = - \operatorname{div} \mathbf{E} = 0,$$

at all points in the dielectric.

At a boundary between a metal and a dielectric (non-conducting) the boundary conditions reduce to the simpler form

(i) $$\mathbf{C}_{n_i} = 0,$$

or $$\left(\frac{\partial \phi}{\partial n}\right)_i = \mathbf{E}_{e_n},$$

wherein i refers to the internal field in the metal.

(ii) $$\phi_i - \phi_o = \phi_{io},$$

ϕ_{io} being a constant for the two surfaces adjutting. This implies

$$\left(\frac{\partial \phi}{\partial t}\right)_i = \left(\frac{\partial \phi}{\partial t}\right)_o,$$

o referring to the external field.

The conditions are still sufficient to determine the problem.

335. In the cases which actually occur in practice the impressed electromotive forces are applied outside the bodies, in which the current flow is to be determined. The electric current is then supplied to these bodies through electrodes. When the latter are composed of much better conducting

* Kirchhoff, *Ann. Phys. Chem.* 64 (1885), p. 497.

material than the substances in which they are imbedded the problem is determinate. If we put $\mathbf{E}_e = 0$ everywhere and $\kappa = \infty$ for the electrodes then we must also have $\mathbf{E} = 0$ at all points in an electrode. This implies that ϕ is constant on the electrodes. The problem in this case may be thus stated as follows:

(*a*) To determine a regular function ϕ which inside the conducting substances satisfies

$$\operatorname{div}(\kappa \operatorname{grad} \phi) = 0,$$

and in the dielectrics

$$\nabla^2\phi = 0,$$

and which assumes given constant values $\phi_A, \phi_B, \phi_C, \ldots$ on the electrodes $A, B, C, \ldots$ and at other parts of the boundaries of the metals

$$\frac{\partial \phi}{\partial n} = 0.$$

If ϕ is determined then $\mathbf{E}$ and $\mathbf{C}$ follow directly.

If the currents supplied through the electrodes are $J_A, J_B, J_C, \ldots$ so that

$$J_A = \int_A \mathbf{C}_n df, \quad J_B = \int_B \mathbf{C}_n df, \ldots$$

then we obtain by multiplication of these equations by $\phi_A, \phi_B, \ldots$ respectively and adding

$$\phi_A J_A + \phi_B J_B + \ldots = \int_A \phi_A C_n df + \int_B \phi_B C_n df + \ldots,$$

and this latter integral sum can be converted into the integra'

$$\int \kappa \mathbf{E}^2 dv,$$

taken over the volume of the conductors. This gives the energy equation.

Direct integration over the outer boundary of the conductor leads to the continuity equation

$$J_A + J_B + J_C + \ldots = 0.$$

If only two electrodes are present, say A and B, then

$$J_A = -J_B = J,$$

and thus the heat developed in the conductor is expressed by

$$H = J(\phi_A - \phi_B),$$

or if we denote by k the effective resistance of the body defined by

$$\phi_A - \phi_B = kJ,$$

then

$$H = kJ^2.$$

The general problem here defined reduces to the analogous electrostatic one when the outer boundary of the conducting substance, where it does not coincide with the charged surfaces, is chosen to be composed of lines of force in the electrostatic field. Now consider the two following analogous problems*.

* Cf. E. Cohn, *Das electromagnetische Feld*, p. 155.

336. (1) Two conductors A and B are kept at a constant potentia difference Φ; the field of this arrangement in air is known and the dis tributions of charge on the conductors. Now suppose that a number o complete tubes of force running between the two have the air dielectri replaced by a homogeneous dielectric of constant ϵ. The solution of th new problem is obviously that

(*a*) The forms of the lines of force and equipotential surfaces ar unchanged.

(*b*) The potential and therefore also the force may be everywher unchanged provided that

(*c*) we make the density at the ends of the tubes of dielectric ϵ times it previous value.

We could interpret this result by saying that the parts of the conductor where the dielectric tubes adjut on them contribute a part to the capacity of the whole arrangement equal to

$$b = \frac{\epsilon}{\Phi}\int \frac{\partial \phi}{\partial n}\, df,$$

when the dielectric is there and

$$\frac{1}{\Phi}\int \frac{\partial \phi}{\partial n}\, df,$$

when it is away; ϕ being the potential in the field in either case and the integral is taken over the parts of the surface of either conductor covered with dielectric.

337. (2) The two conductors A and B are now perfectly conducting electrodes maintained at a potential difference Φ: the space previously filled with dielectric is now filled with some homogeneous conducting sub- stance of conductivity κ. The solution of this problem obviously leads to the same potential function ϕ and thus we deduce that the resistance of the arrangement is

$$k = \frac{\kappa}{\Phi}\int \frac{\partial \phi}{\partial n}\, df,$$

the integral being taken over the same part of the surface of the one conductor as above. We have thus the relation between k and b

$$\frac{b}{\epsilon} = \frac{k}{\kappa}.$$

338. A simple example of this arrangement is provided by the portions of two very long concentric cylindrical electrodes included between two planes through the axis making an angle a with each other. If the radii of the cylinders are r_1 and r_2 the capacity of the part considered is

$$b = \frac{\epsilon a}{\log \dfrac{r_2}{r_1}},$$

and thus
$$\frac{1}{k} = \frac{a\kappa}{\log \frac{r_2}{r_1}}.$$

Many other examples will at once suggest themselves.

339. A slight simplification of the general problem is obtained by the assumption of very small spherical electrodes, which is quite a sufficient approximation in many cases for the physical requirements. If these electrodes are at a distance apart from one another large compared with their radii they may be treated as points, so long as the investigation of the field is not pushed too close up to them. We may in a case like this formulate the general problem in the modified form.

(*b*) To determine the function ϕ which satisfies the equation
$$\operatorname{div}(\sigma \operatorname{grad} \phi) = 0,$$
in the conductors and
$$\nabla^2 \phi = 0,$$
in the dielectrics; with the exception of certain points a, b, ... where it becomes infinite like $\frac{A}{r}, \frac{B}{r}, \ldots$.

In this case $4\pi A$, $4\pi B$, ... are the current supplies at the electrodes and thus
$$A + B + C + \ldots = 0.$$

For the calculation of the heat developed as well as the resistances however we must revert to finite dimensions for the electrodes: as an example we may consider the simple case of an indefinitely extended conductor with one plane face, in which there is a small electrode through which a current J is supplied. The second electrode is presumed to be at infinity. This problem is exactly analogous to an electrostatic problem already solved, and thus the solution is obvious. The potential in the conductor is
$$\phi_i = \frac{J}{4\pi\kappa}\left(\frac{1}{r_1} + \frac{1}{r_2}\right),$$
where r_1 is the distance from the centre of the electrode and r_2 the distance from its image in the boundary plane of the conductor. The resistance of the conductor is thus
$$k = \frac{1}{4\pi\kappa}\left(\frac{1}{a} + \frac{1}{2h}\right),$$
where h is the distance of the electrode from the surface and a its radius.

The potential in external space is
$$\phi_0 = \frac{J}{2\pi\sigma} \cdot \frac{1}{r_1}.$$

The image method can also be used in a similar way when both media are conducting, with specific resistances κ_1 and κ_2. We then get

$$\phi_1 = \frac{J}{4\pi\kappa_1}\left(\frac{1}{r_1} + \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \cdot \frac{1}{r_2}\right),$$

$$\phi_2 = \frac{J}{2\pi(\kappa_1 + \kappa_2)} \cdot \frac{1}{r},$$

and for the resistance

$$R = \frac{J}{4\pi\kappa_1}\left(\frac{1}{a} + \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \cdot \frac{1}{2h}\right).$$

340. As a last example of these principles we will consider a single example of a non-homogeneous conductor. An infinite homogeneous conductor (κ_1) is traversed by a uniform current of density C in the direction of the x-axis. We are required to calculate the disturbance of this uniform flow caused by the introduction of a sphere of conductivity κ_2 and radius a with its centre at the origin of coordinates.

Here again the problem is exactly analogous to the problem of the disturbance caused in a uniform field of electric force by the introduction of a homogeneous dielectric sphere. We can therefore write the results down at once.

The resulting field is obtained by adding to the original potential

$$\phi_0 = -\frac{C}{\kappa_1} x,$$

an additional part which in the sphere is

$$\phi_i = -\frac{\kappa_2 - \kappa_1}{2\kappa_1 + \kappa_2} \cdot \frac{C}{\kappa_1} x,$$

and outside is

$$\phi_0 = -\frac{\kappa_2 - \kappa_1}{\kappa_2 + 2\kappa_1} \cdot \frac{C}{\kappa_1} \cdot \frac{a^3 x}{r^3},$$

r being radial distance from the centre of the sphere.

More general cases can be analysed by harmonic functions*.

341. The thermal relations of an electric current. We have mentioned so far the voltaic cell as the only means of producing by chemical action the necessary energy to drive a steady current in a linear conducting circuit. Experience however has shown that heat is also very effective as a generating agent for electric flow. In fact a current is at once observed in a circuit consisting entirely of pieces of metal if two junctions between different metals

* Various cases have been examined by Helmholtz, *Ann. Phys. Chem.* 89 (1853), pp. 211, 253 (*Wiss. Abhandl.* p. 494); W. M. Hicks, *Messenger of Math.* 12 (1883), p. 183; R. Felici, *Tortolini Ann.* 1854, p. 270; Kirchhoff, *Berlin. Monatsber.* (1882), p. 72 (*Ges. Werke*, p. 66); Greenhill, *Proc. Camb. Phil. Soc.* (1879), p. 293.

or even two points of the same metal are maintained at different temperatures*.

The simplest case is that in which the circuit consists of two pieces of different metal A and B joined at P_1 and P_2 into a complete circuit.

Any difference of temperature between P_1 and P_2 then causes a current to flow in the circuit and the direction of the current is reversed by inverting the temperature difference. If the junctions are at the same temperature there is absolutely no sign of any electric motion whatever.

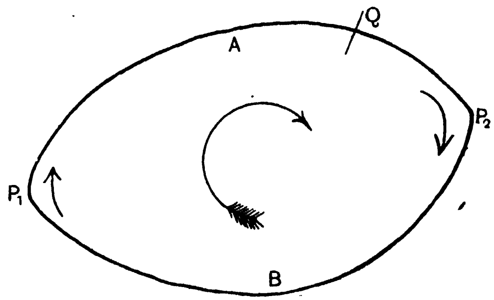


Fig. 65

It is possible to arrange all the metals in a series so that in a circuit of the type described the current flows across the warmer junction in the direction from the metal higher in the series to that which is lower down. A few of the metals arranged in this order are : bismuth, platinum, lead, copper, gold, silver, zinc, antimony.

342. If the circuit just described is broken at a point Q and if the junctions P_1 and P_2 are maintained at the different temperatures θ_1 and θ_2 the same cause which causes the current to flow in the closed circuit will create a potential difference between the ends of the broken circuit even if these ends themselves are at the same temperatures. This potential difference which may be directly measured will then serve as a measure of the electromotive force in the circuit. It appears that for small differences of temperature the electromotive force is proportional to $(\theta_1 - \theta_2)$ but for larger differences the simple proportionality is not even approximately verified.

These thermoelectric currents of course obey Ohm's law as regards the relation between the electromotive force and current and they also obey the series laws relating to compound circuits similar to those formulated for the volta potential difference.

* Seebeck, *Gilbert's Ann.* 73 (1823), pp. 115 and 430; *Pogg. Ann.* 6 (1823), pp. 1, 133, 253.

343. In order to explain these phenomena we can suppose that the heat motion in the junction between the metals A and B drives the electricity towards the former metal (which we may take to be antimony, B is bismuth), with a force which is a function of the temperature. Then as long as the junctions P_1 and P_2 are kept at the same temperature the two electromotive forces at these junctions (indicated by small arrows in Fig. 65) are equal and opposite and therefore neutralise one another. On warming or cooling one of the junctions relatively to the other this equilibrium is disturbed and there is a resulting electromotive force in the circuit.

If in the case exhibited above the warming of the junction P_1 causes a current to flow in the direction of the larger arrow we should expect that at this junction heat would be used up in order to provide energy for the current flux. This is actually the case: Peltier* observed for instance that if a current is driven by an applied electromotive force across a junction between Bismuth and Antimony in the direction $B - A$ the junction is always cooled: whereas if the current is driven in the opposite direction the junction is warmed. This local development or absorption of heat is of course a function of the temperature and if the complete circuit contains two such junctions at the same temperature the development of heat at one junction exactly balances the absorption at the other, whereas if the junctions are at different temperatures so that more heat is absorbed than developed the heat which disappears will appear elsewhere in the circuit either as Joule's heat or as heat of chemical transformations or it may be used up in other more effective ways as purely electrical energy.

344. Next let us take a simple circuit again consisting now of a piece of iron and a piece of copper and steadily heat the one junction up. The electromotive force in the circuit gradually increases, at first proportional to the temperature, but subsequently more slowly until the hot junction reaches the temperature 280°, beyond which point an increase of temperature reduces the electromotive force, finally changing its sign after passing through the zero value†. We might again explain this in terms of our local electromotive forces at the junctions which we may now call ϕ_1 and ϕ_2 at the respective junctions. The electromotive force in the circuit is

$$\phi_{21} = \phi_2 - \phi_1.$$

Now suppose that at ordinary temperatures with $\theta_1 > \theta_2$ then $\phi_1 < \phi_2$ so that ϕ_{21} is positive, and that ϕ_1 decreases as the temperature θ_1 is increased attaining however a minimum value zero at the temperature $\theta_1 = 280°$ after which it gradually increases again, then ϕ_{21} would behave exactly as described, so that so far the explanation is effective.

* *Ann. de Chim. et de Phys.* 56 (1834), p. 371; *Pogg. Ann.* 43, p. 324.

† Cumming, *Annals of Philosophy*, 6 (1823), p. 427.

The temperature $$\theta = 280° \text{ C.}$$
is called the neutral temperature of the copper-iron combination of metals. Similar neutral temperatures exist for all combinations of metals. Moreover it is found that at the neutral temperature there is no Peltier effect, so that if an electric current is passed across the junction at this temperature then no development or absorption of heat takes place. This agrees generally with our explanation for it is only when the local electromotive force at the junction is zero ($\phi_1 = 0$) that there is no development or absorption of heat.

345. Suppose now we have a circuit of the kind described in which the first junction has a temperature equal to the neutral temperature, i.e. $\phi_1 = 0$. The total electromotive force in the circuit is then
$$+\phi_2.$$
The direction of the current will then be that in which the Peltier effect at the upper temperature (θ_1) exists as a heat absorption and at the lower temperature as a heat development. But in the present instance no heat absorption does take place at the temperature θ_1. In spite of this however electrical energy will be transformed into heat energy at the lower junction and developed there and in addition there will be the usual development of Joule's heat. We are therefore driven to the conclusion that at some position in the circuit other than at the junctions heat must be absorbed and transformed into electrical energy. This implies that even in a single wire whose ends are unequally heated an electromotive force may arise as the result of an absorption of heat and if this phenomenon is reversible there must be an additional development of heat in the circuit when a current passes along an unequally heated conductor. This phenomenon was theoretically predicted by Kelvin* and he was soon enabled experimentally to justify the prediction: it is therefore usually known as the Kelvin or Thomson thermoelectric effect.

346. The effectiveness of the explanation here suggested for these phenomena is supported and further substantiated by the application to the transformations of energy of which they are the expression of the general laws of thermodynamics governing all such transformations. There are however difficulties of a fundamental nature involved in any such application in the present case: it may in fact be argued that the whole thermodynamic procedure may be invalid because it is applied to a case in which degradation is continually going on, in the form of conduction of heat, along the same circuit which conducts the current, and of amount depending on the first power of the temperature differences: and it does not appear that this fundamental objection to the procedure can be safely ignored, considering that conductivity for heat is closely connected with conductivity for electricity. It would of course be removed if the heat conduction

* *Phil. Mag.* [4], 11 (1856), pp. 214 and 281.

proceeds in entire independence of the electric current, except as regards the transfer of the electric elements, the influence of which is reversible and is taken into account in the Kelvin effect. The electric cycle can moreover be completed in so short a time that the thermal transfer by ordinary conduction may possibly be neglected. Waiving these difficulties however the argument of Lord Kelvin* may be put in the following form†.

Suppose that in the transfer of the amount $\delta\theta$ of electricity from a place where the temperature is θ to a place where it is $\theta + \delta\theta$ the amount $\sigma\delta\theta\delta Q$ of heat is absorbed by the current and converted into electrical energy of motion of the electricity; σ is called the 'specific heat of electricity' for the conductor and it may be either positive or negative.

347. Let us then—ignoring the finite degradation by heat conduction, but realising that the electric flow may be made so slow that the electric degradation, proportional to the square of the current, is negligible and that therefore the operations are certainly electrically reversible in Carnot s sense—apply the principle of energy and Carnot's principle to a circuit, formed of two metals and including as a part of itself the dielectric of a condenser having these metals for its coatings, the temperature θ varying from point to point along the circuit. When the plates of the condenser are moved closer together without alteration of temperature its charge increases, as the difference of potential ϕ between the plates remains constant: so that there is an electric flow round the circuit and there is at the same time a gain of mechanical work and of available energy each equal to $\frac{1}{2}\phi\delta Q$, or in all ϕ per unit total flow. Thus the plates of the condenser (Fig. 66) being at the same temperature θ_2, we have, by the energy principle and Carnot's principle, considering unit electric flow round the circuit

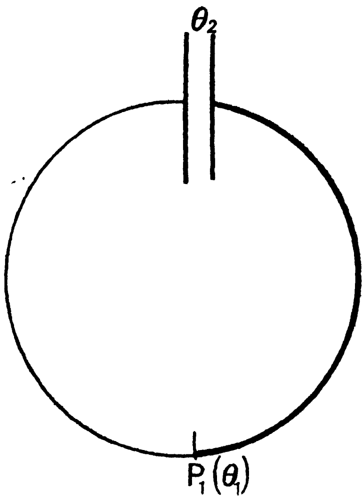


Fig. 66

$$\phi = \Pi_1 + \int_{\theta_1}^{\theta_2} (\sigma - \sigma')\, d\theta,$$

$$0 = \frac{\Pi_1}{\theta_1} + \int_{\theta_1}^{\theta_2} \left(\frac{\sigma}{\theta} - \frac{\sigma'}{\theta}\right) d\theta,$$

* The thermodynamic reasoning was first developed by Clausius (*Pogg. Ann.* 90 (1853), p. 513), but in an incomplete form.

† Cf. Larmor, *Aether and Matter*, p. 306.

where Π_1 is the Peltier effect at the temperature θ_1 of the junction of the two metals, that is the amount of heat absorbed or set free on the passage of unit current across the junction at this temperature; and σ, σ' are the specific heats of electricity in them. Thus

$$\sigma_1 - \sigma_1' = \theta_1 \frac{d}{d\theta_1}\left(\frac{\Pi}{\theta_1}\right),$$

and
$$\phi = \Pi - \int^{\theta_1} \theta \frac{d}{d\theta}\left(\frac{\Pi}{\theta}\right) d\theta = \int^{\theta_1} \frac{\Pi}{\theta}\, d\theta.$$

Hence for a temperature θ of the junction, everything can be expressed in terms of the curve connecting the electromotive force ϕ of the circuit with θ, by the simple relations

$$\frac{\Pi}{\theta} = \frac{d\phi}{d\theta}, \qquad \frac{\sigma - \sigma'}{\theta} = \frac{d^2\phi}{d\theta^2}.$$

The Peltier effect appears in the expression for ϕ, in the form of an electromotive force at the junction, as surmised in the simple explanation offered above. The chemical mutual attractions of the molecules across the interface produce in fact a polar electric orientation of these molecules which gives rise to an abrupt potential difference of contact equal to Π, and each unit of charge passing across the junction thus introduces an energy effect Π which involves absorption or evolution of heat at that place in the Peltier manner. The other term in the potential, viz. $\int (\sigma - \sigma')\, d\theta$ is thermodynamically involved in a convection of heat by the current passing from a warmer to a colder part of the wire: the exact mode in which this arises will appear better in our next discussions on the mechanism of metallic conduction.

348. The mechanism of metallic conduction. Many attempts have been made to develop into further detail the idea of the electric current as a process of diffusion, but before the introduction of the electron theory these attempts could hardly be described as very successful. The difficulty arises in the absence of any sign of transport of matter or of any chemical change accompanying the transport of electricity. Every current circuit must consist of two or more portions composed of different materials which may contain no element in common and we must suppose either that the particles which carry the current can pass from one material to the other or that they cannot. Either supposition lands us in enormous difficulties. If the particles cannot cross the boundary between the different metallic conductors they must remain piled up at these boundaries, even if they are identical with the atoms of the material in which they move, and this piling up must alter the distribution of the mass in the conductor, while, if they are composed of some substance different from that of the rest of the material, some sort of chemical separation should occur. But the most careful experiments on pure metals

and alloys have failed to show the slightest change in any properties of a metallic conductor induced by the passage of a current through it. On the other hand if we suppose that the particles can pass freely from one material to another new difficulties arise. We know that the atoms and groups of atoms of different elements differ markedly in their properties : and we could certainly detect the presence in one substance of atoms derived from a foreign element. Since the properties of the materials forming a non-uniform circuit are unchanged by the passage of the current, if the electricity is conveyed at all by diffusing particles, these particles must be of a nature common to all elements, or at least to all elements forming metallic conductors. Previous to the discovery of the electron however no such electrical elements were known to exist.

The discovery of the electron and the consequent formulation of the so-called 'electron theory' removed in one stroke all the difficulties thus inherent in the earlier theories. According to this theory* there are in every metal a very large number of (negative) electrons freely movable in the spaces between the atoms, and it is the diffusion of these electrons through the metal under the action of the electric force in the external field that is the essence of a conduction current. If there is no external field the velocities of these free electrons will be distributed equally in all directions: there will be no tendency for them to move in one direction rather than in any other; but if the metal is placed in an electric field the electrons are subject to a force in a definite direction (that of the force in the field) in virtue of their charge, and those moving in this one direction will have their velocities increased whilst those moving in the opposite direction will have theirs decreased, there will thus be a slight drift of the electrons in a definite direction and this constitutes a current of electricity : the velocity of drift is however kept in check by the continued encounters of the electrons with the metal molecules and with one another when additional forces come into play tending to deflect the electrons from their forward motion : the essential conditions for a diffusion flux are thus satisfied.

349. Problems relating to the motion of the innumerable number of electrons in a piece of metal are best treated by the statistical method which Maxwell introduced into the kinetic theory of gases, and which may be represented in a simple geometrical form so long as we are concerned only with the motion of translation of the electrons. Indeed it is clear that, if we construct a diagram in which the velocity of each electron is represented in direction and magnitude by a vector OP drawn from a fixed point O, the

* Cf. the original papers by Riecke, *Wied. Ann.* 66 (1898), pp. 353, 545, 1199; Drude, *Ann. der Phys.* (1900), 1, p. 566; 3, p. 369; Thomson, *Rapports de Congrès de Physique* (Paris, 1900), 3, p. 318. The treatment here given follows that given by Lorentz, *The Theory of Electrons*, p. 266, or *Proc. Amsterdam Academy*, 7 (1905), pp. 438, 585.

distribution of the ends of these vectors, the velocity points as we shall say, will give us an image of the state of the motion of the electrons.

If the positions of the velocity points are referred to axes of coordinates parallel to those that have been chosen in the metal itself, the coordinates of a velocity point are equal to the components ξ, η, ζ of the velocity of the corresponding electron.

Let dv be an element of volume in the diagram, situated at the point (ξ, η, ζ) so small that we may neglect the changes of (ξ, η, ζ) from one of its points to another, but yet so large that it contains a great number of velocity points. Then this number may be reckoned to be proportional to dv: representing it by

$$f(\xi, \eta, \zeta)\, dv$$

per unit volume of the metal, we may say that, from a statistical point of view f determines completely the motion of the swarm of electrons.

It is clear that the integral

$$\int f(\xi, \eta, \zeta)\, dv,$$

extended over the whole space of the diagram, gives the total number of electrons per unit of volume. In like manner

$$\int \xi f(\xi, \eta, \zeta)\, dv,$$

represents the stream of electrons through a plane perpendicular to Ox: i.e. the excess of the number passing through the plane towards the positive side over the number of those which go in the opposite direction, both numbers being referred to unit of area and unit of time. This is seen by considering first a group of electrons having their velocity points in an element dv; these may be regarded as moving with equal velocities, and those of them which pass through an element df of area in the said direction between the moments t and $t + dt$, have been situated at the beginning of this interval in a certain cylinder having df for its base, and the height ξdt. The number of these particles is found if one multiplies the volume of the cylinder by the number per unit volume.

Hence if $\int_+$ means an integration over the part of the diagram on the positive side of the η-ζ plane, and $\int_-$ an integration over the part on the opposite side, the number of the electrons which go to one side is

$$df dt \int_+ \xi f(\xi, \eta, \zeta)\, dv,$$

and that of the particles going the opposite way

$$df dt \int_- - \xi f(\xi, \eta, \zeta)\, dv.$$

The expression given above is the difference between these values divided by $dfdt$.

If all the electrons have equal charges e, the excess of the charge that is carried towards the positive side over that which is transported in the opposite direction is given by

$$C = e\int \xi f dv,$$

and it is easily seen that if we use

$$u^2 \equiv \xi^2 + \eta^2 + \zeta^2,$$

for the square of the absolute velocity of an electron, then

$$H = \tfrac{1}{2} m \int \xi u^2 f dv,$$

is the expression for the difference between the amounts of energy that are carried through the plane in the opposite direction.

The function f is to be determined by an equation that is to be regarded as the fundamental equation of the theory, and which we now proceed to establish on the assumption that the electrons are subject to a force giving them an acceleration $\mathbf{F}$ equal for all the electrons in one of the groups considered.

350. Let us fix our attention on the electrons lying, at the time t in an element of volume dv of the metal and having their velocity points in the element dv of the diagram. If there were no encounters of the electrons, neither with other electrons nor with the metallic atoms, these electrons would be found, at the time $t + dt$, in an element dv' equal to dv and lying at the point $(x + \xi dt,\ y + \eta dt,\ z + \zeta dt)$. At the same time their velocity points would have been displaced to an element dv' equal to dv and situated at the point $(\xi + \mathbf{F}_x dt,\ \eta + \mathbf{F}_y dt,\ \zeta + \mathbf{F}_z dt)$. We should have therefore

$$f(\xi + \mathbf{F}_x dt,\ \eta + \mathbf{F}_y dt,\ \zeta + \mathbf{F}_z dt,\ x + \xi dt,\ y + \eta dt,\ z + \zeta dt,\ t + dt)\, dv' dv'$$
$$= f(\xi,\ \eta,\ \zeta,\ x,\ y,\ z,\ t)\, dv dv.$$

The encounters or impacts which take place during the interval of time considered require us to modify this equation. The number of electrons constituting at the time $t + dt$, the group specified by $dv'dv'$, is no longer equal to the number of those which at the time t, belonged to the group $dvdv$, the latter number having to be diminished by the number of impacts which the group of electrons under consideration undergoes during the time dt and increased by the number of impacts by which an electron, originally not belonging to the group, is made to enter it. Writing $adv dv dt$ and $bdv dv dt$ for these two numbers we have, after division by $dvdv = dv'dv'$

$$f(\xi + \mathbf{F}_x dt,\ \eta + \mathbf{F}_y dt,\ \zeta + \mathbf{F}_z dt,\ x + \xi dt,\ y + \eta dt,\ z + \zeta dt,\ t + dt)$$
$$= f(\xi,\ \eta,\ \zeta,\ x,\ y,\ z,\ t) + (b - a),$$

or what is practically the same thing

$$\mathbf{F}_x \frac{\partial f}{\partial \xi} + \mathbf{F}_y \frac{\partial f}{\partial \eta} + \mathbf{F}_z \frac{\partial f}{\partial \zeta} + \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} + \frac{\partial f}{\partial t} = b - a.$$

This is the general equation of which we have spoken. It remains to calculate a and b or at least the difference $(a - b)$ and here the difficulties begin: we can however simplify the problem if we neglect the mutual encounters of the electrons, considering only their impacts against the metallic atoms, whose masses are so great that they may be regarded as immovable: we shall further treat both the electrons and atoms as perfectly elastic spheres so that the velocity of any electron at any two instants separated by at least one collision are wholly independent of one another*.

351. Now† in the absence of any extraneous forces a steady state of perfectly chaotic motion will soon be established among the electrons and one in which $f(\xi, \eta, \zeta, x, y, z, t) \equiv f_0 (\xi, \eta, \zeta, x, y, z, t)$ and on the assumptions just made it seems reasonable to suppose that this distribution will be similar to that which exists under similar circumstances in gas theory and is specified by Maxwell's law so that we may take

$$f = f_0 = Ae^{-qu^2},$$

where
$$A = N \sqrt{\frac{q^3}{\pi^3}}, \qquad q = \frac{3}{2u_m{}^2},$$

where N denotes the number of free electrons per unit volume and $u_m{}^2$ the mean square of their velocities. This is the perfectly chaotic distribution of motions and any departure from it arises as a result of the external forces or condition gradients tending to organise the perfect irregularity which this law specifies.

Now since the velocity of any electron after a collision is independent of that before collision it follows that the distribution of velocities among any set of electrons when taken each just after its next collision succeeding the instant t will be wholly independent of the distribution at the time t and will in general be different from it unless indeed this latter distribution is that specified by Maxwell's law which is specially defined so as to remain unaltered by collision. It follows therefore also that the distribution of velocities among any set of electrons, each taken immediately after its next collision after the instant t will in fact be precisely that specified by Maxwell's law and is therefore the same independently of the state of motion that may exist at the instant t: in other words the collisions completely obliterate

* It has been found possible to generalise the theory to the more probable case where the interaction in collision is like that between centres of force. Cf. Bohr, "Studier over metallernes elektrontheori" (*Dissertation*, Copenhagen, 1911); Richardson, *Phil. Mag.* July, 1912; and various papers by the author in the same magazine during 1915.

† The present form of the argument was sketched by the author *Phil. Mag.* 30 (1915), pp. 112–124. Cf. also pp. 549–559.

any regularity which existed in the electronic motions before collision. Thus the number ($bdvdt$) of electrons which enter the specified group during the small time dt is precisely the same as the number which would enter the same group if Maxwell's law specified the distribution both before and after the collision and it might be calculated on this basis. The number $advdt$ of electrons leaving the group in the same time would then be exactly the same as bdt if there were no external forces or condition gradients to modify the distribution established by the collisions. In the more general case it is however at once obvious that the number

$$(a - b)\, dvdt,$$

can be calculated as the number of electrons removed by collision during the time dt from among the partial group of electrons contained in the specified group at the instant t, which is the excess of the number in this group over and above the number required by Maxwell's law: that is the number

$$(f - f_0)\, dvdt.$$

We thus want to find the number of collisions which the electrons of this group undergo in the small time dt. Each of the electrons in the group is moving along a zig-zag path with the definite velocity u. Let us fix our attention on one of these electrons and calculate the chance of its colliding with an atom at rest in a unit of time. This chance is obviously equal to the number of atoms in a cylinder of base πR^2 and height u, R being the sum of the radii of an atom and an electron; it is therefore equal to

$$n\pi R^2 u,$$

n being the number of atoms per cubic centimetre in the metal.

But in unit time the electron under consideration travels a distance u, hence the chance of a collision of the electron with an atom per unit length of its path is

$$C = \frac{n\pi R^2 u}{u} = n\pi R^2,$$

and thus the mean free path of an electron is, as before,

$$l_m = \frac{1}{C} = \frac{1}{n\pi R^2},$$

and is independent of u.

The chance of a collision in the time dt is thus

$$Cu\,dt = \frac{u\,dt}{l_m} = \frac{dt}{\tau_m},$$

where $\tau_m = \frac{l_m}{u}$ is the mean time in a free path: thus the number of collisions in the group of electrons specified during the time dt is

$$(f - f_0)\, dv \frac{dt}{\tau_m},$$

and thus $$b - a = -\frac{(f - f_0)}{\tau_m}.$$

352. It follows therefore that the equation for the function f in the most general possible case of the present type, is

$$\mathbf{F}_x \frac{\partial f}{\partial \xi} + \mathbf{F}_y \frac{\partial f}{\partial \eta} + \mathbf{F}_z \frac{\partial f}{\partial \zeta} + \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} + \frac{\partial f}{\partial t} = \frac{f_0}{\tau_m} - \frac{f}{\tau_m}.$$

Now the difference between the functions f and f_0 may be shown to be extremely small in any real case, at least compared with f_0 itself and we may therefore use $f = f_0$ on the left-hand side of this equation so that we get

$$f = f_0 - \tau_m \left(\mathbf{F}_x \frac{\partial f_0}{\partial \xi} + \mathbf{F}_y \frac{\partial f_0}{\partial \eta} + \mathbf{F}_z \frac{\partial f_0}{\partial \zeta} + \xi \frac{\partial f_0}{\partial x} + \eta \frac{\partial f_0}{\partial y} + \zeta \frac{\partial f_0}{\partial z} + \frac{\partial f_0}{\partial t}\right),$$

where of course f_0 has the value quoted above: on inserting this we find that

$$f = Ae^{-qu^2}\left[1 + 2q\tau_m (\xi \mathbf{F}_x + \eta \mathbf{F}_y + \zeta \mathbf{F}_z) - \frac{\tau_m}{A}\left(\xi \frac{\partial A}{\partial x} + \eta \frac{\partial A}{\partial y} + \zeta \frac{\partial A}{\partial z}\right)\right.$$
$$\left. + u^2 \tau_m \left(\xi \frac{\partial q}{\partial x} + \eta \frac{\partial q}{\partial y} + \zeta \frac{\partial q}{\partial z}\right)\right],$$

or in vector notation, using $\mathbf{u}$ as the vector velocity of the electron

$$f = Ae^{-qu^2}\left[1 + 2q\tau_m (\mathbf{uF}) - \frac{\tau_m}{A}(\mathbf{u}\nabla) A + \tau_m u^2 (u\nabla) q\right].$$

The density of the electric flux is then determined by its vectors

$$(\mathbf{C}_x, \mathbf{C}_y, \mathbf{C}_z) = e\iiint_{-\infty}^{+\infty} (\xi, \eta, \zeta) f d\xi d\eta d\zeta,$$

whilst the flux of kinetic energy is determined by the vector with components

$$(\mathbf{H}_x, \mathbf{H}_y, \mathbf{H}_z) = \iiint_{-\infty}^{+\infty} \frac{m}{2} u^2 (\xi, \eta, \zeta) f d\xi d\eta d\zeta.$$

The integrals in each of these cases can be directly evaluated by the spherical polar transformation

$$\xi = u\cos\theta, \qquad \eta = u\sin\theta\cos\phi, \qquad \zeta = u\sin\theta\sin\phi,$$

and it is in this way found that

$$\mathbf{C} = \frac{4\pi Ae}{3}\left[i_4\left(2q\mathbf{F} - \frac{1}{A}\operatorname{grad} A\right) + i_6 \operatorname{grad} q\right],$$

whilst
$$\mathbf{H} = \frac{2\pi Am}{3}\left[i_6\left(2q\mathbf{F} - \frac{1}{A}\operatorname{grad} A\right) + i_8 \operatorname{grad} q\right],$$

wherein we have used

$$i_s = \int_0^\infty \tau_m u^s e^{-qu^2} du,$$

but $\tau_m = \frac{l_m}{u}$ where l_m is independent of u so that

$$i_s = l_m \int_0 u^{s-1} e^{-qu^2} du$$
$$= \frac{l_m}{2q^{\frac{s}{2}}} \Gamma\left(\frac{s}{2}\right),$$

and thus
$$\mathbf{C} = \frac{2\pi A e l_m}{3q^2}\left[\left(2q\mathbf{F} - \frac{1}{A}\operatorname{grad} A\right) + \frac{2}{q}\operatorname{grad} q\right],$$
$$\mathbf{H} = \frac{2\pi A m l_m}{3q^3}\left[\left(2q\mathbf{F} - \frac{1}{A}\operatorname{grad} A\right) + \frac{3}{q}\operatorname{grad} q\right]$$

354. Now let us consider the conduction of electricity in a homogeneous bar of the metal which is kept at the same temperature in all its parts: let this metal be acted on by an electric force E in the direction of its length which we may take to be directed along the x-axis. The force acting on each electron will then be eE so that

$$\mathbf{F}_x = \frac{eE}{m}, \qquad \mathbf{F}_y = \mathbf{F}_z = 0,$$

and since the physical conditions of the metal are the same throughout its mass the quantities A and q which depend on these will be constants so that

$$\operatorname{grad} A = \operatorname{grad} q = 0.$$

We have thus in this case a current of electricity defined by its flow per unit area across a section of the bar

$$C = \frac{4\pi A e^2 l_m}{3mq} E,$$

from which we may conclude that the conductivity of the metal under the specified conditions is

$$\kappa = \frac{4\pi A e^2 l_m}{3mq}.$$

This formula was first given by Lorentz.

354. In order however to exhibit all the beauties of this theory it is necessary to consider the question of the conduction of heat in the metal. A bar of metal whose ends are maintained at different temperatures may be likened to a column of gas, placed for example in a vertical position and having a higher temperature at its top than at its base. The process by which the gas conducts heat consists in a kind of diffusion between the upper part of the column, in which we find larger, and the lower one in which we find smaller molecular velocities; the amount of this diffusion and the intensity of the flow of heat that results from it, depend on the mean distance over which a molecule travels between two successive encounters. Now in the present theory of metals it seems at least plausible to assume that the conduction of heat goes on in a way that is exactly similar to that just described, only the carriers by which the heat is transformed from the hotter towards the colder parts of the body, are now the free electrons, and the length of their free path is limited, not, as in the case of a gas by mutual encounters, but by the impacts against the metallic atoms, which we have supposed to remain at rest on account of their comparatively large mass.

Of course this idea seems to imply that some sort of thermodynamical equilibrium exists between the metal molecules and the electrons, so that the latter are partaking of a real heat motion. It is usual to assume that this equilibrium is of the simple type that exists between different kinds of molecules in a compound gas so that the mean kinetic energy of the electron expressed in the above notation by

$$\tfrac{1}{2}mu_m^2 = \frac{3m}{4q},$$

is determined by the simple law of the equality of mean energies. This means that if R is the universal gas constant and θ the absolute temperature of the metal we may write

$$\tfrac{1}{2}mu_m^2 = \frac{3R\theta}{2},$$

so that the relation

$$q = \frac{m}{2R\theta},$$

determines the dependence of q on the temperature.

355. In the general case it is not possible to determine a definite value for the quantity of energy of the irregular or chaotic part of the electronic motions which is transferred during their average flux, because the energy of each separate electron being in part kinetic energy and in part potential energy relative to the metal molecules and the external field is known only to an additive constant. In one special case however when the aggregate flux of the electrons vanishes, so that there is no flow of electricity, will this indefiniteness disappear and the flux of energy through the metal is completely determined by the vector $\mathbf{H}$ given above. As in this case the electric current is zero we must have the condition

$$2q\mathbf{F} - \frac{1}{A}\operatorname{grad} A + \frac{2}{q}\operatorname{grad} q = 0,$$

and thus

$$\mathbf{H} = \frac{2\pi m l_m A}{3q^4}\operatorname{grad} q.$$

We have however from above

$$\operatorname{grad} q = -\frac{2Rq^2}{m}\operatorname{grad}\theta,$$

so that

$$\mathbf{H} = -\frac{4\pi A l_m R}{3q^2}\operatorname{grad}\theta,$$

and the conductivity for heat, as usually defined is therefore in this case given by

$$\gamma = \frac{4\pi A l_m R}{3q^2} = \frac{2\kappa R^2\theta}{e^2},$$

and it depends only on the nature and physical conditions at the point in the metal under consideration.

356. The ratio of the conductivities of heat and electricity is

$$\frac{\gamma}{\kappa} = \frac{2R^2\theta}{e^2},$$

and is therefore the same in all metals at the same temperature. This is the well-known Wiedemann-Franz law which has been successfully verified in numerous cases*: a few typical ones are exhibited below:

Aluminium	$\cdot 706 . 10^{-10}$
Copper	$\cdot 738 . 10^{-10}$
Silver	$\cdot 760 . 10^{-10}$
Iron	$\cdot 890 . 10^{-10}$

and seeing that the specific electrical resistances of these substances range from $\cdot 3 . 10^{-4}$ to $64 . 10^{-4}$ the agreement is remarkable.

The ratio of the two conductivities should also on the present theory vary as the absolute temperature, and this is a law which had been empirically formulated by Lorentz: the temperature coefficients of the ratio in the four cases quoted are respectively 4·37, 3·95, 3·77, 4·32.

But not only are these general qualitative results satisfactorily verified by our theory: the agreement is quantitative as well: in fact it is known from gas theory that

$$\frac{R}{e} = 2\cdot 8 . 10^{-7},$$

where e is the electrostatic charge on the electron and thus

$$\frac{\gamma}{\kappa} = 16 . 10^{-14} . \theta,$$

or taking an absolute temperature of 300° (i.e. 27° C.) this gives

$$\frac{\gamma}{\kappa} = \cdot 48 . 10^{-10},$$

which is of the same order of magnitude as those quoted above†.

This agreement between the theory and experiment, first noticed by Drude, is one of the most beautiful results of this theory and points distinctly to the conclusion that the assumption that both electricity and heat are carried through the metal by the electrons, and that these electrons are in a simple mechanical heat equilibrium with the metal molecules is completely justified.

357. In the preceding paragraphs we have considered two special cases of the transfer of heat and electricity in a homogeneous piece of metal. We shall finally consider briefly the more general case which leads to an

* Full details of the experimental results to 1911 are given by K. Baedeker, *Die elektrischen Erschienungen in metallischer Leitern* (Braunschweig, 1911).

† Still better agreement is obtained in the more general theory when the law of force between the molecules and electrons is the inverse cube law.

explanation of the full circumstances in the thermoelectric phenomena discussed above. For the sake of generality we shall introduce the notion of molecular forces of one kind or another exerted by the atoms of the metal on the electrons and producing for each electron a resulting force along the direction in which the metal is not homogeneous; this is the main point of the idea of Helmholtz's assumption that each substance has a specific affinity for electricity which varies with the temperature: these forces will be assumed to be such that the typical electron will on the average have a potential energy $e\mu$ under the standard conditions relative to the metal molecules surrounding it. Thus if we now assume that the impressed field is derived from a potential ϕ we shall have

$$\mathbf{F} = -\frac{e}{m}\,\mathrm{grad}\,(\phi + \mu),$$

so that we have the currents of electricity and heat

$$\mathbf{C} = -\kappa\left[\mathrm{grad}\,(\phi + \mu) + \frac{R\theta}{Ae}\,\mathrm{grad}\,A + \frac{2R}{e}\,\mathrm{grad}\,\theta\right],$$

and
$$\mathbf{H} = \frac{2R\theta}{e}\,\mathbf{C} - \gamma\,\mathrm{grad}\,\theta.$$

From the first of these expressions we may deduce expressions for the rate of fall of potential at each point and for the difference of potential between the ends of the bar examined in the previous paragraph. It is however more interesting to make the calculations for a more general case. We therefore consider a circuit consisting of a thin curved wire, the dimensions of the normal section at any point of which are small compared with the radius of curvature of the curve of the wire at the point. We may then assume that the nature and temperature of the metal are very nearly the same at all points of a single cross section and we shall therefore only be concerned with the component fluxes tangentially along the wire, the other component being negligibly small. If we use s to denote a coordinate of distance along the wire the equations for these fluxes are

$$\mathbf{C}_s = -\kappa\left[\frac{d}{ds}(\phi + \mu) + \frac{R\theta}{Ae}\frac{dA}{ds} + \frac{2R}{e}\frac{d\theta}{ds}\right],$$

and
$$\mathbf{H}_s = \frac{2R\theta}{e}\,C_s - \gamma\frac{d\theta}{ds}.$$

We now examine special cases of these equations.

358. (1) In an open circuit in which no current is flowing there is a potential difference between the ends which may be regarded as a measure of the electromotive force existing in the circuit when closed. In this case we have

$$\frac{d\phi}{ds} = -\frac{d\mu}{ds} - \frac{R\theta}{Ae}\frac{dA}{ds} - \frac{2R}{e}\frac{d\theta}{ds},$$

so that on integration along the circuit from the point s_1 to the point s_2 we get

$$\phi_1 - \phi_2 = -(\mu_1 - \mu_2) - \frac{R}{e}\int_{s_1}^{s_2} \frac{\theta}{A}\frac{dA}{ds}\, ds - \frac{2R}{e}(\theta_1 - \theta_2).$$

If the temperature is uniform along the circuit this gives

$$\phi_1 - \phi_2 = -(\mu_1 - \mu_2) - \frac{R\theta}{e}\log\frac{A_1}{A_2}.$$

This equation shows at once that the potential difference between any two points of the circuit will depend only on the nature of the metal at these two points and will be zero if these metals are the same, for then $\mu_1 = \mu_2$ and $A_1 = A_2$, these two quantities depending essentially on the character and conditions of the metal and nothing else. The explanation of the volta potential differences and the laws which it obeys is now obvious: the difference in the potential between the ends of a compound circuit may be due either to a difference in the "affinity" potential μ of the electron in the metals at the ends or to a difference in the concentration of the electrons at these ends.

(2) We now consider an open circuit consisting of one kind of metal only but in which the temperature is not uniform. In this case A and μ will be functions of the temperature only so that the expression for $d\phi$ will be an exact differential with respect to θ: again the potential difference between two points of the circuit will only be dependent on the temperature of these points and will be zero if these are the same:

$$\frac{d\phi}{ds} = -\frac{d\mu}{ds} - \frac{R\theta}{Ae}\frac{dA}{ds} - \frac{2R}{e}\frac{d\theta}{ds},$$

$$\phi_1 - \phi_2 = -\int_1^2 \left(\frac{d\mu}{d\theta} + \frac{R\theta}{Ae}\frac{dA}{d\theta} - \frac{2R}{e}\right)\frac{d\theta}{ds}\, ds.$$

359. (3) We finally consider the more general case of a non-homogeneous circuit in which the temperature varies. We shall confine ourselves to the consideration of a circuit such as that described above in the text in which three pieces of metal are joined up in a circuit; the first and third pieces being however of the same material so that if they were joined up the circuit would in reality consist of only two pieces of metal. The junctions are P_1 and P_2 and we shall suppose that they are at temperatures θ_1 and θ_2 and also that the temperatures at the two ends of the circuit are both θ_0. Still retaining the coordinates s_1 and s_2 for these ends we have

$$\phi_1 - \phi_2 = -\int_1^2 \left(\frac{d\mu}{ds} + \frac{R\theta}{Ae}\frac{dA}{ds} + \frac{2R}{e}\frac{d\theta}{ds}\right) ds.$$

Now $$\int_{s_1}^{s_2} \frac{\theta}{A}\frac{dA}{ds}\, ds = \Big|\, \theta\log A\, \Big|_1^2 - \frac{R}{e}\int_{s_1}^{s_2} \log A\, \frac{d\theta}{ds},$$

and the integrated term vanishes because the metal and temperature at the ends 1 and 2 are the same: thus

$$\phi_1 - \phi_2 = -\int_1^2 \left(\frac{d\mu}{ds} + \frac{R\theta}{Ae}\frac{dA}{ds} + \frac{2R}{e}\frac{d\theta}{ds}\right) ds$$

$$= -\int_1^2 \left(\frac{d\mu}{d\theta} - \frac{R}{e}\log A + \frac{2R}{e}\right)\frac{d\theta}{ds}\, ds$$

$$= \int_{\theta_0}^{\theta_1} \left(\frac{d\mu}{d\theta} - \frac{R}{e}\log A + \frac{2R}{e}\right) d\theta$$

$$+ \int_{\theta_1}^{\theta_2} \left(\frac{d\mu}{d\theta} - \frac{R}{e}\log A + \frac{2R}{e}\right) d\theta$$

$$+ \int_{\theta_2}^{\theta_0} \left(\frac{d\mu}{d\theta} - \frac{R}{e}\log A + \frac{2R}{e}\right) d\theta$$

$$= \int_{\theta_2}^{\theta_1} \left(\frac{d\mu}{d\theta} - \frac{R}{e}\log A + \frac{2R}{e}\right) d\theta$$

$$- \int_{\theta_1}^{\theta_2} \left(\frac{d\mu}{d\theta} - \frac{R}{e}\log A + \frac{2R}{e}\right) d\theta,$$

and the first integral now refers to the one type of metal and the second to the other so that the integrands are proper functions of the temperature. Using suffices a, b to denote quantities referring to the different metals we see that

$$\phi_{12} = \phi_1 - \phi_2 = \int_{t_1}^{\theta_2} \left(\frac{d\mu_a}{d\theta} - \frac{d\mu_b}{d\theta} - \frac{R}{e}\log\frac{A_a}{A_b}\right) d\theta.$$

The potential difference between the ends of the circuit depends therefore merely on the temperatures at the junctions of the two different metals and vanishes if these are equal.

360. Let us now examine the development of heat which takes place in the same circuit when the current is allowed to flow round it. To do this we shall assume that the conditions at each point of the circuit are stationary, the temperature being suitably maintained constant by conduction (if the circuit consists of a thin wire this may be done without appreciably altering the conditions under which we are treating these questions).

We now consider a small element of the circuit between the cross sections at distances s and $s + ds$ from the origin on the circuit and we find the quantity of heat dH which must be extracted from such an element to maintain its temperature constant. This quantity will of course be equal to the difference between the amount of energy supplied to the electrons in the element on account of the extraneous forces and the amount which is brought into the element as a result of the electrons diffusing into it from other parts of the circuit. The state of the flow being assumed to be stationary the same number of electrons will flow into the element on one side as will flow out at

the other and the amount of heat can be calculated from the expression for $\mathbf{H}_s$ which determines the amount of electronic energy, consisting in part of kinetic energy and in part of potential energy, which is transferred through unit area of a normal section of the circuit per unit time. If we take the area of the cross section at s to be f then we shall have

$$dH = -\left(f\mathbf{C}_s \frac{d\phi}{ds} + \frac{d}{ds}(f\mathbf{H}_s)\right) ds,$$

or introducing the total current flow J determined by

$$J = f\mathbf{C}_s = -f\kappa \frac{d\phi}{ds},$$

we get
$$dH = \left[-\frac{J^2}{f\kappa} + J\left(\frac{d\mu}{ds} + \frac{R\theta}{\epsilon A}\frac{dA}{ds}\right) - \frac{d}{ds}\left(f\gamma\frac{d\theta}{ds}\right)\right] ds.$$

The first term in this expression, which is proportional to the square of the current strength, indicates of course the Joule's heat developed in the element. The last term, which is independent of the current, indicates the heat supply to the element on account of true thermal conduction in the circuit. The middle term is the important one: it denotes a development of heat in the circuit

$$dH' = J\left[\frac{R\theta}{eA}\frac{dA}{ds} + \frac{d\mu}{ds}\right] ds,$$

which is proportional to the strength of the current and which therefore changes sign when the direction of the current is reversed. It is this term which contains the expression of the Peltier and Thomson effects, as we see by examining two simple cases.

361. (*a*) We first consider a part of the circuit in which the temperature is constant and in which a transition from the metal a to the metal b takes place; this transition is assumed to be gradual and on integration of the expression for dH' across it we find the corresponding amount of heat developed per unit of time by the passage of the current across the junction is equal to

$$J\left[\frac{R\theta}{\epsilon}\left(\log\frac{A_b}{A_a}\right) + (\mu_b - \mu_a)\right],$$

so that the Peltier effect as defined in the text is

$$\frac{R\theta}{e}\left(\log\frac{A_b}{A_a}\right) + (\mu_b - \mu_a).$$

(*b*) Now consider a portion of the circuit within which the metal is the same but in which the temperature varies. We then find in a similar manner that the amount of heat developed per unit of time in a small part of the circuit in which the temperature changes uniformly from θ to $\theta + d\theta$ will be

$$J\left[\frac{R\theta}{Ae}\frac{dA}{d\theta} + \frac{d\mu}{d\theta}\right] d\theta,$$

so that the Thomson effect as defined is

$$-\left(\frac{R\theta}{Ae}\frac{dA}{d\theta}+\frac{d\mu}{d\theta}\right).$$

The expressions thus found for the quantities denoting the potential in a circuit of the type under consideration and the Peltier and Thomson effects are fully consistent with the relations between these quantities deduced by Kelvin from thermodynamical considerations and given above in the text.

This theory moreover effectively accounts for many features of the phenomena which it is rather difficult to explain on any other basis and in particular the phenomena associated with the Thomson effect, which on the present basis becomes almost self-evident.

CHAPTER VIII

ELECTRIC CURRENTS IN LIQUID AND GASEOUS CONDUCTORS

362. Electrolysis. So far we have been dealing with the relations of electric flow without troubling much about the actual generation of that flow. We have, it is true, been led to certain conditions which are essential to the flow, but the exact way in which they are satisfied did not appear. We shall now attempt an exposition of this other side of the subject and explain how a current is actually generated. As a preliminary we must first explain in detail some important facts connected with the flow of currents through liquid conductors*.

Chemical compounds which can exist either as salts or acids, or have the same general characteristics of these bodies, can conduct an electric current when in the liquid form or in solution. In order to observe this it is merely necessary to insert two conductors as electrodes into the liquid or solution and connect them with the poles of a voltaic cell. A current will be found to flow round the circuit and if this experiment is carefully examined it will be found that

(i) No substance can conduct an electric current, without being resolved into two constituents of which the one appears at the positive electrode and the other at the negative. This resolution is called *electrolysis*: the substances which admit of such, *electrolytes*, and the two constituents, the *ions*.

(ii) The metal or the stuff which takes its place in the compounds (Hydrogen in the acids) is always one of these constituents and it always appears at the electrode where the current leaves the electrolyte (the negative electrode or cathode). We call it the *cation*, the other part the *anion*.

(iii) Secondary actions may follow the deposition of the ions on the electrodes, but they are not connected with the current.

If for example copper sulphate ($CuSO_4$) is in solution between copper electrodes; when the current flows one electrode is covered with copper but from the other a portion of the copper combines with the free SO_4 ion again. The total quantity of dissolved copper sulphate thus remains the same but the one electrode increases in weight and the other decreases by the same amount.

* A complete account of the theory of conduction and the correlated phenomena in liquids is given with references by Whetham, *The Theory of Solutions* (Cambridge, 1902).

363. We can obtain a very simple picture of this phenomenon in the following way. Suppose AB and CD are the electrodes and suppose the current flowing from left to right. We can now imagine that one constituent P of the substance remains at rest but that the other moves with the current. There will thus appear a definite quantity of this latter constituent in the free state at the electrode CD and an equivalent amount of P left free at AB; whilst each volume element in the middle of the liquid contains just as at first equal amounts of both constituents.

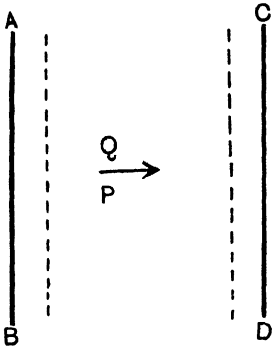


Fig. 67

As far as the resolution of the substance is concerned we might have imagined that the constituent Q was at rest and the part P moved to the left. The phenomenon really only depends on the relative motion of the two constituents and both P and Q might move, which will in general be the case.

Of course the actual motions in the electrolytes are very complicated. Before the current is sent through, each molecule has its irregular heat motion and the constituents may have relative motion inside the molecule. Single molecules may even be already dissociated into atoms or atomic groups and capable of existence as such for a short time. But in all of these phenomena no direction can be chosen for which the motion has a preference. All this however alters as soon as the current crosses the liquid. The atoms of the constituent P will then move to the left in larger numbers or with larger velocities than to the right. Similarly the atoms of Q, although moving in reality in every direction, will have a slight preference for going to the right.

364. In a careful examination of all the circumstances governing this phenomenon of electrolysis Faraday found the following laws to be always true*.

(i) The quantity of a substance which is resolved by an electric current, and thus also the quantity of each constituent which is liberated, is proportional to the current strength.

(ii) If different substances are traversed by equally strong currents, then the quantities resolved in each case are chemically equivalent: the same is true also of the different constituents.

If for example solutions of zinc sulphate and copper sulphate are included in the same circuit, then the same number of molecules of both substances will be deposited and also the same number of atoms of each metal.

* *Experimental Researches*, Ser. 7 (1834), § 662 *et seq.*

On the other hand from dilute sulphuric acid a quantity of hydrogen is liberated which contains twice as many atoms as the quantity of copper which the same current would separate from a solution of copper sulphate. But then one atom of divalent copper is chemically equivalent to two atoms of univalent hydrogen.

365. All of these facts tend to show that in electrolytes the motion of the electricity is invariably connected with the motion of the matter, in as far as that motion is conceived as a displacement of the constituent ions relative to one another. There is only one explanation of this connection; we must assume that of the two parts into which a molecule can be separated (the ions) the metal or corresponding part has a positive charge and the other an equal negative charge. It is thus clear how under the influence of a potential difference in the liquid the metal is driven to the cathode and the other constituent to the anode. We must in addition assume that the ions give up their charges as soon as they reach the electrodes, i.e. they are uncharged as soon as they appear in a free state.

This hypothesis means that the motion of electricity in electrolytes is a convection but with the characteristic that the small particles which convey the current, need not receive any charge to convey, they merely give up what they already have. If this convection is the only kind of electrical motion involved then the quantity of electricity which goes over from the cathode through a connecting wire, is equal to the sum of the charges of all the metal particles which reach the cathode. The first Faraday law is thus explained by assuming that in a definite electrolyte each metal-atom is combined with a definite invariable charge.

As far as the second law is concerned it leads to a consequence which is more than a connection between the chemical and electrical phenomena. For example when two voltameters, the one with copper-sulphate and the other with zinc-sulphate are connected in the same circuit, then equal quantities of electricity go to the cathode in each voltameter. Since this is accomplished by the deposition of the same number of atoms we see that the copper atom in the one salt and the zinc in the other must have equal charges. The negative charges which belong to the SO_4 ion in each molecule are also equal and equal to the positive charge on a copper or zinc atom. Consequently the same group in sulphuric acid has exactly the same charge, so we see that an atom of hydrogen in this acid has a charge equal to half that of a copper atom. Thus the atom of a divalent element carries a charge twice as large as that of a univalent element.

It is these Faraday laws which suggest that electricity like matter has an atomic structure and that the ions are combinations of chemical and electrical atoms; and they were first publicly interpreted in this light by

von Helmholtz*. The suggestion however was not well received at first chiefly owing to the apparent lack of the necessity for it, and it was left for the modern theory of electrons definitely to adopt the conception into electrical theory. The remarkable success which has attended the developments on both the theoretical and experimental side of this theory now hardly leaves any doubt as to the fundamental basis on which it is constructed.

366. The explanation of the action of a voltaic element. We can now attempt an explanation of the action previously ascribed to the voltaic element, viz. that of being able to supply a current. We know that in any such element as soon as a current is flowing there is a chemical action and we must therefore regard these actions or rather the forces which are in play with them as the origin of the electrical motion. We are strengthened in this view by the law of the conservation of energy. This connects the heat developed in the current circuit with the decrease in the energy of the cell. Since new chemical combinations are formed and thereby energy lost, there is no doubt that we must seek for the source of the heat developed in the circuit in this direction and experiment proves that the one perfectly accounts for the other. We thus conclude that the chemical attraction between the atoms, which combines them together, is the origin of the electromotive force which drives the electricity in the element to the electrodes.

How can these chemical attractions create the potential difference in the circuit? How also is it that the chemical actions depend on the presence of a wire outside the cell connecting the two electrodes? Or how is it that a heat development can be found at a place different from that where the process giving rise to it is operative?

Without being able to give a perfect answer to this question we can make a fairly good representation of how the phenomena work. We shall assume that in the electrolytes the ions are electrically charged and that in consequence the motion of the ions and the motion of the electricity are invariably connected with each other. When decomposed these ions can be driven by the electric force; they may however also be moved by forces of another kind: in any case then there is a motion of electricity.

When we dip a copper plate and a zinc plate in sulphuric acid solution without completing the circuit the zinc attracts the SO_4 ions; some molecules of SO_4 will give way to the attraction and combine with the metal. This cannot however proceed very far for the molecules as they combine give up their negative charge to the zinc and so there gradually arises a repelling force between the zinc and the next arriving SO_4 molecules. A condition is in fact very soon attained in which the chemical attraction is in equilibrium

* *Faraday Lecture*, 1881.

with the electric repulsion. A similar process goes on at the copper plate, although the SO_4 ions are attracted much more by the zinc than the copper, so that the copper plate will have a much less negative charge than the zinc when equilibrium is attained.

It is then clear that between the zinc and the copper there will be a potential difference such as is actually observed.

367. We can now if we like restart the chemical action which ceases when equilibrium is attained as above. We have only to put a positive charge on the zinc to neutralise its negative charge. This is accomplished by connecting it by a wire to the copper plate. If this is done the equilibrium is broken. The negative charge on the zinc reduces and thus the attraction between the zinc and the SO_4 ion is greater than the electric repulsion. Thus the SO_4 ion moves towards the zinc and communicates its negative charge to it and this is continually being neutralised by positive charge supplied from the copper through the wire.

It would thus appear that the resultant difference in the attraction of the zinc for the SO_4 and the copper for the SO_4 is the moving force in the circuit.

This idea also provides us with an explanation of the heat development phenomena. When a particle of SO_4 moves towards the zinc, only under the influence of the chemical attraction, it acquires a kinetic energy which we would notice as heat. But in the cell the attraction is practically balanced by the electric repulsion so that the molecule of SO_4 reaches the zinc without a large velocity, i.e. the combination takes place with only a small heat production.

Similar considerations are also true for cells of other types. A chemical attraction between one electrode and the negatively charged element of the surrounding electrolyte always tends to create a current in a definite direction, whilst an opposed current can arise through an attraction between the other electrode and the positive ion. If the electrode attracts both ions it is only a question as to which is the preponderating action. In the general case we should also have to take into account the forces exerted by the second electrode on the ions, so that the phenomena can be and is in reality, more complicated than our above description. We have however been able to illustrate the underlying essential physical principles and our object is thus accomplished.

368. The effectiveness of the explanation of the action of the voltaic cell is further substantiated by results of the application of the general laws of thermodynamics to the thermal transformations which take place in them*.

* Cf. W. Thomson, *Phil. Mag.* [4], 2 (1881), pp. 429, 551. The theory is originally due to Helmholtz and W. Gibbs. Cf. Whetham. *Theory of Solutions*, p. 235.

There are several of the galvanic cell elements in which the chemical transformations consequent on the passage of a current are perfectly reversible, i.e. elements in which the reverse current produces exactly the reverse chemical changes; there are others where secondary actions of a purely chemical character accompany the passage of a current through the element so that the actions are not properly reversible.

We shall confine our present considerations to the former type of element, for which alone the general thermodynamical procedure is directly applicable.

If a reversible element at a temperature θ has the electromotive force ϕ we may extract current from it until the total quantity δQ of electricity has passed round the circuit. The mechanical work gained in this process from the chemical reactions is $\phi\delta Q$. In this case a definite quantity of metal from one electrode has dissolved and an equivalent quantity of the other metallic electrode has been deposited. The amount of heat corresponding to these transformations we shall suppose to be $H\delta Q$. If then $\phi < H$ the whole of the energy obtained from the chemical processes is not all transformed into available electrical energy; part of it has been dissipated as heat in the circuit. Conversely if $\phi > H$ heat must have been absorbed from the circuit to account for all the electrical energy.

Now increase the temperature of the cell to $\theta + \delta\theta$: its electromotive force will in general be different, say $\phi + \delta\phi$. Then at this higher temperature join the element to a cell of another type and pass in the opposite direction the quantity δQ of electricity through it. In doing this we use up electrical energy from the second element of amount $\delta Q\,(\phi + \delta\phi)$ and an amount $\delta Q\,(H + \delta H)$ of energy will be absorbed on account of the reverse chemical reactions taking place in the cell; finally the amount of energy

$$\delta Q\,\{(H + \delta H) - (\phi + \delta\phi)\},$$

has been developed in the cell as purely thermal energy.

Next cool the cell to the original temperature θ; the initial conditions in it are then completely reestablished*. We have thus performed a completely reversible Carnot cycle with the cell and since electrical and mechanical energy are completely transformable and can therefore be regarded as equivalent we may apply the ordinary principle of Carnot to the process. But in the cycle we have absorbed the quantity of heat $\delta Q\,(\phi + \delta\phi - H - \delta H)$ at the higher temperature and given up the quantity $\delta Q\,(\phi - H)$ at the lower temperature and the excess of electrical energy gained is $\delta Q\,.\,\delta\phi$. Since this process is reversible we must have

$$\frac{\delta\phi\,.\,\delta Q}{\delta Q\,(\phi - H)} = \frac{d\theta}{\theta},$$

or

$$\theta\frac{d\phi}{d\theta} = \phi - H.$$

* This of course assumes that the thermal capacity of the cell is so large as not to be appreciably affected by the slight changes taking place as described.

Thus when the electromotive force of an element increases with the temperature the element must be cooled when a current is passed through it. Conversely when heat is developed in the cell by the passage of a current the electromotive force decreases with the temperature. These conclusions have been completely verified by experiment.

369. Electrolytic dissociation*. The assumption made above that the molecules of an electrolyte split up into ions before any current can pass has been confirmed in a most remarkable manner by very different sets of phenomena. It has in fact been found, for instance, that the addition of a certain amount of salt or acid to a given quantity of water causes a greater depression in the vapour pressure and freezing temperature than is to be expected from the number of molecules thus added to the water. This fact can be explained by assuming that at least some of the molecules of the salt or acid ionise, that is split up into the two constituent ions, which then move about separately among the molecules of the water. If we then admit this explanation it is possible to calculate the extent of the ionisation in any given solution simply by a measurement of the vapour pressure or freezing point of the solution: it is found in this way that the proportionate ionisation is bigger the more dilute the solution, and that it becomes perfect in infinitely dilute solutions.

These hypotheses were first made by Arrhenius. According to his views the ions of a salt or acid in a very dilute solution are no longer connected with one another but are flying about in the spaces between the molecules of the solvent, each with its own ionic charge, as above. If there is no electric field present the motions of these separated ions are directed equally in all directions so that there is no resultant flux of electricity in any direction. But in the presence of an electric field impressed from without there will be a tendency for each ion to move in the direction of this field, the positive ones in the direction of the lines of force and the negative ones in the opposite direction. The result will be a flux of electricity through the liquid. What is the velocity of this average drift of the ionised molecules through the liquid?

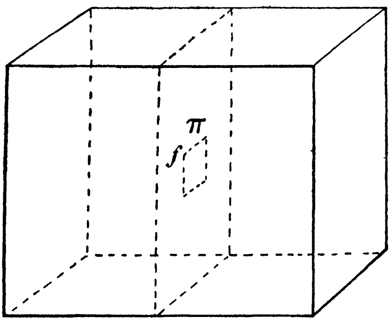


Fig. 68

370. We imagine that a vessel in the form of a rectangular parallelopiped contains a very dilute solution, say, of sodium chloride and also that two parallel electrode plates are placed against opposite sides of the vessel.

* Cf. Whetham, *Theory of Solutions*, ch. XII.

If the electrode on the left is kept at a higher potential than the other, the sodium atom (the positive ion) will be driven towards the right side of the vessel and the chlorine atoms will be attracted towards the left side as a result of the electric field in the solution. Of course the charges on the two ions are equal, merely differing in sign, so that the forces on them as a result of the action of the electric field will be equal (and opposite). For simplicity we assume that the solvent (water) is on the whole at rest. The atoms of sodium will then diffuse through the water towards the right with an average velocity which we may call v_1 and the chlorine atoms will diffuse to the left with an average velocity v_2.

The first question arises: is $v_1 = v_2$? The answer is of course in the negative. In fact when we recognise that the velocity of diffusion of any given type of atom under the action of a given force depends essentially on the size, shape and mass of that atom it is seen to be hardly likely that this velocity will be the same for any two different atoms.

371. Now let π be a plane in the fluid, parallel to the electrodes and f an area on this plane which lies wholly in the solution. Now let n be the number of sodium atoms per unit volume in the solution; the number of chlorine atoms will of course also be n: the mass of a sodium atom is taken to be m_1 and that of a chlorine atom m_2. Then nv_1f sodium atoms pass per unit of time through the plane f going towards the right and nv_2f chlorine atoms pass through to the left in the same time. Thus if at the beginning there were in all N atoms of each ion to the right of the plane π, then at the end of the first second there would be $N + nv_1f$ sodium atoms and $N - nv_2f$ chlorine atoms. These latter remain neutralised by an equal number of particles of the former in the solution but there is the number

$$n(v_1 + v_2)f$$

of sodium atoms in excess. Thus a mass

$$mn_1(v_1 + v_2)f$$

of sodium has appeared at the cathode.

Now if we know the resistance of an electrolyte, then we know also how strong the current will be for a given potential gradient and thus also with the help of a knowledge of the electro-chemical equivalents of sodium, we can calculate how many atoms of sodium will be set free at the cathode per unit time. We know therefore

$$nm_1(v_1 + v_2)f,$$

and since we know nm_1 we can also deduce from this fact the value of $(v_1 + v_2)$.

After a certain time we can also analyse the solution on one side of the plane (say the right) and thus find out how much of the dissolved sodium

chloride has disappeared. According to what we have said above the initial quantity on the right is

$$N (m_1 + m_2),$$

and after the first second it is

$$(N - nv_2 f) (m_1 + m_2).$$

We can therefore calculate from the reduction in this quantity, the value of v_2 and therefore also of v_1. The mean velocity of each ion is thus completely determined.

372. Suppose that these measurements have been made for two dilute solutions, one of sodium chloride the other of potassium chloride, both with the same potential gradient. Now according to the above hypotheses the two constituent ions of any salt or acid in a very dilute solution are completely separated so that their motions will be perfectly independent: if this is the case we ought therefore to find the velocity of the chlorine atom in the two above-mentioned solutions to be the same. This is actually found to be the case and generally observation has shown that in all very dilute solutions the mean velocity of diffusion of a given ion with the same potential gradient, is always the same independently of the character of the other ion.

It may perhaps be worth mentioning that these mean velocities are not big: for a potential gradient of 1 volt per centimetre they are not more than a small fraction of a millimetre per second.

373. Currents arising from variations of concentration in the solution. It is obvious from what we have just said that, when v_1 and v_2 are unequal, different quantities of the electrolyte will disappear in a given time on the two sides of the plane and a difference in the concentration of the solution on the two sides is caused by the current. Experience has in fact shown that the passage of an electric current through an electrolyte often does cause a variation in the concentration of the solution round both electrodes.

Conversely a difference of concentration may give rise to a potential difference and therefore also a current. To see exactly how this arises let us suppose that in a solution of hydrochloric acid the concentration decreases as we go down into the liquid and that in the lowest layer of solution it is so small that complete dissociation of the hydrogen and chlorine atoms exists. The two different ions will then begin independently of each other and with different velocities, to diffuse upwards through the rest of the solution. They will do this merely as a consequence of their heat motions. The velocities of diffusion are not the same so that more of the one ion (hydrogen) will pass upwards through a horizontal plane than of the other. A small excess of hydrogen atoms is thus soon obtained above the plane, and it of course carries with it a positive charge, and leaves an excess of negatively charged

chlorine atoms below the plane. That these charges cannot increase indefinitely is easily seen : for as soon as a difference of potential is established in this way the electric force in the field which arises from it and which is directed downwards will tend to hinder further hydrogen atoms from coming up, but will on the other hand help the chlorine atoms by pulling them. The diffusion of the hydrogen atoms is thus decreased whilst that of the chlorine atoms is increased : when there are as many hydrogen as chlorine atoms passing upwards through the plane the limit is reached and the potential gradient which their relative motion tends to establish has attained its maximum. If we put two electrodes in the solution one at the top and one at the bottom a potential difference between them will be observed and if they are joined up by a wire a current will flow. Of course the potential differences between the electrodes and the liquid in their neighbourhood will also have an influence in the same direction and these cannot be exactly equal and opposite, even if the electrodes are of identical constitution, on account of the different concentrations of the solution in their neighbourhood.

374. The conduction of electricity through gases. We must finally consider the question of the conduction of electricity through gases. The electrical conductivity of a gas in its normal state and at atmospheric pressure is extremely small, and it is only within the last few years that it has been established irrefutably that gases conduct at all; it can however be increased by subjecting the gas to certain influences : for instances in the neighbourhood of a red hot body a gas conducts quite well*, as also do the gaseous products of combustion proceeding from a red-hot flame† : again when the so-called Roentgen rays pass through a gas they render it a good conductor of electricity, or to speak more accurately they increase its conductivity; if the rays cease to act the conductivity dies away rapidly and after a few seconds the gas is no more conducting than before exposure to the rays: the conductivity imparted by these rays can also be removed by passing the gas through a plug of cotton wool.

The relation between the potential difference between the boundaries of the gas and the current through it, when it is rendered and maintained conducting by the uniform action of some agent as above described, is deserving of attention : it is shown by the diagram in Fig. 69. As the potential difference is increased from zero the current gradually increases, but it does not increase so fast as the potential difference (as it would in a conductor obeying Ohm's law) : the increase of current for a given increase of potential difference, decreases as the potential difference is increased until finally no increase in the current is produced by an increase in the potential difference.

* Becquerel, *Ann. de Chimie et de Physique* [3], 39, p. 355 (1853).

† Cf. Wiedemann, *Die Lehre der Elektrizität*, IV. B; Giese, *Wied. Ann.* 17 (1882), p. 519.

This limiting current, the strength of which is independent of the electric intensity and depends only on the volume and nature of the gas and on the rays acting is termed the 'saturation' current. It is represented by the part of the curve SS'. If the electric field be increased still further a stage will be reached at which a spark will pass through the gas and the current will increase once more. These facts were described and an explanation of them offered by J. J. Thomson and Rutherford in 1896*. They supposed that when a gas is rendered conducting there are introduced into it, by some means or other, a number of particles charged, some with positive and some with negative electricity. When the gas is subjected to an electric field the positive particles move to the negative boundary and the negative to the positive: this motion of the charge constitutes the current in the gas. But, if the gas is left to itself under the action of no external field, the oppositely charged particles attract each other and coalesce in pairs, or *recombine*. When once the particles have all recombined and their charges are neutralised, they move no longer under the action of a field and the gas loses its conductivity. However if the gas is kept under the action of the rays, fresh charged particles are produced continuously: the number of particles in the gas at any time is such that the number recombining in

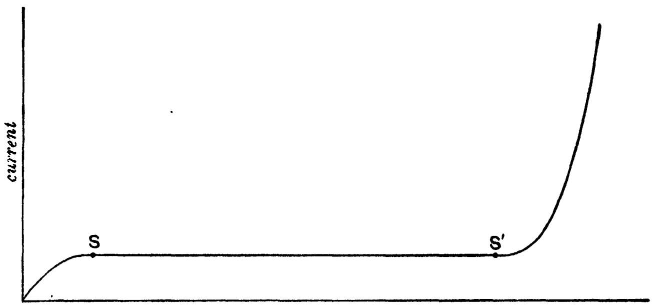


Fig. 69

one second is equal to the number produced in the same time: the gas is then in a steady state, and the number of particles in the gas does not change with the time. In this condition the gas shows no sign of possessing a charge as a whole, and hence the total charge on all the positive particles must be equal to the total charge on all the negative.

When the gas is passed through a plug of cotton wool, the charged particles, which differ from the molecules of the gas, are retained, while the molecules pass through. It is not necessary to conclude that the particles are larger than the molecules, for the fact that they are charged, while the molecules are neutral, might account for their retention.

* *Phil. Mag.* [5], 42 (1896), p. 392.

375. Consider now the variation of the current with the potential difference in such a mixture of molecules and charged particles as has been imagined. The greater the strength of the field, the greater is the velocity with which the particles move and the shorter the time that is required for them to get from any part of the field to the boundary. But while any particle is moving to the boundary, it is liable to encounter a particle of the opposite sign and coalesce with it, losing thereby its conducting power. The shorter the time of passage, the less likely will such an encounter be, and the greater the number of particles which start from any part of the field and arrive, still charged, at the boundary: hence the current, which is measured by the number of charged particles arriving at the boundary, will increase with the potential difference. But it is clear that the current cannot increase indefinitely: it must reach a limit when the time occupied by the particles in reaching the boundary is so short that none recombine and the number arriving at the boundary per second is the number produced in the gas in the same time. This number multiplied by the charge on each particle gives the *saturation current* through the gas. The further increase in the current observed with still stronger fields can only be attributed to an increase in the rate of production of the particles.

According to this explanation, the process of conduction is essentially the same, whether it takes place in a metal or in an electrolyte or in a gas. In each case the current consists of a stream of charged particles, and the great difference between the laws governing the conduction in different states of matter is due only to a difference in the nature of the charged particles, in their mode of production and in their relation to the medium that surrounds them. In view of the similarity between gaseous and electrolytic conduction, the nomenclature of the latter has been applied to the former. The moving charged particles are called *ions* and a gas, when it is rendered conducting, is said to be *ionised*: the process of ionising a gas is named *ionisation*, a word that is also used quantitatively to express the concentration of the ions in the gas or the number per unit volume. The negative electrode is called the kathode and the positive the anode.

376. The charge and velocity of the ions. The charge on each ion in a gas can be ascertained if (1) the total charge on all the ions present in a gas at any time, and (2) the number of such ions can be measured. The first quantity presents no difficulty. Let the gas be subjected to an ionising influence, such as Roentgen rays, until a state of equilibrium is reached between the number of ions produced per second by the rays and the number disappearing in the same time through combination. Let the action of the rays be stopped suddenly and the gas immediately exposed to the action of a strong electric field. All the ions of either sign present in the gas will be driven to the boundary of the field, and, if the field be sufficiently strong, the

time occupied in reaching the boundary will be so short that no appreciable recombination will have taken place in the meanwhile: the number reaching the boundary is equal to the number present in the gas before the field was put on. If the charge received by the boundary is measured by connecting it to an electrometer, the total charge on all the ions of one sign present in the gas can be ascertained. It may be remarked that if the ionising agent consists of Roentgen rays it is found that the charge on all the ions of either sign is the same.

377. The measurement of the number of ions present is a much more difficult matter, but has been effected by a most ingenious method devised by J. J. Thomson and based on a discovery due to C. T. R. Wilson*.

It is known that the quantity of water that can exist in the state of vapour in a given volume of gas depends greatly on the temperature of the gas, and that, if a volume of gas saturated with water vapour is cooled, the excess of the water, or the amount representing the difference between that which the gas could hold at the higher temperature and that which it can hold at the lower, is deposited in the form of rain or mist. It was found by Aitken that this deposition of the superfluous water depends on the presence in the gas of solid particles, or dust, which acts as nuclei for the formation of liquid drops: if a gas is rendered perfectly free from dust by filtering it through cotton wool, it can be cooled to a temperature very much lower than that at which a cloud would form in the presence of dust without any consequent condensation. The necessity for the presence of nuclei in the formation of drops can be shown readily to be a consequence of a surface tension in liquids, and it appears that the larger the particle of dust the more efficient is it as a nucleus and the smaller the supersaturation of water vapour that it is possible to produce in its presence.

C. T. R. Wilson found that ions possess peculiar properties with respect to the formation of clouds in gases supersaturated with moisture. In his investigations he cooled the gases by expanding them adiabatically, thus producing a fall in temperature which could be calculated from the known values of the initial and final volume and the ratio of the specific heats of the gas. He found that it requires less supersaturation to produce a cloud in air, when it is ionised, than when it is not ionised: the ions act as nuclei for condensation. This does not imply necessarily that the ions are larger than the molecules which are always present and available as nuclei, for it can be shown on theoretical grounds that a charged body should act as a more efficient nucleus than an uncharged body of the same size. In the absence of ionisation eightfold supersaturation† is required to produce

* *Phil. Trans.* A, 189 (1897), p. 265; 192 (1899), p. 403.

† By 'supersaturation' is meant the ratio of the mass of water actually present in the gas to that which it could hold without condensation in the presence of large nuclei.

condensation, whereas fourfold supersaturation is sufficient for the same purpose in the presence of negatively charged ions, and sixfold supersaturation in the presence of positively charged ions. These values are independent of the degree of ionisation and also, with few exceptions, of the means by which it is produced. Further, no increase in the amount of condensation is caused by increasing the supersaturation between fourfold and sixfold.

These experiments show that there is a difference in the properties of positive and negative ions; that the properties of the ions are not determined solely by the method by which they are produced, and that the properties of all the ions of the same sign are the same.

378. Wilson's discovery was used by Thomson to determine the number of ions in a gas. The ionised gas, in which the total charge on all the ions had been determined, was expanded adiabatically and cooled to such an extent that a supersaturation between sixfold and eightfold was produced: a cloud formed round all the ions of the gas, but not on the molecules. As soon as the cloud formed it began to fall, because the drops were heavier than the air surrounding them: owing to viscosity of the air they soon attained a small and constant velocity which could be measured by observing the rate at which the upper boundary of the cloud moved down the vessel. Now Stokes has calculated the steady velocity with which a body of known dimensions moves through a medium of known viscosity under the action of a constant force, such as its own weight. He finds that if F is the force, a the radius of the body (supposed to be spherical), ρ its density, μ the coefficient of viscosity of the medium, v the steady velocity is given by

$$v = \frac{F}{6\pi a\mu},$$

or if F is the weight of the body

$$v = \frac{2}{9}\frac{ga^2\rho}{\mu}.$$

By applying this formula to the case under consideration, where the velocity, the force and the viscosity are known, a, the radius of the drop, can be determined. The total mass of all the drops is the mass of the water set free by the cooling, for the mass of the ion contained in each drop is so small compared with the whole mass of the drop, that it may be neglected. The calculation of this total mass is somewhat complicated, but it depends on well known principles, and for further details the reader may be referred to Thomson's account. If then, the total mass of all the drops is M, the number of the drops n, which is the number of ions, can be calculated from the relations

$$n = \frac{M}{m}, \qquad m = \tfrac{4}{3}\pi\rho a^3, \qquad a = \sqrt{\frac{9\mu v}{2g\rho}}.$$

The charge on each ion, e, is the total charge on all the ions divided by the number of the ions.

As a final result of this research the charge on each ion in a gas is found to be in electrostatic units

$$e = 3{\cdot}4 \,.\, 10^{-10}.$$

Experiments made in air and hydrogen gave results identical within experimental error.

379. H. A. Wilson* has used a slight modification of the method, which obviates the necessity for the cumbrous calculation of the total quantity of water set free. The total charge on the ions is measured as before. The gas is then expanded and cooled so as to give a supersaturation between four and six and to cause condensation on the negative and not on the positive ions. The velocity v with which the cloud falls is observed. The experiment is then repeated, with the difference that the falling cloud is exposed to the influence of a vertical electric field E, which exerts on the charged drops a force Ee, accelerating or retarding the fall. The new velocity v' is measured.

Since the velocity of the drop is proportional to the force on it, we have

$$\frac{v'}{v} = \frac{Xe + mg}{mg} \quad \text{where} \quad m = \tfrac{4}{3}\pi\rho a^3,$$

but from above

$$v = \frac{2}{9}\frac{ga^2\rho}{\mu},$$

hence

$$e = \frac{9\pi\sqrt{2}}{E}\sqrt{\frac{\mu^3 v^3}{g\rho}}\,\frac{v' - v}{v}.$$

By this method Wilson found $e = 3{\cdot}1 \,.\, 10^{-10}$. A few drops were found to bear a double and triple charge but their numbers were too small to have exerted any influence in Thomson's experiments.

Since Thomson measured the average charge on both positive and negative ions and Wilson that on the negative ions alone the coincidence between their results shows that the charge on an ion is independent of its sign.

380. The velocity of the ions can be measured in various ways, of which the most convenient depends on the following principles. Suppose that two plates A and B, parallel to each other, are maintained at a constant difference of potential, and that ions are produced by some means close to the plate A. Under the action of the electric field the ions of one sign will give up their charges to A while the others will move towards B. Just as they are about to settle on B let the direction of the field be reversed, so that the plate which was formerly positive is now negative: these ions will then be driven back towards A while those of opposite sign move towards B: as these are about to touch B let the direction of the field be again reversed. If then

* *Phil. Mag.* (6), 5 (1903), p. 429.

the direction of the field is reversed at constant intervals and these intervals are so short that they do not permit the ions to travel from one plate to the other before the field is reversed, the plate B will receive no charge: but if the interval is longer, so that the ions have time to get across before the reversal of the field, B will receive a charge. By measuring the time of reversal for which B just begins to receive a charge, the time required for the ions to pass from one plate to the other is known, and hence the velocity of the ions ascertained. In this way it is found that the velocity of an ion is proportional to the strength of the field in which it moves, i.e. to the force acting on it: the essential condition for a diffusive motion is thus satisfied. The velocity of the negative ions is slightly greater in all cases than those of the positive ions.

The evidence thus far points to the fact that the carriers of the current in gases are more like those in liquids than the electrons in metals. But a study of the discharge of electricity through gases at low pressures has thrown a new light on this question.

381. Suppose that a long cylindrical glass vessel is taken and filled with a gas at atmospheric pressure, and suppose the potential difference between the kathode (K) and the anode (A) is gradually increased. Only an extremely small current will pass through the gas, but at a certain high limit a 'spark' passes through the gas and a very considerable current begins to flow. If the pressure of the gas is lower the limit at which the spark passes is also lower and the spark itself is broader and more diffuse than before. As the pressure is reduced the diffuseness of the spark increases until finally the luminosity of the discharge fills the whole volume of the tube between the electrodes; but it is seen on careful inspection that it is not continuous. Covering the kathode is a thin luminous layer, the 'kathode layer': next comes a dark space, the 'Crookes dark space': a luminous layer follows, the 'negative glow,' then another dark space, the 'Faraday dark space,' and lastly, a region of light, which is sometimes divided into striae, extending up to the anode and known as the positive column. As the pressure is still further reduced, the Crookes dark space grows at the expense of all the other regions, which diminish in extent, except the kathode layer, which remains practically the same. When the dark space has become so large that its boundaries touch the glass of the walls a curious green phosphorescence is excited in the glass. At first this phosphorescence appears on a few isolated patches near the kathode, but by decreasing the pressure sufficiently the whole of the tube may be filled by the Crookes dark space, with the exception of thin layers on the kathode and anode, and then all the glass walls of the tube with the exception of the parts behind the kathode and anode glow with the green phosphorescence: it is in this condition that the tube is emitting Roentgen rays plentifully.

If the pressure be still further reduced, the current diminishes and finally vanishes: at sufficiently high vacua no current can be made to pass.

382. In 1859 Pluecker* discovered that if a magnet is brought near to the tube in which the luminous discharge is observed all the luminous portions are distorted in some measure from their original positions. But the most marked distortion was produced in the patches of phosphorescence near the kathode. This discovery at once directed the attention of physicists to these patches. Soon afterwards Hittorf discovered that if a solid body is placed in the tube, near the cathode, the phosphorescence ceases at all points which were shielded from the kathode by the solid body: a shadow of the body is thrown on the walls of the tube. It thus appears that the influence which caused the phosphorescence on the walls must proceed in a straight line from the surface of the kathode to the walls of the tube, and it was therefore described as due to certain kathode rays. It may also be noted that it was found that the nature of the shadow was such as to prove that the rays do not proceed in all directions from the kathode but are emitted at right angles to its surface.

It was surmised by Goldstein† that these kathode rays were a type of aethereal radiation, like light; but the view of Sir William Crookes‡ that they were streams of particles electrically charged has been proved to be the more correct of the two. Before it can be definitely asserted that the rays consist of charged particles some further information is required: the term particle connotes a finite mass and a finite number of carrying agents and to settle the point it is necessary to determine these two quantities. The number is known if we can measure the charge carried by each particle: hence we want to determine the mass of each particle and the charge carried by it. Unfortunately no direct method has been devised for dealing with individual particles in the kathode rays and neither of these quantities has been determined directly, but the ratio of the mass to the charge of a kathode particle can be ascertained by a method devised by Thomson§.

383. Suppose that a particle of mass m carrying a charge e is moving along parallel to the x-axis of a conveniently chosen rectangular coordinate system with a velocity v under the action of a magnetic field of intensity H parallel to the axis of y. Then as we shall see later the particle will be subject to a force $\frac{Hev}{c}$ perpendicular to both H and v, i.e. along the z-axis. The acceleration of the body parallel to the axis of z will be $f = \frac{Hev}{cm}$, and if it

* *Pogg. Ann.* 103 (1859), p. 88.

† *Wied. Ann.* 1880–1884.

‡ *Phil. Trans.* 1879–1885.

§ *Phil. Mag.* [5], 44, p. 293 (1897). Cf. also Wiechert, *Sitzungsber. der phys. ökon. Gesellsch. zu Königsberg*, 38 (1897), p. 1; *Wied. Ann. Beiblätter*, 21 (1897), p. 443.

moves for a time t it will have travelled a distance $\frac{1}{2}ft^2 = \frac{Hevt^2}{2mc}$ in that direction. In the same time it will have travelled a distance vt parallel to the x-axis. Hence the particle while travelling a distance l parallel to the x-axis is deflected a distance δ parallel to the z-axis, where

$$\delta = \frac{l^2}{2c} \cdot H \cdot \frac{e}{m} \cdot \frac{1}{v};$$

if we know the dimensions of the discharge tube we can thus find e/m.

We are not able to measure directly the charge on the kathode ray particle; in fact it is only when the carriers of the current are atoms or molecules of matter that it is possible for us to determine the charge on the separate carrier; but since it is found that the value of this charge in all cases where it can be determined is always identical, it is natural to assume that the charges on the kathode particles have the same constant value, viz

$$4{\cdot}69 \,.\, 10^{-10},$$

so that the mass is

$$2 \,.\, 10^{-27} \text{ gms.},$$

this is much smaller than any known molecule of matter: a molecule of hydrogen has for instance a mass of about

$$3 \,.\, 10^{-24} \text{ gms.}$$

The kathode particle whose existence is thus presumed is the negative electron: it is identical with the negative particle thrown off by radio-active substances and these facts combined with certain other evidence to be subsequently discussed confirms us in the view that these electrons are common constituents of all matter.

384. While it thus appears that in the low pressure discharge through gases the most important part is played by negative electricity there is no doubt that positively charged particles also play some part in the mechanism of the phenomena. The complicated appearance of the discharge and the difference which it shows in different gases suggests that some agent other than the universal negative electron must have an influence on the phenomena; but for some time after the discovery of kathode rays no such agent had been directly detected. However in 1886 Goldstein working with a discharge tube in which the kathode was a plate perforated by several holes, observed faintly luminous streaks stretching out from the holes in the kathode into the space remote from the anode. Where these streaks meet the walls of the tube, they excite a slight phosphorescence, usually of a mauve colour, but always totally different from the green phosphorescence excited by the kathode. He imagined that these streaks represented the path of rays, similar in some respects to the kathode rays, to which he gave the name Kanalstrahlen. Since these rays are travelling in a direction from the anode

to the kathode, it was natural to suppose that they are positively charged, but at first no experiment could detect any sign of charge. However in 1898 Wien* showed that, if fields of sufficient magnitude be employed, a deflection of the rays could be obtained, and its direction being in the direction opposite to that of the electrons indicates that the charge is positive. The results of the experiments were however otherwise somewhat indefinite and it remained for Thomson† to attack the problem with an improved apparatus. By using very low pressures Thomson was able to determine distinctly the types of particles involved and he found that they were in almost every case molecules of the gas in the tubes carrying the monovalent charge equal to that of a single electron. They are therefore merely the atoms or molecules of the substance from which an electron has been extracted.

These experiments unfortunately still leave us in the dark concerning the nature of the positive electricity in the atoms and so far no definite evidence is forthcoming on this point from any other branch of the work‡. Crowther sums up the state of affairs in his excellent little book§ by the statement that 'the term positive electrification' remains a not too humiliating method of confessing ignorance. And this is where we must leave it for the present.

* *Wied. Ann.* 65 (1898), p. 440. Cf. also *Ann. der Phys.* (4), 8 (1902), p. 244.

† *Phil. Mag.* 13 (1907), p. 561; 16 (1908), p. 657; 18 (1909), p. 821.

‡ Cf. however p. 44, § 48.

§ *Molecular Physics* (London, 1914).

CHAPTER IX

THE ELECTROMAGNETIC FIELD

385. First fundamental notion : Ampère's circuital relation. We have now discussed the fundamental principles underlying the theory of electric flow and also of magneto-statics, but so far independently of each other. In 1829 Oersted* discovered however that a current exerted an action on a magnet, in a transverse manner as he described it. This is the first fundamental effect connecting currents and magnetism.

The current, as Faraday would say, possesses a magnetic field, i.e. in the space surrounding the wire carrying the current a magnet is acted on by forces dependent on the position of the wire and strength of the current. This magnetic field may be studied in the same way as we examined the fields of ordinary magnets, by tracing the course of the lines of magnetic force at every point. This is the rational method adopted by Faraday as long ago as 1837.

Having already strongly insisted that every current flows in complete circuits Faraday then showed that a small closed circuit carrying a steady current possessed a magnetic field identical with that of a little magnetic needle stuck normally through its middle. At least he showed that it was possible to compensate the magnetic field of the current by a small magnetic needle in this way.

Accepting this as a fundamental fact we must first enquire as to the moment of the little magnet which is equivalent to the current. It has been shown by numerous experiments, of which the earliest are those of Ampère and the most accurate those of Weber†, that it is proportional to the strength of the current flowing and to the area of the elementary circuit assumed to be plane. Thus if the small circuit be supposed to be filled up by a surface bounded by the circuit and thus forming a diaphragm, and if a magnetic shell of strength proportional to the current coinciding with this surface be substituted for the current, its magnetic action on all distant points will be identical with that of the current.

* "Experiments on the Effect of a Current of Electricity on the Magnetic Needle," *Annals of Philos.* (1820), 16, p. 273. Oersted clearly recognised the fact that a current is surrounded by a magnetic field.

† *Elektrodynamische Maasbestimmungen* [*Abhandl. der königlich Sächs. Gesellsch. zu Leipzig*, 1850–1852].

So far we have supposed the dimensions of the circuit to be small compared with the distance of any part of it from the point of the field examined. The extension to finite circuits is however easily obtained in the manner introduced by Ampère*.

386. Conceive any surface f bounded by the circuit and not actually passing through the point P at which the field is examined. On this surface draw two series of lines crossing each other so as to divide it into elementary portions, the dimensions of which are small compared with their distance from P and also with the radii of curvature of the surface. (Cf. fig. 5, p. 14.)

Round the boundary of each of these elements conceive a current of strength equal to that of the original current to flow, the direction of circulation being the same in all the elements as it is in the original circuit.

Along every line forming the division between two contiguous elements two equal currents flow in opposite directions. But the effect of two equal and opposite currents in the same place is absolutely zero, in whatever aspect we consider them. Hence their magnetic effect is zero. The only portions which are not neutralised in this way are those which coincide with the original circuit. The total effect of the elementary circuits is therefore equivalent to that of the original circuit.

Now since each of the elementary circuits may be considered as a small plane circuit whose distance from P is great compared with its dimensions we may substitute for it an elementary magnetic shell of strength proportional to that of the current whose bounding edge coincides with the elementary circuit. But the whole of these elementary shells constitute a magnetic shell of equal strength coinciding with the surface f and bounded by the original circuit and thus the magnetic action of the whole shell at P is equivalent to that of the circuit.

The magnetic force due to the circuit carrying a current J at any point is therefore identical in magnitude and direction with that due to a uniform magnetic shell bounded by the circuit and not passing through the point, the strength of the shell being numerically proportional to J, say $\frac{J}{c'}$. The constant $\frac{1}{c'}$ is an absolute constant depending on the units adopted. If we define the current by its magnetic effect we can take $c' = 1$. This would give us the absolute electromagnetic unit of current. The equivalent magnetic shell has then a strength numerically equal to the strength of the current.

It is necessary to include one further remark. In order to be able completely to specify the shell which is equivalent to any current we must

* "Mémoire sur la théorie mathématique des phénomènes électrodynamiques," *Mem. de l'institut.* 6 (1820). Cf. also *Ann. de Chem.* 15 (1820).

decide which is to be the positive side of the shell. Experiment shows that in the small circuit the positive direction of magnetisation of the shell bears to the direction of the current the same relation as advance to rotation in a right-handed screw.

387. There is however an important limitation to the general statement of the equivalence of shell and current. The potential function of a magnetic double sheet being obtained from a sum

$$\Sigma \frac{m}{r},$$

is essentially a single-valued function. It is true that as we go round a circuit as shown by the line from P to P' (near the sheet one on each side on the same normal) we find that there is a drop of potential equal to $4\pi \frac{J}{c'}$ but this drop is recovered in going through the sheet from P' to P. The potential changes very rapidly in going through the sheet. In the case of the current however there is no place where the change in potential produced in going round a circuit as from P to P' (which are ultimately very close together) can be recovered.

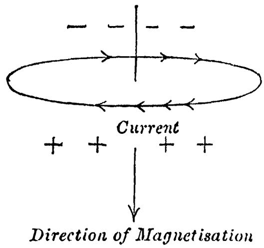


Fig. 70

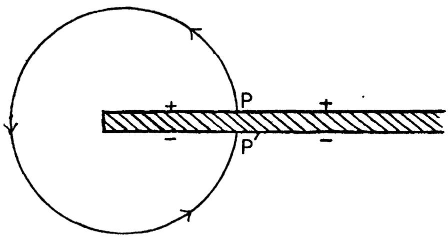


Fig. 71

The magnetic field of a closed current is thus a cyclic one and agrees with that of the equivalent magnetic shell only at points outside the substance of the shell. The agreement does not hold for points inside the shell.

388. The potential of a uniform magnetic shell of strength $\frac{1}{c'} J$ can easily be found; it is $\psi = -\frac{J\Omega}{c'}$, where Ω is the solid angle subtended by the shell at the point in the field. Thus the magnetic field due to a closed current of strength J has the multiple-valued potential function

$$\psi = -\frac{J\Omega}{c'},$$

where Ω is the solid angle subtended by an area closing the current circuit at the point.

Thus if we take a path linked with the current circuit and, starting from place where the potential is ψ, go right round, we find that the potential on arriving at the same place again is now $\psi \pm \frac{4\pi J}{c'}$, a different value. On going round again it becomes $\psi \pm \frac{8\pi J}{c'}$; and so on. If the path is not linked with the current there is no total change at all in the potential on going round it; along paths not linked with the current the potential is a single-valued function. The mathematical distinction between the two cases is involved in the fact that the current forms a line singularity in the magnetic field. The singularity is of the nature of a circulation round the wire and if we consider a small circuit close up to and encircling the wire the potential must change very rapidly in going round it because the total change in a complete circuit is finite, viz. $4\pi J/c'$. This means that the magnetic force gets bigger and bigger as we approach the current. If we actually get on the current itself the force is indeterminate; but this merely means that the problem needs closer specification before we can examine this point. There is in reality no indeterminacy in physical affairs. Of course from another point of view it really means that the current cannot be confined to a geometrical line.

389. This result provides us with the integral form of the characteristic property of such fields; it says that if $\mathbf{H}_s$ is the component of the magnetic force in the field along the element ds of the path drawn round the circuit carrying a current J and threading it in the positive direction, then

$$\int_s \mathbf{H}_s ds = \frac{4\pi J}{c'}.$$

In other words if we have a single magnetic pole of strength m, and starting from a point P go right round a circuit threading the current, eventually arriving back at P, the potential changes by $\frac{4\pi J}{c'}$ and an amount of work has been done equal to $\frac{4\pi J m}{c'}$. The magnetic pole could thus be used to do work at the expense of the energy of the system. This work must of course come from the energy of the system or, in other words, the moving of the magnetic pole must affect the strength of the current. This is Faraday's idea, by moving a magnet in the neighbourhood of a current it can be made to drive an engine, the power being derived from the current. The interaction between a magnet and a current is a source of power; this is the principle underlying the fact that an electric current can be used to drive a motor*.

* We have spoken of moving a magnetic pole and not a magnet. We consider however that a magnet has two poles and we should get no work by moving them together unless some means were devised to take them along different paths relative to the current. Faraday's idea of sliding contacts breaking the circuit enabled him to get a single pole through the circuit by itself.

Whether the work in the complete path is zero or not depends only on whether the path encircles the current or not. This can be expressed in a rather different manner. Having chosen a path we can always imagine it closed over by a barrier surface. If this surface is chosen as simply as possible the current will always cut through it when the path links with the current; and at the place where this happens a quantity of electricity J will flow across the surface per unit time. On the other hand if the path does not link with the current the surface closing it may be so chosen that the current circuit does not cut through it and no electricity will flow across it. We may thus state our rule in the different form

$$\int_s (\mathbf{H}d\mathbf{s}) = \frac{4\pi}{c'} \text{ (total quantity of electricity flowing through the circuit } s \text{ per unit time)}$$

and now it appears to be true for the very general case if only we measure the electricity flowing through the circuit by putting a barrier across and finding the current flowing across the barrier; it is the amount of electricity flowing across the chosen barrier per unit time. In this form we can apply the rule to a continuous current distribution of density $\mathbf{C}$ at any point and it is

$$\int_s \mathbf{H}_s ds = \frac{4\pi}{c'} \int_f \mathbf{C}_n df,$$

where f is the barrier surface bounded by the path s. Kelvin described* this relation as a circuital relation between $\mathbf{H}$ and $\mathbf{C}$. It will hereinafter be described as Ampère's circuital relation, as Ampère was the first one to obtain it although in a rather complicated form. We shall return to this relation later.

390. On closed and open circuits: displacement currents. It is of importance to notice that the mode of expression of Ampère's circuital relation adopted at the end of the previous paragraph can have no meaning whatever unless the quantity of electricity flowing through the circuit measured as there specified is the same for all possible barriers; this can only be true if the current is, so to speak, a stream, so that there is no accumulation in the space between two barriers. This amounts to the same as saying that all currents must flow in complete circuits and the fundamental significance of the relation is based on this idea of closed current circuits.

If we define a current as a flow of electricity (or electrons) the discharge of a condenser with a vacuum dielectric must constitute a current with a break in it, so that in a case of this kind the current is an open one and the preceding ideas could not possibly be applied to treat it; such cases would

* Cf. Larmor, *Aether and Matter*, p. 76.

be beyond a theory involving solid angles and such like, for we can attach no meaning to Ω for open circuits.

Faraday insisted however from the beginning that all currents however produced were circuital. Maxwell followed the hint and discovered the wonderful simplicity thereby introduced into the analysis. Helmholtz and his school, however, not willing to accept the fundamental difficulty in the assumption of closed currents, constructed a theory of unclosed currents, but it is terribly complicated.

In order to surmount the difficulty mentioned above, in cases like the discharge of a condenser, Maxwell made the hypothesis that even in such cases there must be in the aether in the gap some sort of release of strain taking place which possesses the electrodynamic properties of a current or a movement of electricity. It is of course *not* a movement of electricity.

391. The point here involved has been discussed at length in a previous chapter*, but the following example will provide the general analytical form of the hypothesis. Consider the process involved in charging a conductor existing alone in the field. As we charge the conductor a state of 'polarisation' is gradually established in the surrounding dielectric medium (and aether) being accompanied by a 'displacement' in the Maxwellian sense away from the conductor. It is the essence of Maxwell's hypothesis that, in addition to the true electric displacement involved in the polarisation of the dielectric medium, there is some effect in the aether, an aethereal displacement he calls it, which is *not* an electrical displacement but for some reason or other its rate of change has the properties of a true electric current. The actual measure of this current is obtained as follows: if $\mathbf{C}'$ denote the current intensity of the true electric flow supplying the charge to the conductor then the integral

$$-\int_f \mathbf{C}_n' df,$$

taken over any closed surface f enclosing the conductor indicates the rate at which the charge on the conductor is increasing; or if we use Q for the charge on the conductor at any time t

$$-\int_f \mathbf{C}_n' df = \frac{dQ}{dt}.$$

When however there is a charge Q on the conductor the conditions in the surrounding field are such that if $\mathbf{D}$ denotes the totality of displacement in the dielectric (aethereal and true electrical polarisation) at any point, then

$$\int_f \mathbf{D}_n df = Q,$$

* Cf. p. 221.

so that we have

$$-\int_f \mathbf{C}_n' df = \frac{d}{dt}\int_f \mathbf{D}_n df,$$

or

$$\int_f (\mathbf{C}_n' + \dot{\mathbf{D}}_n)\, df = 0,$$

where we use $\dot{\mathbf{D}}_n$ for the time rate of change of $\mathbf{D}_n$. This indicates that the vector

$$\mathbf{C}' + \dot{\mathbf{D}},$$

is always circuital, that is always flows in closed circuits. Thus if we add to the true current of electrons the time rate of change of the total displacement we obtain a total current vector which is always circuital. But

$$\mathbf{D} = \mathbf{P} + \frac{\mathbf{E}}{4\pi},$$

and thus the total displacement current

$$\dot{\mathbf{D}} = \dot{\mathbf{P}} + \frac{\dot{\mathbf{E}}}{4\pi}$$

consists of a part $\dot{\mathbf{P}}$ depending on the presence of the dielectric which is a real displacement of electric charge in the molecules of the medium. The part $\frac{1}{4\pi}\dot{\mathbf{E}}$ is the part of the displacement current which must be ascribed to some action in the aether; it is Maxwell's aethereal current. The only way of explaining the existence of this current is by a theory of the constitution of the aethereal medium, about which however we know so little. A dynamical theory of its mode of action can however be described which gives some idea of what sort of thing this displacement current is and of how it simulates an electric flow*.

392. We must make a distinction between the true current

$$\mathbf{C}' + \dot{\mathbf{P}},$$

which is a true flow of electricity and this fictitious current in the aether. The contrast is in reality between the total current

$$\mathbf{C}' + \dot{\mathbf{P}} + \frac{\dot{\mathbf{E}}}{4\pi},$$

and the true current. The total current always contains a part $\left(\frac{\dot{\mathbf{E}}}{4\pi}\right)$ which is not electric flow at all but is a something possessing the electrodynamic properties of a true electric flow.

The important thing is that now every current is effectively circuital. Looking at it from the practical side the only case where the distinction comes in is that of the electrostatic discharge. All currents of conduction are in

* Cf. Larmor, *Aether and Matter*.

themselves complete. In technical electrodynamics electrostatic discharges are of little account although of late years the phenomena of wireless telegraphy has imparted a technical aspect even to this side of the subject.

It took over thirty years before anything like a decisive test of this hypothesis of Maxwell's was obtained. At the beginning there were no phenomena in which the dynamical relations of an electric discharge could be investigated. Hertz however finally succeeded in discovering the electric waves whose existence and theoretical relations had been deduced by Maxwell as an essential consequence of his theory and thereby provided the necessary experimental proof of the hypothesis on which the theory is based.

393. Second fundamental notion: Faraday's circuital relation. We now come to the consideration of the second fundamental notion concerning the interaction of magnetic fields and currents. This is the notion that under certain circumstances the magnetic field may excite electric currents in the closed conducting circuit: such currents are described as induction currents.

Since Oersted's discovery of the mutual action of a magnetic system and a current it was a problem to find out whether magnetism could make a current; and numerous attempts were made to solve it. Faraday* was the first to succeed. He found that no stationary field however great could make a current, but that as soon as the field was varied the conditions were right.

Faraday's notion of representing a magnetic field was by lines of force; as a field of flux in tubes. He then soon found that the essence of the induced electromotive force necessary to drive the current observed was really the change in the flux through the circuit. The rule deduced experimentally by him was as follows:—

Whenever the total magnetic flux through any circuit varies there is an induced electromotive force created in the circuit and the amount of it is proportional to the rate of diminution of the total number of tubes of induction threading the circuit.

This law is found to be universally true when either or both the magnetic system and the current circuit are moving. It can be shown to be a direct consequence of the previous fundamental notion, if the energy principle applies to these things. The deduction will be given later.

The translation of this law into mathematics is in reality the whole subject of electrodynamics. If $\mathbf{E}$ denotes the electric force measured in electromagnetic units the electromotive force in any closed circuit s is

$$\int_s \mathbf{E}_s ds,$$

* *Exp. Res.* II. p. 127.

and according to Faraday this is equal to

$$-\frac{1}{c}\frac{d}{dt}\int_f \mathbf{B}_n df,$$

where **B** denotes the vector of magnetic induction, the integral being extended over any surface f bounded by the circuit s and c is a constant, depending on the units adopted. We thus have

$$\int_s \mathbf{E}_s ds = -\frac{1}{c}\frac{d}{dt}\int_f \mathbf{B}_n df,$$

another circuital relation forming the second fundamental equation of our subject. Faraday's name is usually attached to it.

This relation of course implies that there is a definite value for the magnetic induction flux through the circuit, i.e. it must be the same on whatever barrier surface f it is calculated. This implies of course that **B** is a stream vector or that

$$\operatorname{div} \mathbf{B} = 0,$$

which is of course a relation always satisfied by this vector; so that the equation so interpreted is quite consistent with our former ideas and needs no extension as in the previous case.

394. The units in the electromagnetic equations. In the process here employed of the construction of a mathematical theory of electricity and magnetism definite sets of units have been introduced as associated with the different classes of phenomena. As we now see that all the various phenomena are in the most general case correlated with one another there must be some definite relation between the various systems of units thus adopted: the theory would otherwise not be consistent with itself.

The chief object in the choice of a system of units is to obtain the simplest expression for our purposes of the fundamental physical relations on which the particular theory is based.

In any relation connecting physical quantities of fundamentally different characters, certain constants must occur in order to secure that the dimensions of the fundamental quantities correlated are the same. For example in the expression of the mechanical action between two point charges q_1 and q_2 concentrated at a distance r apart we say that the force between them is proportional to $\frac{q_1 q_2}{r^2}$ a quantity of a fundamentally different kind. We therefore write

$$F = \gamma \frac{q_1 q_2}{\gamma^2},$$

and choose the dimensions of γ so as to make the dimensions of both sides of this equation the same. This constant γ is then an absolute constant of the theory, whose value and dimensions depend however on the choice of

units for the other quantities involved in the relation. In the simple electrostatic theory as we have developed it we choose to measure a quantity of electricity so that the constant γ in this expression is a simple number (without dimensions) numerically equal to unity. This means that with the same notation as before the dimensions of a quantity of electricity are given by the symbolical equation

$$[Q] = [m^{\frac{1}{2}} l^{\frac{3}{2}} t^{-1}].$$

The dimensions of the electric displacement D then follow as the quotient of charge by a surface; those of the force intensity E as the quotient of force by charge; those of the potential being then the product of the dimensions of E and a length; and finally the dimensions of a current density are those of a charge divided by a time and a surface. In symbols

$$\begin{aligned}
[Q] &= [m^{\frac{1}{2}} l^{\frac{3}{2}} t^{-1}],\\
[E] &= [m^{\frac{1}{2}} l^{-\frac{1}{2}} t^{-1}],\\
[D] &= [m^{\frac{1}{2}} l^{-\frac{1}{2}} t^{-1}],\\
[\phi] &= [m^{\frac{1}{2}} l^{\frac{1}{2}} t^{-1}],\\
[C] &= [m^{\frac{1}{2}} l^{-\frac{1}{2}} t^{-1}],\\
[J] &= [m^{\frac{1}{2}} l^{\frac{3}{2}} t^{-2}].
\end{aligned}$$

These are the quantities that occur in the specification of the electric part of the general scheme. The magnetic quantities can be similarly examined and the results are identically the same as those just given for the analogous electric quantities or in symbols

$$\begin{aligned}
[H] &= [m^{\frac{1}{2}} l^{-\frac{1}{2}} t^{-1}],\\
[B] &= [m^{\frac{1}{2}} l^{-\frac{1}{2}} t^{-1}].
\end{aligned}$$

395. But we have in this chapter introduced relations connecting these two classes of quantities here involved: these relations however contain certain universal constants c and c' so that as usual a definite choice of units is not implied in the form of their expression adopted. If however we interpret the equations in terms of units as defined in the earlier part of this work it is found that both these constants have the dimensions of a velocity. Their actual magnitudes could then be obtained by measurements of the relative magnitudes of the quantities involved. We could for instance determine the constant c' in the relation

$$\int_s \mathbf{H}_s ds = \frac{4\pi J}{c'},$$

connecting the magnetic field of a linear current with the strength of that current by examining the field of a given current and evaluating directly

the integral on the left. The exact method of doing this will appear later. Again, the constant c in the relation

$$-\frac{1}{c}\frac{d}{dt}\int_f \mathbf{B}_n df = \int_s \mathbf{E}_s ds,$$

connecting the electromotive force in a circuit with the rate of change of induction through it might be deduced by moving a circuit in a known magnetic field and determining the current produced in it. If measurements of this type, or others involving the same principles, are made it is found that the two constants c and c' are exactly the same in actual magnitude, viz.

$$c = c' = 3 \,.\, 10^{10} \text{ cm./secs.}$$

We shall therefore in our future analysis always use the equations with $c = c'$ and imply the value just given, the fundamental importance of which will subsequently appear.

396. Differential form of fundamental relations: continuous current distributions. Our previous discussions have been almost entirely confined to the mathematical abstractions of linear currents and the laws of the magnetic field were given in a form suitable to this method of expression. We have however in the previous chapters already prepared ourselves for the discussion of volume distributions of current flow and we must now discuss the extensions of the present ideas to such cases. The notion of a linear current is obtained from the idea of a current flowing in a conducting wire whose cross section dimensions are infinitely small compared with the other lengths involved in the analysis: this condition was illustrated when we saw how such an assumption led to difficulties in the analysis of the magnetic field when we got too near the conducting circuit. The analysis obtained above can however be very easily extended to continuous volume distributions of current by dividing such currents into single current elements, of small cross-section, each of which can be regarded as a linear current in the ordinary way. The combination of all these elementary currents laid side by side constitutes the finite current.

We shall now assume that in all stationary or quasi-stationary conditions the total current is a closed one and thus each of the elementary currents may be regarded as closed and the previous general laws apply to it.

397. The first fundamental notion above states that in any space in which stationary electric currents are flowing the line integral of the magnetic force round any closed path is equal to $\frac{4\pi}{c}$ times the algebraic sum of all the currents flowing through the path. This can be interpreted in terms of continuous analysis, for if we use $\mathbf{C}$ as the total current density at any

point then the sum of all the elementary currents flowing through the curve is

$$\int_{e} \mathbf{C}_n\, df,$$

taken over any surface abutting on the curve. Our rule thus gives

$$\int_{s} \mathbf{H}_s\, ds = \frac{4\pi}{c}\int_{f} \mathbf{C}_n\, df.$$

The integral on the left can be converted by Stokes's rule and we get

$$\int_{f}\left(\text{curl}\, \mathbf{H} - \frac{4\pi\mathbf{C}}{c}\right)_n df = 0,$$

and as the curve s and surface f have been arbitrarily chosen we must have

$$\text{curl}\, \mathbf{H} = \frac{4\pi\mathbf{C}}{c},$$

a vector equation expressing Ampère's circuital relation in differential form. The essential condition is that the current should be a stream vector, it should always flow in complete circuits, analytically

$$\text{div}\, \mathbf{C} = 0.$$

398. The second fundamental principle is equally easily adopted to the present scheme. Consider for example any closed circuit drawn in the conducting material and count the number of lines of magnetic induction which pass through it. Then according to Faraday's law there will be an electromotive force in the circuit which is proportional to the rate of diminution of this number of lines of induction. The electromotive force is

$$\int_{s} \mathbf{E}_s\, ds,$$

and according to Faraday this is

$$= -\frac{1}{c}\frac{dN}{dt},$$

where $N = \int_{f} \mathbf{B}_n\, df$, so that we have

$$\int_{s} \mathbf{E}_s\, ds = -\frac{1}{c}\frac{d}{dt}\int_{f} \mathbf{B}_n\, df,$$

a form already quoted for the case of linear conductors. It is here regarded as applying to any circuit drawn in the conducting substance, whether that circuit is the path of one of the elementary currents or not. By again using Stokes's theorem we can write this in the form

$$\int_{f} (\text{curl}\, \mathbf{E})_n\, df = -\frac{1}{c}\frac{d}{dt}\int_{f} \mathbf{B}_n\, df,$$

so that if the circuit is not moving we have

$$\int_{f}\left(\text{curl}\, \mathbf{E} + \frac{1}{c}\frac{d\mathbf{B}}{dt}\right)_n df = 0,$$

or owing to the arbitrary nature of the circuit

$$-\frac{1}{c}\frac{d\mathbf{B}}{dt} = \text{curl}\,\mathbf{E},$$

which is the differential form of Faraday's circuital relation.

399. These are the fundamental equations of the generalised electromagnetic theory. It is however convenient to interpret them in terms of certain auxiliary vectors and scalars. In calculating N as the number of lines of induction through the circuit we take the barrier surface f and sum up all over it

$$N = \int_f \mathbf{B}_n df,$$

and of course this implies that the result depends only on the circuit and not on the particular barrier surface f taken to measure it on. The condition for this is

$$\text{div}\,\mathbf{B} = 0,$$

and is of course satisfied in our case. We should therefore be able to express N in terms of the circuit alone; this is done in Stokes's theorem, for if

$$\mathbf{B} = \text{curl}\,\mathbf{A},$$

then

$$N = \int_f \mathbf{B}_n df = \int_s \mathbf{A}_s ds.$$

This quantity $\mathbf{A}$ is the vector potential of the field; it enables us to abolish the idea of barrier surfaces; all results can be interpreted in terms of the circuit alone.

400. Now let us examine the electromotive forces. The electromotive force round a circuit is, by Faraday's rule, equal to

$$-\frac{1}{c}\frac{dN}{dt} = -\frac{1}{c}\frac{d}{dt}\int_s (\mathbf{A}\,d\mathbf{s}),$$

and if the circuit is fixed this is

$$= -\frac{1}{c}\int_s \left(\frac{d\mathbf{A}}{dt}\cdot d\mathbf{s}\right).$$

But by definition this is the same as

$$\int (\mathbf{E}\,d\mathbf{s}),$$

and thus

$$\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{A}}{dt},$$

is a particular solution for the electric force at a point: to generalise it we must add the gradient of any acyclic function ϕ which would give nothing on integration round the closed circuit

$$\mathbf{E} = -\frac{1}{c}\cdot\frac{d\mathbf{A}}{dt} - \text{grad}\,\phi.$$

This is the general expression for the electric force in the field.

The first term in this expression for $\mathbf{E}$ is the electrodynamic part and the second the electrostatic part: the first is motional and the second statical or elastic.

401. The three differential equations involved in the vector relation

$$\mathbf{B} = \operatorname{curl} \mathbf{A},$$

are however not sufficient to determine $\mathbf{A}$ because if $\mathbf{A}$ is one solution then obviously

$$\mathbf{A} + \operatorname{grad} \chi,$$

is another, χ being any function of the coordinates. To define $\mathbf{A}$ more completely we may therefore impose another condition. Maxwell takes

$$\operatorname{div} \mathbf{A} = 0,$$

and this appears to be the most convenient although it still leaves a certain amount of indefiniteness. From this definition of $\mathbf{A}$ we deduce at once that

$$\begin{aligned}\operatorname{curl} \mathbf{B} &= \operatorname{curl} . \operatorname{curl} \mathbf{A} \\ &= -\nabla^2 \mathbf{A} + \operatorname{grad} \operatorname{div} \mathbf{A},\end{aligned}$$

and if we take $\operatorname{div} \mathbf{A} = 0$ this gives

$$\operatorname{curl} \mathbf{B} = -\nabla^2 \mathbf{A}.$$

This equation being analogous to that of Poisson we may consider $\mathbf{A}$ to be the potential of a distribution of matter of density $\dfrac{\operatorname{curl} \mathbf{B}}{4\pi}$, attracting according to the inverse squares law. We thus see that

$$\mathbf{A} = \frac{1}{4\pi} \int \operatorname{curl} \mathbf{B} \frac{dv}{r},$$

the integral extending to the whole of space occupied by the electromagnetic field.

Now $$\mathbf{B} = \mathbf{H} + 4\pi \mathbf{I},$$

and $$\operatorname{curl} \mathbf{H} = \frac{4\pi \mathbf{C}}{c},$$

so that $$\mathbf{A} = \frac{1}{c} \int (\mathbf{C} + c \operatorname{curl} \mathbf{I}) \frac{dv}{r}.$$

In the case of a linear conductor carrying a current of strength J and when there is no magnetic matter about this reduces to

$$\mathbf{A} = \frac{J}{c} \int \frac{d\mathbf{s}}{r},$$

where $d\mathbf{s}$ is the vectorial element of the conducting line along which the integral is taken.

402. In the general case we have also

$$\rho = \text{div } \mathbf{D} = \frac{1}{4\pi} \text{div } \mathbf{E} + \text{div } \mathbf{P},$$

or on inserting the value of $\mathbf{E}$ in terms of $\mathbf{A}$ and ϕ and noticing that div $\mathbf{A} = 0$ we find that

$$\begin{aligned} \nabla^2 \phi &= -4\pi (\rho - \text{div } \mathbf{P}) \\ &= -4\pi (\rho + \rho'), \end{aligned}$$

where ρ' is the Poisson density of ideal electrification which is equivalent for some purposes of the electric polarisation. It follows then that

$$\phi = \int \frac{\rho + \rho'}{r} dv,$$

so that ϕ is the static electric potential of a distribution of density $(\rho + \rho')$.

403. Most subsequent writers* have adopted a slightly different definition of the vector potential which has certain advantages over that given by Maxwell. Starting from the definition of the magnetic induction in terms of a vector potential and the consequent derivation of the electric force in terms of this same potential and a scalar potential ϕ as above, it is assumed that these two potentials are connected by the equation

$$\text{div } \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0.$$

It is then easy to verify that $\mathbf{A}$ and ϕ satisfy respectively the equations

$$\nabla^2 \mathbf{A} = \frac{1}{c^2} \frac{d^2 \mathbf{A}}{dt^2} - \frac{4\pi}{c} (\mathbf{C}_1 + c \text{ curl } \mathbf{I}),$$

and

$$\nabla^2 \phi = \frac{1}{c^2} \frac{d^2 \phi}{dt^2} - 4\pi (\rho + \rho'),$$

where in the former equation $\mathbf{C}_1$ is used to denote the density vector of true electric flux.

Under ordinary circumstances and in finite regular fields the appropriate solutions of these equations follow immediately from the analytical theorem discussed in the introduction (§§ 25–29). When the conditions of the field vary in both space and time we have in fact

$$\phi = \int [\rho + \rho'] \frac{dv}{r},$$

and

$$\mathbf{A} = \frac{1}{c} \int [\mathbf{C}_1 + c \text{ curl } \mathbf{I}] \frac{dv}{r},$$

the integrals being extended over the whole of the field; r as usual denotes the distance from the typical element of integration to the field point at

* Cf. Lorentz and others in the *Encyk. d. math. Wissensch.* Bd. v.; Macdonald, *Electric Waves*; Lorentz, *Theory of Electrons*; Schott, *Electromagnetic Radiation*.

which the functions are calculated and the square brackets serve to indicate that the values of the functions affected are to be taken for the instant $t - \frac{r}{c}$, t being the instant at which the functions themselves are evaluated.

The advantage of this form of definition is that it expresses both potentials directly in terms of the positions and motions of the actual charge elements. The potentials themselves are usually called the *retarded potentials* because the contributions to them due to charges at a distance r away is not due to the instantaneous value of these charges but to their values at the previous time $\left(t - \frac{r}{c}\right)$. This means of course that the effect of any change in the charge distribution is not felt at points a distance r away until a time r/c after it has occurred, which is interpreted as implying that effects of electric changes are propagated outwards through space with the uniform velocity c in all directions.

404. The retarded potentials* are to be strongly contrasted with the instantaneous potentials of Maxwell's theory, and although they are perhaps more consistent with a propagation theory it is not to be inferred that the instantaneous potentials of Maxwell's theory necessarily implies the instantaneous propagation of effects from all parts of the field. It must be remembered that on both forms of the theory the potentials have been introduced primarily for analytical simplification and they do not necessarily represent directly definite physical quantities, although it may in certain circumstances be convenient to regard them as so doing. The ultimate procedure in either case involves the elimination of these potentials and the expression of all necessary relations in terms of the physical quantities that are propagated, without the aid of any auxiliary mathematical conceptions.

405. The retarded potentials are the most useful for the determination of the field of specified charge and current distributions but in their above form they are not directly suitable for numerical calculation, inasmuch as the elements of charge $[\rho]\,dv$ or current $[\mathbf{C}_1]\,dv$† which enters in their expression are not all present in the volume element dv at the same effective instant. To render them more useful we must express the integrals explicitly as functions of the instantaneous distributions of the charge and current elements, which are the data usually specified in any problem.

Regarded as functions of t the densities ρ and $\mathbf{C}_1$ for a given point of space may in the limit be discontinuous, as for instance when the boundary of a charged conductor crosses the point; but from the nature of the case the number of discontinuities or infinities which occur during any finite interval

* Cf. Larmor, *Aether and Matter*, pp. 111–112.

† We shall for the present drop the term in the vector potential due to the magnetisation. It can be replaced quite easily at any stage or may be included in the current $\mathbf{C}_1$.

of time is necessarily finite, and for each such irregularity the aggregate variation is also finite. Hence the quantities $[\rho]$ and $[\mathbf{C}_1]$ which occur in the integrals for the potential functions can always be expressed as Fourier integrals. Doing this and supposing that the values of both functions at any point (x_e, y_e, z_e) are prescribed for all values of t we get

$$\phi = \frac{1}{2\pi}\int dv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\mu\left(t-\frac{r}{c}-\tau\right)}\rho\frac{d\tau\, d\mu}{r},$$

and
$$\mathbf{A} = \frac{1}{2\pi c}\int dv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\mu\left(t-\frac{r}{c}-\tau\right)}\mathbf{C}_1\frac{d\tau\, d\mu}{r},$$

where in both integrals dv is the typical element of volume round the point (x_e, y_e, z_e) and

$$r^2 = (x-x_e)^2 + (y-y_e)^2 + (z-z_e)^2,$$

and ρ and $\mathbf{C}_1$ are now regarded as functions of τ with (x_e, y_e, z_e) as parameters.

Although these integrals appear to require a complete knowledge of the future history of the field for its present determination, they in reality do effectively define the field at the present instant independently of such knowledge. We may in fact choose ρ and $\mathbf{C}_1$ quite arbitrarily as far as future time is concerned; but when these values have been chosen the values of $[\rho]$ and $[\mathbf{C}_1]$ and hence also of ϕ and $\mathbf{A}$ are quite determinate for all time: they have the proper value for all past time whatever values may be selected for the future. If it is desired to express the integrals explicitly in terms of specified quantities only then the sine or cosine integrals must be used, but this seems to be unnecessary in the present instance.

We may now in these expressions rearrange the order of integration in the triple integrals; for this only amounts to a rearrangement of the terms of a triple sum which is for physical reasons known to be absolutely convergent. We may therefore effect the integration with respect to v first and then the difficulties of the kind mentioned above do not present themselves, because the summation is that of instantaneous contributions at the time τ from elements all over the field.

406. The whole of the circumstances in the surrounding field can now be determined from these forms of the potentials. To determine the electric force and magnetic induction vectors we use the relations

$$\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{A}}{dt} - \operatorname{grad}\phi,$$
$$\mathbf{B} = \operatorname{curl}\mathbf{A},$$

so that
$$\mathbf{E} = \frac{1}{2\pi c}\frac{\partial}{\partial t}\int dv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{i\mu\left(t-\frac{r}{c}-\tau\right)}}{r}\left(\rho\mathbf{r}_1 - \frac{1}{c}\mathbf{C}_1\right) d\tau\, d\mu$$
$$+ \frac{1}{2\pi}\int dv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{i\mu\left(t-\frac{r}{c}-\tau\right)}}{r^2}\rho\mathbf{r}_1\, d\tau\, d\mu,$$

and
$$\mathbf{B} = \frac{1}{2\pi c^2}\frac{\partial}{\partial t}\int dv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{i\mu\left(t-\frac{r}{c}-\tau\right)}}{r}[\mathbf{C}_1\mathbf{r}_1]\, d\tau\, d\mu$$
$$+ \frac{1}{2\pi c}\int dv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{i\mu\left(t-\frac{r}{c}-\tau\right)}}{r}[\mathbf{C}_1\mathbf{r}_1]\, d\tau\, d\mu,$$
wherein $\mathbf{r}_1$ denotes the unit vector in the line joining the typical element of integration to the field point where the functions are calculated so that
$$\nabla \mathbf{r} = \mathbf{r}_1.$$

407. Ampère's hypothesis on the origin of magnetism. Before closing this discussion reference can be made to an important consequence of an obvious analogy in the specification of the magnetic vector potential of a magnetic distribution and that of a continuous distribution of currents just deduced above.

It has already been seen that the vector potential $\mathbf{A}$ of a distribution of magnetic polarity of intensity $\mathbf{I}$ is given at points outside the magnetism by the equation
$$\mathbf{A} = \int [\mathbf{I} \,.\, \nabla] \frac{dv}{r},$$
but that at places inside the magnetism
$$\mathbf{A} = \left| \int [n \,.\, \mathbf{I}] \frac{df}{r} \right|_1^2 + \int \text{curl}\, \mathbf{I} \frac{dv}{r},$$
the previous formula being then plainly inapplicable because it integrates to a quantity whose differential coefficients are infinite where r can vanish. Analogously, it may be recalled that, in the ordinary statical theory of magnetism, the magnetic force is derived from the potential of the actual magnetic polarity only at places outside the magnets, but at places in its interior is derived from the potential of the Poisson volume and surface distributions of an ideal continuous magnetic substance. At a point in the interior of the magnetism the magnetic force should be in fact defined as the part of the force, acting on a unit pole there situated, that is independent of the local polarity at the spot, it being then shown how the definite value of this part can be determined.

This formula indicates that such a magnetic distribution is equivalent at all points to an electric current distribution of density equal to $c \,.\, \text{curl}\, \mathbf{I}$ throughout the volume of the magnet together with sheets of current given by $c\,[\mathbf{n}, \mathbf{I}]$ flowing along the interface.

This is the origin of Ampère's hypothesis that the magnetism of a substance has its origin in some intrinsic distribution of currents throughout the substance. Of course these currents must circulate within the molecules or molecular aggregates which is the ultimate element of the aggregate magnetic

polarity; they are minute current whirls not involving continuous flow in any direction.

The equivalence between these two systems, one a magnetic and the other a current system, includes *ex hypothesi*, that of the vector potential and therefore of its curl, that is of the magnetic induction **B** which is always a stream vector. They are not however equivalent as regards magnetic force; for in the one case the curl of the magnetic force is $\frac{1}{c}$ times the current, in the other it is null. In treating of a current system devoid of magnetism, the only quantity that occurs is the magnetic induction due to the currents; the portion of the expression for this induction which forms the contribution of the part of the current arising from contiguous molecules or elements of volume being always negligible compared with the induction as a whole.

408. For some analytical purpose it is convenient to convert in the manner indicated all the magnetism associated with the medium of the electromagnetic field into a volume distribution of electric currents and if the surfaces of the magnetic media are replaced by continuous transition layers the surface integrals disappear and thus

$$\mathbf{A} = c\int \text{curl}\,\mathbf{I}\,\frac{dv}{r},$$

and the distribution of current density c curl **I** effectively replaces the magnetism by itself.

Supposing now this current distribution is added to that specified in the Ampèrean circuital equation

$$\frac{4\pi\mathbf{C}}{c} = \text{curl}\,\mathbf{H},$$

then it is seen that the equation assumes the form

$$\frac{4\pi\mathbf{C}}{c} = \text{curl}\,(\mathbf{H} + 4\pi\mathbf{I}) = \text{curl}\,\mathbf{B},$$

where the current **C** is now supposed to include the distribution which replaces the magnetism. This form is convenient as it eliminates the magnetic force from the fundamental equations and thus leaves only the specification of the electric current to be effected in the complete theory, and in the modern theory of electrodynamics which explains the magnetism as the result of the motion of electrons this seems to be the only consistent course to adopt.

If we thus complete the current **C** the definition of the vector potential by the integral

$$\frac{1}{c}\int\frac{\mathbf{C}dv}{r} \text{ or } \frac{1}{c}\int\frac{[\mathbf{C}]\,dv}{r},$$

is completely effective in the most general case without the necessity for the introduction of the extra term in the magnetic polarisation.

409. Maxwell's generalised theory of the electromagnetic field. So far our discussions have been limited to electrostatic and electromagnetic phenomena which are either actually stationary or at least so slowly variable that they admit of treatment along similar lines. The restriction thus implied will be more fully discussed in a more appropriate place: all that is inferred is that the state of the motion adjusts itself so quickly at each instant that it is practically an equilibrium one under the conditions pertaining in each instant. For cases in which this restriction is satisfied the laws of Ampère and Faraday provide a sufficient and satisfactory foundation but a generalisation is needed to extend it beyond these cases. What are the fundamental laws of non-stationary electromagnetic fields in general?

The most successful attempt to answer this question was made by Maxwell. With Faraday he saw all the obvious phenomena of electromagnetics merely as the terminal aspects of a variation of condition in the space (or field) surrounding the apparatus. The observed actions are transmitted through and by a something, the aether, in this field which is capable of varying its condition. Although the aether is thus merely regarded as the definite something to which we can attach the vector quantities of the electromagnetic field, it is convenient to have a definite representation of its mode of action. Of course any such consistent representation is simply in set terms a dynamical or analytical theory of the activity of this aether.

The essential characteristics of a theory of this kind is of course that the actions must be transmitted by the aether with a finite velocity. This corresponds to a representation of the phenomena by means of differential equations connecting the time and space variations of the vectors with one another. The values of the vectors at a definite point of space are directly connected only with their values at infinitely near points, and only indirectly with the conditions at finitely distant points. The new departure instituted by Maxwell comes, when expressed mathematically, to a statement that the generalised inter-relation between the essential vectors in any electromagnetic field are always exactly expressed by the two fundamental differential relations already established for the field of closed currents. This statement of course involves the assumption that all *electric discharges* are effectively of the nature and possess the properties of systems of closed currents, being completed when necessary by the so-called displacement current in free space (i.e. in the aether) and in dielectric media; in fact the consideration of unclosed circuits never arises.

The extension here implied is assumed to apply not only to systems in which a part or the whole of electrical motions are those that take place in ordinary conduction currents but to every conceivable phenomenon which involves real electric motions, i.e. of course, motions of electrons. The true electric current of the complete Ampèrean equation will thus include all

possible types of coordinated or averaged motions of electrons, namely currents arising from conduction, from material polarisation and its convection and from the convection of charged bodies.

410. The vectors necessary for a complete specification of the conditions in the electromagnetic field at any point are:

(i) **E**, the intensity of the electrostatic field measured generally by the ratio of the resultant force to the charge on a small conductor placed at the corresponding point.

(ii) **D**, the total electric displacement of Maxwell consisting partly of a true displacement measured by the polarisation intensity **P** and partly by the aethereal displacement $\frac{\mathbf{E}}{4\pi}$ characteristic of Maxwell's theory.

(iii) **H**, the intensity of the magnetic force in the field measured in a method similar to that adopted for **E**.

Regarding the measures of **E** and **H** at points inside the ponderable bodies it is merely necessary to refer back to the critical discussions given on their introduction.

(iv) **B**, the magnetic induction vector which always satisfies the circuital condition

$$\operatorname{div} \mathbf{B} = 0.$$

It is connected with the magnetic force and magnetic polarisation intensity **I** by the general vector equation

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I},$$

so that in free space outside the magnetic matter

$$\mathbf{B} = \mathbf{H}.$$

(v) **C**, the total current of Maxwell's theory. In the general case this consists essentially of several distinct parts which require very careful specification. These are separately examined below.

411. The vectors thus specified in the most general case are now assumed to be correlated by the two fundamental equations of the theory, viz.

(i) Faraday's law

$$-\frac{1}{c}\frac{d\mathbf{B}}{dt} = \operatorname{curl} \mathbf{E}.$$

(ii) Ampère's law

$$\frac{4\pi\mathbf{C}}{c} = \operatorname{curl} \mathbf{H},$$

the two vectors **B** and **C** being however restricted to satisfy the conditions

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{div} \mathbf{C} = 0.$$

These two relations are quite independent of the nature of the medium filling the space of the electromagnetic field. They are in the nature of fundamental dynamical relations between the quantities involved and for this reason were adopted by Maxwell as the fundamental equations of the more generalised theory. They are of course self-consistent only so long as the current **C** and the magnetic induction **B** are circuital; it is therefore necessary to adopt Maxwell's hypothesis if we wish to use these equations for the general case; the one condition is involved in the other.

412. There are however four vector quantities involved in these two relations so that for a complete specification of the scheme two more relations are required. These are easily obtained but before proceeding to this we must examine the expression for the complete current density. This consists of several distinct parts which are best examined separately.

(*a*) $\mathbf{C}_1$, the current of conduction, is, as we have already seen, made up of a drift of electrons or ions, the positive ones travelling in one direction, the negative ones in the opposite direction, under the influence of the electric force.

(*b*) $\mathbf{C}_2$, the polarisation current associated with the material medium. We have already seen how the establishment of a condition of polarisation in a dielectric medium is accomplished on the Larmor-Lorentz view of the subject, by an electric displacement which may be caused either by turning round the little bi-polar molecules, or by an actual displacement of the charges relative to one another in the molecule. A time variation of such a state of affairs would therefore involve an electric current in the dielectric. The displacement in any position is the same as if the positive pole started from the final position of the negative pole and moved up to its final position. The total amount can thus be estimated and is such that

$$\mathbf{C}_2 = \frac{d\mathbf{P}}{dt}.$$

(*c*) A material medium moving with the velocity equal at the (x, y, z) point to $\mathbf{u}$ and having in the neighbourhood of that point a charge of amount ρ per unit volume, clearly contributes a convection current of density $\rho\mathbf{u}$. The elements of the moving medium may be the molecules or ions as in electrolysis, so that conduction currents are merely particular examples of convection currents. The convection current may however also consist simply of a convection of free electrons so that on such a theory a conduction current may in some respects be regarded as a convection current. It is therefore in reality rather difficult to distinguish between conduction and convection currents, although it is usual and convenient to use the distinction provided by the applicability of Joule's and Ohm's laws.

The convection of a material medium merely polarised to intensity **P** also supplies a part to the volume distribution of electric currents : but its determination requires more refined analyses. Consider in the first place the convection of a simple type of polar molecule involving a single electron $+q$ for one pole and $-q$ for the other pole. The transfer of these two electrons in company, as in the diagram, is equivalent to the transfer of a positive electron round the long narrow circuit in the direction of the curved arrow : and this circuit can be divided up into sub-circuits of ordinary form in the Ampèrean manner by partitions represented by the dotted lines. The distance between the two poles of the molecule is absolutely negligible compared with the distance that the molecule is carried by the convection, in a time which is effectively infinitesimal for the analytical theory of continuous currents even in its optical applications : so that the circumstance

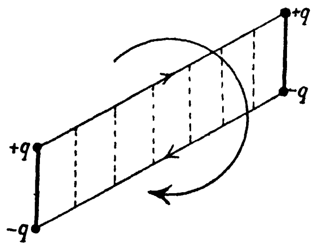


Fig. 72

that convection round the ends of the elongated circuit is not really effected is immaterial. It follows that the convection with velocity **u** of a medium, containing such molecules polarised or orientated to intensity **P** gives rise to an Ampèrean system of currents round minute circuits, which forms effectively a magnetic polarisation, or quasi-magnetism of the material, of intensity $\frac{1}{c}$ [**Pu**] per unit volume. This result will clearly not be disturbed when the distribution of polarity in the molecule is more complicated than that here assumed.

It will be convenient in most cases to retain this mode of specification by means of a distribution of magnetisation, as it will enable us to take direct advantage of the known principles governing such distributions. But we can restore it directly to the form of a distribution of currents by the transformation explained in the last section and this shows that the distribution is generally equivalent to a current of density

curl [**Pu**],

the surface of the medium being replaced by a gradual transition to avoid the introduction of the surface distribution.

(*d*) $\mathbf{C}_3$, the displacement current in the aether introduced by Maxwell as an aethereal current of such amount as to complete into a single circuital stream all the types of true electric flux which are associated with the matter. This is shown, as above, to be measured by the density

$$\frac{1}{4\pi}\frac{d\mathbf{E}}{dt},$$

in the general case.

The total current density in the general case is therefore

$$\mathbf{C} = \mathbf{C}_1 + \frac{d\mathbf{P}}{dt} + \frac{1}{4\pi}\frac{d\mathbf{E}}{dt} + \rho\mathbf{u} + \text{curl}\,[\mathbf{P}\,.\,\mathbf{u}]$$

and this is always circuital, or $\quad \text{div}\,\mathbf{C} = 0$

always. This is the essence of Maxwell's assumption and when once granted the general theory is much simplified. All the consequences of this theory, some of which will occupy our attention later, confirm us in the view that this hypothesis of Maxwell's is the correct one to make for the general theory.

413. The significance of the various terms in the complete expression for the true electric flux density which depend on the electric [and magnetic] polarisations can also be exhibited in a more analytical form, if we assume that all such polarisations result entirely as the average aspects of a more or less complex distribution of electrons or point charges of both signs. In such a case the part of the current density depending on these electrons at any point in the medium is the limiting ratio of the volume of the physically small element δv to the sum $\quad \Sigma q\mathbf{v},$

extended over all the electrons inside it; $\mathbf{v}$ is the velocity of the typical electron. In effecting the summation care must be taken to refer each electron to its proper position in the matter. Having fixed on a definite point in the medium moving with the velocity $\mathbf{u}$ we notice that the electron attached to it is displaced from that point through a distance $\mathbf{r}$ on account of the polarisation, and has therefore a velocity

$$\mathbf{u} + \dot{\mathbf{r}} = \mathbf{u} + \frac{d\mathbf{r}}{dt} + (\mathbf{u}\nabla)\,\mathbf{r}.$$

The electron which is actually at the point in the matter under review and which properly belongs to some other point has therefore the velocity

$$\mathbf{u} + \dot{\mathbf{r}} - (\mathbf{r}\nabla)\,\mathbf{u} - (\mathbf{r}\nabla)\,\dot{\mathbf{r}},$$

it being assumed that the velocity of an electron is a continuous function of its position in the matter. This is the value that has to be taken for $\mathbf{v}$ in effecting the summation as above.

Again we must notice that the summation is to be taken only over those electrons actually in the volume element under consideration, the number

of which is a function of their displacements. In fact in setting up the displacements typified by $\mathbf{r}$ the charge displaced across any surface is

$$\int_f \Sigma q\,(\mathbf{r}_n df),$$

Σ denoting a sum for the electrons in the volume element $\mathbf{r}_n df$. A simple application of Green's lemma shows that the statistical effect of this displacement is the same as if each electronic charge inside the surface were reduced in the ratio

$$1 - \operatorname{div}\mathbf{r} : 1,$$

the number remaining unaltered.

414. It is thus the summation of

$$\Sigma q\,(1-(\nabla\mathbf{r}))\left[\mathbf{u}+\frac{d\mathbf{r}}{dt}+(\mathbf{u}\nabla)\,\mathbf{r}-(\mathbf{r}\nabla)\,\mathbf{u}-(\mathbf{r}\nabla)\frac{d\mathbf{r}}{dt}-(\mathbf{r}\nabla)\,(\mathbf{u}\nabla)\,\mathbf{r}\right],$$

or, neglecting quantities of higher order in the displacement, of

$$\Sigma q\left[\mathbf{u}-\mathbf{u}\,(\nabla\mathbf{r})+(\mathbf{u}\nabla)\,\mathbf{r}-(\mathbf{r}\nabla)\,\mathbf{u}+\frac{d\mathbf{r}}{dt}-\frac{d\mathbf{r}}{dt}(\nabla\mathbf{r})-(\mathbf{r}\nabla)\frac{d\mathbf{r}}{dt}\right],$$

that is to be effected. Owing to the smallness of the volume element δv we may assume that $\mathbf{u}$ and its space gradients are constant throughout, whilst

$$\Sigma q=\rho\delta v,\qquad \Sigma q\mathbf{r}=\mathbf{P}\delta v,$$

where ρ is the density of the free charge and $\mathbf{P}$ the polarisation intensity of the element. Thus

$$\Sigma q\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\mathbf{P}\delta v)=\left\{\frac{d\mathbf{P}}{dt}+\mathbf{P}\,(\nabla\mathbf{u})\right\}\delta v,$$
$$\Sigma q\,(\mathbf{u}\nabla)\,\mathbf{r}=(\mathbf{u}\nabla)\,\Sigma q\mathbf{r}=(\mathbf{u}\nabla)\,\mathbf{P}\delta v,$$
$$\Sigma q\mathbf{u}\,(\nabla\mathbf{r})=\mathbf{u}\,(\nabla,\ \Sigma q\mathbf{r})=\mathbf{u}\,(\nabla\mathbf{P})\,\delta v,$$
$$\Sigma q\,(\mathbf{r}\nabla)\,\mathbf{u}=(\Sigma q\mathbf{r},\ \nabla)\,\mathbf{u}=(\mathbf{P}\nabla)\,\mathbf{u}\delta v.$$

Due to these terms we have therefore a current of density

$$\rho\mathbf{u}+\frac{d\mathbf{P}}{dt}+(\mathbf{u}\nabla)\,\mathbf{P}-(\mathbf{P}\nabla)\,\mathbf{u}-\mathbf{u}\,(\nabla\mathbf{P})+\mathbf{P}\,(\nabla\mathbf{u})=\rho\mathbf{u}+\frac{d\mathbf{P}}{dt}+\operatorname{curl}[\mathbf{P}\mathbf{u}],$$

which agrees with the result obtained above.

415. The remaining terms have another significance. They are

$$-\Sigma q\left\{\frac{d\mathbf{r}}{dt}(\nabla\mathbf{r})+(\mathbf{r}\nabla)\frac{d\mathbf{r}}{dt}\right\}=-\Sigma q\,(\nabla\mathbf{r})\frac{d\mathbf{r}}{dt}=-\tfrac{1}{2}\Sigma q\frac{d}{dt}\{(\nabla\mathbf{r})\,\mathbf{r}\}$$
$$+\operatorname{curl}\tfrac{1}{2}\left\{\Sigma q\left[\mathbf{r},\ \frac{d\mathbf{r}}{dt}\right]\right\},$$

where in these equations the operator ∇ is presumed to affect all quantities immediately following it.

The second term on the right-hand side of the last equation receives its main contribution from those electrons which are executing rapid orbital

motions about their mean positions in the matter, the contribution of any one electron being proportional to the moment of its momentum about the equilibrium position. Moreover if the orbital motions are to any extent permanent the sum
$$\Sigma q\,(\nabla \mathbf{r})\mathbf{r},$$
for these electrons will be practically independent of the time. Thus when some or all of the electrons are executing motions of the type considered there is an additional term in the complete expression for the current density which to the first approximation is equal to
$$c \operatorname{curl} \mathbf{I},$$
where
$$\mathbf{I}\delta v = \frac{1}{2c}\Sigma q\left[\mathbf{r}, \frac{d\mathbf{r}}{dt}\right].$$
In this form we recognise that the additional term in the current is the same as would be contributed by the medium magnetically polarised to intensity $\mathbf{I}$, if the magnetism is regarded as equivalent to a distribution of minute current whirls. This is a tentative suggestion of an electron theory of magnetism, which will be further discussed in the sequel. For the present we shall usually disregard terms of this type in the current expression.

416. In the whole of the above discussion it has been assumed that the matter extends continuously throughout and beyond the element under direct observation. If as is sometimes the case it is necessary to include the effects of discontinuities in the material distribution these can always be estimated by regarding such discontinuities as continuous rapid transition regions throughout which the definitions given above remain effective. In this way it is easily seen that a surface of discontinuity in the material medium is to be regarded as the seat of a current sheet of density
$$c\,[\mathbf{n}_1\mathbf{I}_1]\Big|_1^2,$$
where
$$\mathbf{I}_1 = \mathbf{I} + \frac{1}{c}[\mathbf{P}\mathbf{u}],$$
and $\mathbf{n}_1$ is the unit vector normal to the surface in the direction from the side 1 to the side 2. The notation implies that it is the difference of the values of the function on the two sides that is to be taken as the current density.

Thus, for example, in any continuous piece of matter with a definite boundary the electronic motions in it may be specified in their average aspect as being equivalent to a distribution of body currents of density
$$\mathbf{C}_1 + \rho\mathbf{u} + \frac{d\mathbf{P}}{dt} + c \operatorname{curl} \mathbf{I}_1$$
throughout its mass together with the surface current sheet of density
$$-c\,[\mathbf{n}_1\mathbf{I}_1] = c\,[\mathbf{I}_1\mathbf{n}_1],$$
at any point of its outer boundary.

These surface currents are important when we consider the dynamical aspects of electromagnetic phenomena.

417. Returning to our fundamental equations we now want the relations connecting the vectors involved. We know already that there will be some relation connecting the current intensity with the electric force, something in the nature of Ohm's law only rather more general. There is also a relation connecting the magnetic force and magnetic induction. These two constitutive relations depend essentially on the nature of the ponderable matter involved and must therefore be obtained by experiment. The first two relations being of the nature of dynamical principles must be exact, but this second pair obtained by experiment can only be approximate. As to the actual form of these relations we may state the following, referring back to the respective previous chapters for a discussion of their relative merits.

418. (i) In any given material medium devoid of hysteretic quality, the intensity of electric polarisation **P** must be a mathematical function of the electric force **E** which excites it. In ordinary cases, certainly in all cases in which the exciting force is small the relation between **P** and **E** is a linear one: thus in the general problem of an aeolotropic medium there will be nine static dielectric coefficients. The principle of negation of perpetual motions requires this linear relation to be self-conjugate and so reduces the nine coefficients to six. In the special case of isotropy there is only one coefficient and the relation may be expressed in the usual form

$$\mathbf{P} = \frac{\epsilon - 1}{4\pi} \mathbf{E},$$

where ϵ is the single dielectric constant of the medium.

In problems relating to moving material media the question may naturally arise whether the value of ϵ for the medium is sensibly affected by its movement through the aether. When it is considered that each molecule that is polarised by the electric force has effectively two precisely complementary poles, positive and negative, it becomes clear that a reversal of the motion of the material medium cannot alter the polarity induced: hence the influence of the motion on ϵ can only depend on square and higher even powers of the velocity.

419. (ii) In cases in which the magnetisation induced in the medium is of sufficient magnitude to be taken into account, similar statements will apply to it. In the general crystalline medium there are six independent coefficients of magnetisation: these reduce for an isotropic medium to a single coefficient and the relation between induction and force is

$$\mathbf{B} = \mu \mathbf{H}.$$

A simple equation of this kind, representing linear and reversible magnetisation applies to substances such as iron only when the field is of small intensity.

(iii) Finally the relation between the current of conduction $\mathbf{C}_1$ and the electric force may be taken as a linear one involving nine independent coefficients of conductivity: in the case of isotropy these reduce to a single one

$$\mathbf{C}_1 = \kappa \mathbf{E}.$$

It has been found by experiment that coefficients of electric conduction, unlike the other coefficients above considered, remain constant for all intensities of the current up to very high limits, so long as the temperature and physical condition of the conducting substance are not altered. This is what was perhaps to be anticipated from the circumstance that conduction arises from the filtering of the simple non-polar electrons or ions through the conducting medium under the directing action of the electric force, not from orientation of polar complex molecules which may originate hysteretic changes in their cohesive grouping in the substance.

420. For a body of compound nature at rest and in which both polarisation and conduction currents can coexist the relation between the total current of Maxwell's scheme and the electric force is more complicated than those given above. In fact

$$\mathbf{C} = \mathbf{C}_1 + \dot{\mathbf{D}}$$

$$= \mathbf{E} + \frac{\epsilon}{4\pi} \dot{\mathbf{E}},$$

and this and the relation

$$\mathbf{B} = \mu \mathbf{H}$$

are the two directly required in the theory involving only media at rest.

With these four relations we have then a complete electromagnetic scheme, which if presumed to apply in the whole range of electrodynamic phenomena, is a sufficient basis for the mathematical development of the subject with regard to media at rest.

For media in motion the additional terms arise which are directly connected with the other vectors in the field by relations more of a dynamical than of a constitutional character and for this reason a full discussion of their significance is postponed to another chapter.

The last four chapters of this book are mainly confined to the consideration of various aspects of this theoretical scheme as a basis for all electromagnetic and electrodynamic phenomena and further discussion of its import will be reserved for those chapters.

CHAPTER X

SOME SPECIAL ELECTROMAGNETIC FIELDS

421. The magnetic field of special current distributions. It is often desirable to have a complete knowledge of the magnetic field associated with certain simple types of current distribution, not only for the purpose of testing the validity of the theory, but also with a view to obtaining the most suitable arrangements for practical purposes in electrotechnics. The method is to work out the cases which are obviously susceptible of rigorous mathematical treatment and then to construct the practical instruments to agree with the theoretical arrangement or to attempt by a method of successive approximation to obtain some idea of the working of the practical case from the behaviour of the nearest representative in the theoretical cases. We must therefore examine the fields of certain simple current distributions with a view to obtaining for reference as many workable cases as possible.

422. (*a*) *The magnetic field of an infinitely long straight current in a linear conductor.* If we consider the wire in the axis of z from $-\infty$ to $+\infty$ then the z-component of the magnetic vector potential of the field is the only one that exists and in the case of a positive current J this is

$$\mathbf{A}_z = J\int_{-\infty}^{+\infty} \frac{dr}{\sqrt{r^2 + z^2}} = C - \frac{2J}{c}\log r,$$

where r is the distance of the point in the field from the wire.

The magnetic potential in the field is therefore

$$\psi = -\frac{2J}{c}\,\theta,$$

where θ is the angle measured round the axis of z. This can also be deduced in the elementary manner for the solid angle subtended at any point P in the field is the area of the lune of the sphere of angle θ.

We have thus

$$\mathbf{H}_z = \mathbf{H}_r = 0, \qquad \mathbf{H}_\theta = \frac{2J}{c}\cdot\frac{1}{r}.$$

The force is therefore of magnitude $\frac{J}{2\pi c} \cdot \frac{1}{r}$ and is perpendicular to the radius r and the current direction, in the sense which is associated with the direction of the current in the same way as rotation to advance in a right-handed screwing motion.

The field of any straight current of finite length at distances small compared with its length is similarly determined and gives the same result. The magnetic lines of force are in circles round the wire; we should then have on integrating round one of them

$$\int \mathbf{H}_s ds = 2\pi r \mathbf{H} = \frac{4\pi J}{c},$$

and this is another reason for the form of $\mathbf{H}$ taken.

The same result can be applied even to a closed curved current circuit. If we have a current in a curved wire then very near the wire at any point it may be treated as practically straight and thus the magnetic force is as before. It is an interesting problem to work out the first order correction due to the curvature in such a case.

423. In the above argument the current is treated as infinitely thin. But very near the wire these results do not hold, the shape of the wire and the distribution of the current in it being then of fundamental importance. To treat cases of this nature we have merely to regard the current as divided up into a large number of elementary current filaments, each of which is straight. The total field is then the superposition of the fields of each of these filaments. At any point therefore

$$\mathbf{A}_z = C - \frac{2}{c}\int \delta J \log r,$$

r now denoting the distance of the point in the field from the axes of the typical current filament δJ. Of course if we knew the shape of the cross section and the distribution of the current over it we could find $\mathbf{A}_z$ at each point of the field. As a first approximation however in slowly varying fields* we may treat the current as uniformly distributed over the section so that

$$\mathbf{A}_z = C - \frac{2J}{c}\int_f df \log r,$$

and this latter integral can be interpreted as a logarithmic potential integral.

If the cross section is circular and of radius a we get:

(i) at external points

$$\mathbf{A}_{0_z} = C - \frac{2J}{c} \log R,$$

* Cf. below, p 559.

R being now the distance from the axis; and thus

$$\mathbf{H}_z = \mathbf{H}_r = 0, \qquad \mathbf{H}_\theta = \frac{2J}{c} \cdot \frac{1}{R}.$$

(ii) at internal points

$$\mathbf{A}_{i_z} = C - \frac{J}{c}\frac{R^2}{a^2},$$

and thus
$$\mathbf{H}_z = \mathbf{H}_r = 0, \qquad \mathbf{H}_\theta = \frac{2J}{c}\frac{R}{a^2}.$$

Notice that there is no magnetic potential inside corresponding to $\psi_0 = -\frac{2J\theta}{c}$ outside.

424. If there are several parallel wires present in the fields and carrying currents $J_1, J_2, \ldots J_n$ then the vector potential of the field A is still such that $\mathbf{A}_x = \mathbf{A}_y = 0$ but outside the wires

$$\mathbf{A}_{0_z} = - \sum_{p=1}^{n} \frac{2J_p}{c} \log r_p + \text{const.},$$

a result which is still true right up to the wires if the wires have circular cross sections and carry their currents uniformly.

The internal field of the pth wire under similar conditions is obtained by superposing on its own internal field the external one due to all the others and is thus determined by

$$\text{const.} - \mathbf{A}_{i_z} = \frac{J_p}{c}\frac{r^2}{a_p^2} + \sum_{q=1}^{p-1} \frac{2J_q}{c} \log r_q + \sum_{q=p+1}^{n} \frac{2J_q}{c} \log r_q.$$

From these the magnetic field strength can be determined as previously. The results confirm those already obtained in the limiting case.

As a special case we may quote the results for two parallel wires of equal circular section (a) carrying equal currents J in opposite directions. For these in the external field

$$\mathbf{A}_z = -\frac{2J}{c} \log \frac{r_1}{r_2},$$

and in the interiors of the wires

$$\mathbf{A}_{i_z} = -\frac{2J}{c}\left(\frac{1}{2}\frac{r_1^2 - a^2}{a^2} - \log \frac{r_2}{a}\right),$$

in the first and

$$\mathbf{A}_{i_z} = +\frac{2J}{c}\left(\frac{1}{2}\frac{r_2^2 - a^2}{a^2} - \log \frac{r_1}{a}\right),$$

in the second. The field in this case is far more concentrated into the space near the wires than when the currents flow in the same direction along the wires.

425. (*b*) *The magnetic field of a solenoid.* A current J in a solenoidal circuit can be replaced by a lot of parallel circular currents round the solenoid and a uniform current J uniformly distributed round the solenoid and flowing along its length. If the coil is very long a uniform current flowing in its length is equivalent to the same current flowing up a wire in the middle. Thus if we complete the circuit by a wire down the middle the upward current would practically be cancelled and we could then neglect it. It is completely cancelled if the coil is very long and practically so in most other cases.

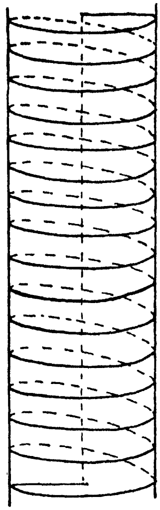


Fig. 73

We can then treat the coil as a series of parallel currents; in any case the outstanding part is merely of the order of a single convolution and is therefore negligible in a closely wound coil. If we replace each circular current by its equivalent magnetic shell we see that the coil is equivalent to a bar magnet. The current circulating is of strength nJ per unit length of coil, n being the number of turns per unit length: and this flow is equivalent to uniform magnetisation inside the solenoid of intensity $4\pi nJ$. The components of the magnetisation would then be

$$\mathbf{I}_x = 0, \qquad \mathbf{I}_y = 0, \qquad \mathbf{I}_z = \frac{4\pi nJ}{c}.$$

The coil is thus in all equivalent to:

(1) an ideal volume density

$$\frac{\partial \mathbf{I}_z}{\partial z} = \frac{4\pi J}{c}\frac{\partial n}{\partial z},$$

(2) and magnetic polarity

$$\sigma = \frac{4\pi nJ}{c},$$

at the ends.

If the winding is uniform $\dfrac{\partial n}{\partial z} = 0$ and thus the magnet is simple, with its poles right at the ends.

This of course applies only to outside space. The current in the solenoid is equivalent as regards its magnetic action at external points to the magnetic distribution specified.

426. The same method will however determine the internal field at any point. We have merely to separate the two faces of adjacent equivalent magnets so that there is empty space between them in which the force can be calculated as before. The force is equal to $4\pi\sigma$, where σ is the strength

of the polarity on one side of an equivalent magnetic shell. If t is the thickness of this shell, then σt is the strength of the shell and this is pro-

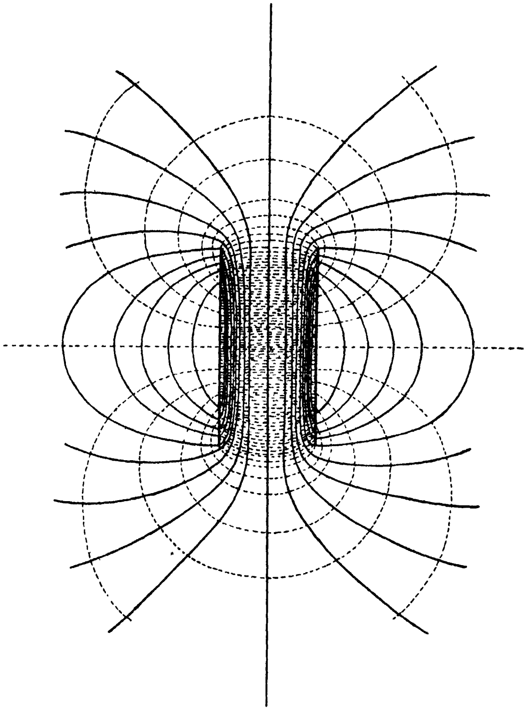


Fig. 74

portional to the total current in the breadth of the shell; but the number of convolutions in the breadth is

$$nt,$$

and if J is the current flowing, the total current round the shell is

$$ntJ,$$

and thus
$$\sigma t = 4\pi \frac{ntJ}{c}.$$

Thus the strength of the field inside the solenoid is

$$\mathbf{H} = 4\pi \frac{nJ}{c},$$

outside we have practically

$$\mathbf{H} = 0.$$

All this is correct as long as we remain well inside the shell; as soon as we get up near the ends irregularities enter and we are no longer able to apply the results. Down inside the coil the effect of the ends is very small and the

force is as calculated, so that the lines of force are straight up, near the ends they begin to bend out and they complete themselves round in outside space. The field is very intense inside compared with what it is outside, which if the winding is uniform is merely due to the uncompensated polarities at the ends.

We can obtain a first approximation to the behaviour of the field at the ends, inside the coil; for there the field is compounded of the part $4\pi nJ$ with that due to the end polarity (uncompensated). We thus see that if the coil is densely wound the uniformity in the field remains good very close up to the ends, and practically all the lines of force go right up and out at the ends without cutting through the sides.

A more fundamental aspect can be given to these results. If we assume that the magnetic force is uniform inside the solenoid and very small outside, the integral

$$\int \mathbf{H}_s ds,$$

taken along any line of force is approximately equal to

$$l\mathbf{H},$$

where l is the length of the coil and $\mathbf{H}$ the constant value of $\mathbf{H}$.

But this integral is equal to $\frac{1}{c}$ (current which threads the circuit s) or $\frac{JN}{c}$, where N is the total number of turns in the coil: thus

$$\mathbf{H} = 4\pi \frac{JN}{cl} = 4\pi \frac{nJ}{c},$$

where n is the number of turns per unit length.

427. In this discussion we have confined ourselves to the consideration of the magnetic force. This is however the same as the magnetic flux unless there is iron present. When there is no iron present there is no difference between the magnetic force and magnetic flux or induction.

Suppose however that the interior of the coil is filled up with an iron core. This core would be magnetised by the field, the intensity of the magnetisation being

$$\kappa \,.\, 4\pi \frac{nJ}{c},$$

due to the field of the current alone. If we could neglect the field due to itself (which is very improbable in the case of iron) this is all.

The end polarities would now be $\pm \frac{\kappa nJ}{c}$ and thus the external field is increased κ times; the external magnetic induction is increased κ times.

Inside the magnetic induction is

$$\mathbf{B} = (1 + 4\pi\kappa)\,\mathbf{H} = \mu\mathbf{H},$$

and is increased in a much greater ratio. For ordinary iron undeı usual circumstances κ is large and so μ is about 12κ*.

We thus see that iron concentrates the magnetic flux without altering the magnetic force much. This is the essential thing in dynamos, where it is a great flux intensity that is required.

428. (*c*) *The magnetic field of a circular current.* The general nature of the field for the case of a circular current is obvious from the analyses of the preceding cases. The lines of force in any plane through the centre and the axis of the circuit (i.e. the line through the centre perpendicular to the plane) will be exactly similar. For this purpose it is more convenient to use cylindrical polar coordinates (ϖ, θ, z) the first two being taken in the plane of the circuit and the second along its axis. There are then only the two coordinates of the magnetic force, viz. H_ϖ and H_z since $H_\theta = 0$. The vector potential may thus be taken to have only the one component $\mathbf{A}_\theta$ and this is given at once by

$$\mathbf{A}_\theta = \frac{2Ja}{c}\int_0^\pi \frac{\cos\theta\, d\theta}{\sqrt{z^2 + \varpi^2 + a^2 - 2a\varpi\cos\theta}},$$

(z, ϖ, 0) being the coordinates of the point at which it is calculated, and a the radius of the circle.

By the substitution

$$\xi = \cos\frac{\theta}{2},$$

this integral is transformed at once into the usual form for the complete elliptic integrals of the first and second kind, viz. F and E. We obtain in fact

$$\mathbf{A}_\theta = \frac{4J}{c}\sqrt{\frac{a}{\varpi}}\frac{1}{k}\left[\left(1 - \frac{k^2}{2}\right)F(k) - E(k)\right],$$

wherein the modulus k of the functions is given by the formula

$$k^2 = \frac{4a\varpi}{z^2 + (a + \varpi)^2}.$$

On the axis of the circle and at infinity $k = 0$ and on the wire itself $k = 1$. For values of $k \ll 1$ we can use the expansions of the elliptic integrals in power series and we then get

$$\mathbf{A}_\theta = \frac{\pi J}{8c}\sqrt{\frac{a}{\varpi}}\,k^3\left[1 + \frac{3k^2}{4} + \frac{75}{128}k^4 + \dots\right].$$

* Really $4\pi\kappa + 1$.

From this function we obtain at once by differentiation the components of the magnetic force intensity in the field

$$\mathbf{H}_\varpi = -\frac{\partial \mathbf{A}_\theta}{\partial z}, \qquad \mathbf{H}_z = \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi \mathbf{A}_\theta),$$

so that the lines of force in any plane through the axis are formed by the lines $\mathbf{A}_\theta = \text{const.}$ They are exhibited graphically below in the figure.

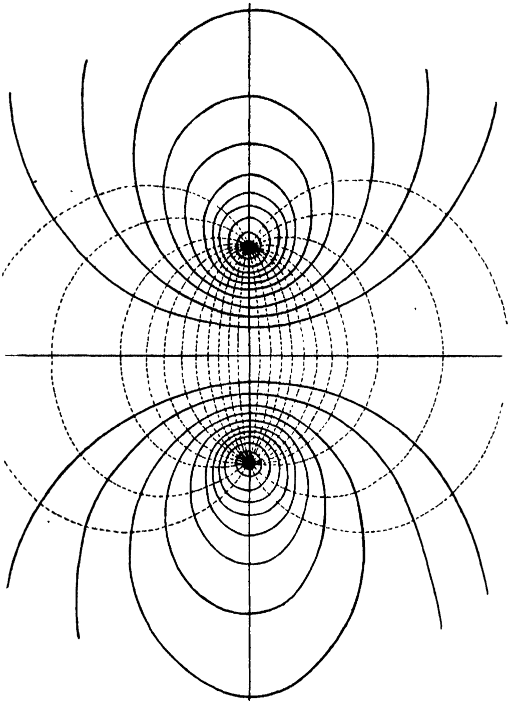


Fig. 75

To a first approximation very near to the axis we shall have

$$k^2 = \frac{4a\varpi}{(z^2 + a^2)^{\frac{3}{2}}},$$

and then also

$$\mathbf{A}_\theta = \frac{\pi a^2 J}{c} \frac{\varpi}{(z^2 + a^2)^{\frac{3}{2}}},$$

so that the magnetic force here has components which to the same order of approximation are given by

$$\mathbf{H}_z = \frac{2\pi J}{ca} \frac{a^3}{(a^2 + z^2)^{\frac{3}{2}}},$$

and

$$\mathbf{H}_\varpi = -\frac{3\pi a^2 J}{c} \frac{\varpi z}{(a^2 + z^2)^{\frac{5}{2}}}.$$

The field in the neighbourhood of the centre of the circle is thus to a first approximation normal to the circuit and of strength*

$$\mathbf{H}_z = \frac{2\pi J}{ac}.$$

429. This result is made use of for the construction of the instruments† for the measurement of currents which are known as galvanometers: the more simple of these arrangements is that known as the tangent galvanometer, which consists of a circular coil of the wire placed with its plane in the magnetic meridian, i.e. the vertical plane containing the resultant direction of the earth's magnetic field at the point. At the centre of the coil there is a very small magnet which can turn freely about a vertical axis. When the magnet is in equilibrium its axis will lie along the horizontal component of the magnetic force at the centre of the coil, thus when no current is flowing through the coil the axis of the magnet will be in the plane of the coil. A current flowing through the coil will produce in the neighbourhood of the magnet a magnetic force at right angles to the plane of the coil, proportional to the intensity of the current. Let this magnetic force be equal to GJ where of course

$$G = \frac{2\pi}{ac},$$

then G is called the *galvanometer constant.*

Let H denote the horizontal component of the earth's magnetic force at the centre of the coil. Then the resultant magnetic force at the centre of the coil has a component H in the plane of the coil and a component GJ at right angles to it, hence if θ is the angle which the resultant magnetic force makes with the plane of the coil

$$\tan\theta = \frac{GJ}{H},$$

so that

$$J = \frac{H\tan\theta}{G}.$$

When the magnet is in equilibrium its axis will be along the direction of the resultant magnetic force, hence the passage of the current will deflect the magnet through an angle $\tan^{-1}\frac{GJ}{H}$ and a knowledge of this angle enables us to determine J. As the current is proportional to the tangent of the angle of deflection, this instrument is called a *tangent galvanometer.* The larger we can make G the greater will be the sensitiveness of the galvanometer. If

* When two equal parallel coils are employed the field between them is much more uniform especially if they are at a distance apart equal to their common radius.

† Cf. Maxwell, *Treatise*, II. ch. XV.

the galvanometer consists of n terms of wire placed so close together that the distance between any two terms is a very small fraction of the radius, then the field of the current at the centre will be practically

$$\frac{2\pi n J}{ac},$$

so that
$$G = \frac{2\pi n}{ac}.$$

430. The same arrangement can also be used to measure the total quantity of electricity that passes through its coil, provided the electricity passes so quickly that the magnet of the galvanometer has not time appreciably to change its position while the electricity is passing. Let us suppose that, when no current is passing, the axis of the magnet is in the plane of the coil, then if J is the current passing through the plane of the coil, G the galvanometer constant, i.e. the magnetic force at the centre of the coil when unit current passes through it, m the moment of the magnet, the couple on the magnet while the current is passing is

$$GJm.$$

If the current passes so quickly that the magnet has not time sensibly to depart from the magnetic meridian while the current is flowing, the earth's magnetic force will exert no couple on the magnet. Thus if I is the moment of inertia of the magnet, θ the angle the axis of the magnet makes with the magnetic meridian, the equation of motion of the magnet during the flow of the current is

$$I\frac{d^2\theta}{dt^2} = GJm.$$

Thus if the magnet start from rest the angular velocity after the small time t is given by

$$I\frac{d\theta}{dt} = Gm\int_0^t J\,dt.$$

If the total quantity of electricity which passes through the galvanometer is Q and the angular velocity communicated to the magnet ω, we have therefore

$$I\omega = GmQ.$$

This angular velocity communicated almost instantaneously before the magnet alters its position, now makes it swing out of the plane of the coil: if H is the external magnetic force at the centre of the coil, the equation of motion of the magnet is, if there is no retarding force,

$$I\frac{d^2\theta}{dt^2} + mH\sin\theta = 0,$$

whence
$$I(\dot{\theta}^2 - \omega^2) + 2mH(1 - \cos\theta) = 0.$$

If α is the angular swing of the magnet then $\dot{\theta} = 0$ when $\theta = \alpha$, so that

$$I\omega^2 = 2mH(1 - \cos\theta)$$

$$= 4mH \sin^2\frac{\alpha}{2}.$$

Thus
$$Q = 2 \sin\frac{\alpha}{2} \frac{\sqrt{mH} \cdot I}{mG}$$

and thus a measurement of α would determine Q.

Galvanometers which are used for the purposes of measuring quantities of electricity are called 'ballistic galvanometers.'

There are other types of magnetic galvanometer which are frequently used but the principles of their action are the same as those described. A common form is that known as the Desprez-d'Arsonval galvanometer: its action is exactly similar to that of the tangent galvanometer but in it the magnets are fixed and the coil movable, and it is the motion of the coil that is observed and measured.

431. The field of a system of linear conductors—the coefficients of self and mutual induction. We have already mentioned the difficulties to be met with in considering a finite current in a very thin wire. At a point close up to the wire the magnetic field, consisting of a local part due to the near element (which can be considered as straight and very long compared with the distance of the point from it, if it is near enough) and that due to the rest can be approximately calculated as of strength

$$\mathbf{H} = -\frac{2J}{cr},$$

r being the small distance from the wire of the point. If r is small this is very big. The number of tubes through a small closed curve near the wire can be very big.

This of course only means that the cross section of the wire is fundamental; the field in reality depends on the cross section of the wire and then of course on the distribution of the current over it. Thus in order to obtain the exact relations of such fields, particularly as regards the reaction between the field and the current, we must proceed in a more fundamental manner; we must divide each current up into elementary filaments and estimate the field more closely as due to these filaments laid side by side to form the finite current.

432. Now consider one of the currents so divided into small current filaments of cross sections df_1, df_2, ... so that the current in the first is

$\frac{J_1'df_1}{f}$, that in the second $\frac{J_2'df_2}{f}$, and so on ..., where $f = \Sigma df$. The total current in the wire is

$$J = \frac{1}{f}\Sigma J'df = \frac{1}{f}\int J'df.$$

Now apply Faraday's circuital relation to the closed curve forming the axis of the typical current filament $\frac{J'df}{f}$ and we have

$$\int_s \mathbf{E}_s ds = -\frac{1}{C}\frac{dN}{dt} + \int_s \mathbf{E}_{e_s} ds,$$

where $\mathbf{E}$ is the electromotive force intensity in the whole field, $\mathbf{E}_e$ that in the applied external field, and N the flux of induction through the curve taken. But if κ is the specific conductivity of the material of the conductor at any place then Ohm's law gives

$$\frac{J'}{f} = \kappa \mathbf{E}_s,$$

so that
$$\int_s \frac{J'}{\kappa f} ds = \int_s \mathbf{E}_{e_s} ds - \frac{1}{C}\frac{dN}{dt}.$$

Thus even if the wire is uniform this equation shows that a uniform distribution of the current over the cross section is possible only in very special conditions for the distribution of the impressed electromotive force in the field. In all the ordinary cases however for which $\int_s \mathbf{E}_{e_s} ds$ has a value independent of the curve s inside the wire along which it is integrated we can assume the uniform distribution as a first approximation *provided the time variations of the field are not too rapid**. Under such circumstances J' is equal to J, the total current in the wire, and thus

$$\int_s \frac{J\,ds}{\kappa f} = \int_s \mathbf{E}_{e_s} ds - \frac{1}{C}\frac{dN}{dt},$$

which can also be written, using $\delta J = \frac{J\delta f}{f}$,

$$\delta J \int_s \frac{ds}{\kappa \delta f} = \int_s \mathbf{E}_{e_s} ds - \frac{1}{C}\frac{dN}{dt},$$

or using δk for the resistance of the filament in which the typical current δJ flows

$$\delta J \,.\, \delta k = \int_s \mathbf{E}_{e_s} ds - \frac{1}{C}\frac{dN}{dt},$$

whence on integration over the cross section for the average

$$Jk = \bar{\mathbf{E}}_e - \frac{1}{C}\frac{d\bar{N}}{dt},$$

* See below page 559.

where $\bar{\mathbf{E}}_e$ denotes an average value of $\int_s \mathbf{E}_{e_s} ds$ taken for the various filaments, and $\bar{N}$ an averaged value of N for the closed circuits formed by these filaments; k is the resistance of the wire so that

$$\frac{1}{k} = \Sigma \frac{1}{\delta k}.$$

This is the more exact form of Faraday's relation as applied to a linear current, when the dimensions of the wire are accounted for.

433. We now want to calculate $\bar{N}$ and this is done in the following manner for the particular case under review when the field is that due entirely to a system of currents. We can consider the most general case of n linear conductors in a field in which there are no strongly magnetic bodies about. We then know that $\bar{N}$ for each circuit must be a linear function of the currents in the various circuits; or

$$\bar{N}_r = a_{r1} J_1 + a_{r2} J_2 + \dots \quad a_{rn} J_n \; (r = 1, \; 2, \dots n)$$

$a_{r1}, a_{r2}, \dots$ being geometrical constants independent of the strengths of the currents, and $J_1, J_2, \dots$ these current strengths in the various circuits.

The coefficients a_{pp} are called the *coefficients of self-induction* of the respective circuits and the coefficients a_{pq}, those of *mutual induction*.

A complete determination of the field is however not necessary for a determination of the coefficients a_{pq}, if there are no magnetic substances about, an assumption we shall henceforth adopt. We know in fact that if $\mathbf{A}_r$ is the part of the vector potential $\mathbf{A}$ arising from the current in the rth conductor

$$\mathbf{A} = \Sigma \mathbf{A}_r$$

and

$$c\mathbf{A}_r = J_r \int df_r \int \frac{d\mathbf{s}_r}{r f_r},$$

or if the cross section is constant all round

$$c\mathbf{A}_r = \frac{J_r}{f_r} \int df_r \int \frac{d\mathbf{s}_r}{r},$$

wherein $d\mathbf{s}$ is the vector element of length of a current filament taken in the conductor and r the distance of the point of integration from the point in the field.

But then we know that N_{rs}, i.e. the induction through the typical filament of the sth conductor due to this current in the rth conductor is given by

$$N_{rs} = \int_s (\mathbf{A}_r d\mathbf{s}_s),$$

and then also the average value of N_{rs} for the current is

$$\bar{N}_{rs} = \frac{1}{f_s}\int df_s \int (\mathbf{A}_r d\mathbf{s}_s)$$

$$= \frac{1}{c}\frac{J_r}{f_r f_s}\int df_s \int df_r \int \frac{(d\mathbf{s}_r \,.\, d\mathbf{s}_s)}{r_{rs}},$$

and thus we see that

$$ca_{rs} = \frac{1}{f_r f_s}\int df_s \int df_r \int \frac{(d\mathbf{s}_r \,.\, d\mathbf{s}_s)}{r_{rs}}.$$

434. This result applies of course equally well for the coefficient of self-induction of the rth circuit on itself, but in the form

$$ca_{rr} = \frac{1}{f_r^2}\int df_r \int df_r' \int \frac{(d\mathbf{s}_r \,.\, d\mathbf{s}_r')}{r},$$

where each integration is taken once over the various filaments in the wire itself.

In the case of the mutual induction coefficients, when the cross sections of the wires are small compared with their distance apart we may write the formula in the very approximate form

$$ca_{rs} = \int \frac{(d\mathbf{s}_r d\mathbf{s}_s)}{r_{rs}},$$

whence we see that $a_{rs} = a_{sr}$; another view of this relation will subsequently appear.

This is Franz Neumann's formula for a_{rs}*.

435. The various coefficients of induction between circuits as thus defined of course only have a meaning when applied to closed current circuits; but the formulae obtained show that they are composed of contributions from each element of the circuit, and thus calculations of the values of the integrals for unclosed portions of the circuits would enable us to estimate the relative effects of the various parts as regards their contribution to the total coefficients under consideration. For example we see at once that the thinnest parts of the wires contribute the greatest part to the integrals for the self-induction of a circuit, so that we might even in some cases consider the effects of these parts as approximately the same as that of the whole circuit. The greatest contribution to the integrals in any case arise from those parts of the current filaments concerned which are nearest together. This suggests working out the integrals for corresponding portions of parallel filaments of finite but ultimately large length.

* "Die mathematische Gesetze der induzierten elektrischen Ströme," *Berlin Abhandl.* (1845). Cf. also *Vorlesungen über elektrische Ströme.*

Firstly consider the integral

$$\int \frac{(d\mathbf{s}_r d\mathbf{s}_s)}{r_{rs}}$$

taken along two parallel filaments of length l and distance ξ apart. Its value can easily be calculated as the mutual potential of two uniformly charged rods placed along the same filaments and is

$$l \log \frac{\sqrt{l^2+\xi^2}+l}{\sqrt{l^2+\xi^2}-l} - 2\sqrt{l^2+\xi^2} + 2\xi,$$

or, expanding in powers of the ultimately small quantity ξ/l,

$$= 2l\left[\log\frac{2l}{\xi} - 1 + \frac{\xi}{l} - \frac{1}{4}\frac{\xi^2}{l^2} + \dots\right].$$

Now apply this result to the calculation of the mutual induction between two parallel currents in the wires of length l, the cross sections of the wires being small compared with their distance apart, and we get approximately

$$ca_{rs} = \frac{1}{f_1 f_2}\int df_1 \int df_2 \left[2l\left(\log\frac{2l}{\xi} - 1\right)\right]$$

$$= 2l\log 2l - \left(\frac{2l}{f_1 f_2}\iint \log \xi\, df_1 df_2 + 2l\right).$$

The first term depends on the shape at a distance but the second term is a purely local part at each point of the wires. The first term in the value of $a_{rs}/2l$ is altered by altering the length of the wires, but the second term remains constant.

Thus if we have two currents in very long parallel wires there is a local inductance between them which can be reckoned as so much per unit length at the place. There is another part in the inductance which depends on the distant configuration, i.e. how the currents get round to complete their circuits. This latter part is nearly the same all along the finite parts of the wires and we could call it a constant. The distant part is so far off that its influence is the same at any point in the neighbourhood. In many cases it is only the local influence that is of any importance.

The same argument is still true if the two conductors are not quite parallel or even if they are not quite straight; if the radius of curvature is very large and if ξ does not change too rapidly along the wires we could reckon the inductance as a constant with another part which is calculated according to the above formula as so much per unit length.

436. As to the actual determination of the coefficients we notice that if

$$\iint \log \xi\, df_1 df_2 = f_1 f_2 \log R,$$

then R is the geometric mean distance between two points one on either cross section and then

$$ca_{rs} = 2l\left(\log\frac{2l}{R} - 1\right),$$

R of course lies between the greatest and least distances between points on either cross section.

Now the mean distance R_0 of a point from a cross section f_1 is given by

$$f_1 \log R_0 = \int_f \log \xi\, df_1,$$

and the integral on the right can be interpreted as the logarithmic potential of the cross section coated with electricity of density 2π. We can therefore use results calculated for such cases. When the cross section is circular (radius a) we have

(i) $r > a$ external points

$$f_1 \log R_0 = f_1 \log r,$$
$$r = R_0,$$

(ii) $r < a$ internal points

$$f_1 \log R_0 = \frac{f_1}{2a^2}(r^2 - a^2) + f_1 \log a,$$
$$R_0 = ae^{-\frac{a^2 - r^2}{2a^2}},$$

r being in each case the distance of the point from the centre.

When the cross section is a ring whose radii are a and b ($> a$) we have

(i) Outside $r > b$

$$\log R_0 = \log r,$$
$$r = R_0.$$

(ii) In ring $a < r < b$

$$\log R_0 = \frac{b^2 \log b - a^2 \log r}{b^2 - a^2} - \frac{1}{2}\frac{b^2 - r^2}{b^2 - a^2}.$$

(iii) Inside hollow $r < a$

$$\log R_0 = \frac{b^2 \log b - a^2 \log a}{b^2 - a^2} - \frac{1}{2}.$$

From these formulae it follows at once that the mean distance between the points on two rings is equal to the distance between their centres.

The logarithm of the mean distance R of a circular ring on itself follows by integration of (ii) above,

$$\log R = \log b - \frac{a^2}{b^2 - a^2}\log\frac{b}{a} + \frac{1}{4}\frac{3a^2 - b^2}{b^2 - a^2},$$

which for a circular surface ($a = 0$) gives

$$R = be^{-\frac{1}{4}},$$

and for a single ring ($b = a$)

$$R = b.$$

* Maxwell, *Trans. R. S.* (Edinb.), 1871–2. Cf. also *Treatise*, II. ch. xiii.

437. We thus conclude that the mutual induction for two straight wires of length l with equal circular sections is

$$4\pi c a_{rs} = 2l\left(\log\frac{2l}{d} - 1\right)$$

if d is the distance between the axes of the wires.

The same argument and results, of course, apply to the self-induction of wire on itself. In the case of the straight wire with a circular section the coefficient of induction is given by

$$a_{rr} = 2l\left[\log\frac{2l}{a} - \frac{3}{4}\right],$$

or is of amount $\left[\log\frac{2l}{a} - \frac{3}{4}\right]$

per unit length.

438. An important case slightly different from the above is provided by the previous example of two long parallel wires, but traversed by the same current in opposite directions, so that they form part of the same circuit. In calculating the coefficient of self-induction for this arrangement we proceed just as before, but each integral with respect to s is taken up one wire and down the other. The result is easily deduced that

$$c a_{rr} = 2l \log\frac{R_{12}{}^2}{R_1 R_2},$$

wherein R_1 and R_2 are the mean geometrical distances for the two cross sections in themselves and R_{12} the same quantity for the one relative to the other. In the case when the wires have circular cross sections of radii a, b and are at a distance d apart this is

$$c a_{rr} = 2l\left(\log\frac{d^2}{ab} + \frac{1}{2}\right).$$

The importance of this result is that the local part in the inductance alone appears. The non-local part due to the distant parts of the circuits is zero, or at least negligible. Thus the inductance can be reckoned to a very good approximation as so much per unit length.

The return circuit localises the field, which is thus far more concentrated when the return circuit exists than when it does not. The distant parts of the current which are practically equivalent to equal and opposite currents very near to one another do not give any effect for the field at a finite distance. The field at any point in such circumstances is entirely local and depends only on the conditions in parts of the conductors just near.

439. As a final example of great practical importance we must now calculate the mutual inductance of two parallel circular circuits on the same axis. The calculation in this case is best made directly from the formula

$$a_{12} = \frac{1}{c}\iint\frac{(d\mathbf{s}_1\, d\mathbf{s}_2)}{r_{12}}.$$

We introduce parallel polar coordinate systems (ϖ_1, θ_1), (ϖ_2, θ_2) in the plane of each circle, whose radii will be taken to be (a_1, a_2) and distance apart d: we shall have then

$$r_{12} = a_1^2 + a_2^2 + d^2 - 2a_1a_2 \cos(\theta_1 - \theta_2),$$

$$(d\mathbf{s}_1 \,.\, d\mathbf{s}_2) = a_1 d\theta_1 \,.\, a_2 d\theta_2 \cos(\theta_1 - \theta_2),$$

so that
$$ca_{12} = \int_0^{2\pi}\int_0^{2\pi} \frac{a_1a_2\cos(\theta_1 - \theta_2)\,d\theta_1 d\theta_2}{\sqrt{a_1^2 + a_2^2 + d^2 - 2a_1a_2\cos(\theta_1 - \theta_2)}},$$

and this again is easily shown to transform in terms of the complete elliptic integrals so that

$$c\,.\,a_{12} = -\,4\pi\sqrt{a_1a_2}\left\{\left(k' - \frac{2}{k}\right)F(k) + \frac{2}{k}E(k)\right\}$$

where the modulus k is given by the equation

$$k^2 = \frac{4a_1a_2}{(a_1 + a_2)^2 + d^2}.$$

This is the general result which includes all the cases but is of rather a complicated form. The particular problem of real practical importance concerns the case when the coils are so placed that the distance between their arcs is small compared with the radius of either circle. In this case k is nearly equal to unity and we might deduce the expansion of the elliptic integrals appropriate to this case; the following alternative method is given by Maxwell as a more direct application of electrical principles.

440. Let a, $a + b$ be the radii of the circles and d the distance between their planes, then the shortest distance between their circumferences is

$$\sqrt{b^2 + d^2} = \rho.$$

We have to find the magnetic induction through the one circle due to unit current in the other.

We shall begin by supposing the two circles to be in one plane. Consider a small element δs of the circle whose radius is $a + b$. At a point in the plane of the circle, distant r from the centre of δs, measured in a direction making an angle θ with the direction of δs, the magnetic force due to δs is perpendicular to the plane and equal to

$$\frac{\sin\theta\,\delta s}{r^2}.$$

To calculate the surface integral of this force over the space which lies within the circle of radius a we must find the value of

$$2\delta s\int_{\theta_1}^{\pi/2}\int_{r_2}^{r_1}\frac{\sin\theta}{r}\,d\theta\,dr,$$

where r_1 and r_2 are the roots of the equation

$$r^2 - 2r(a + b)\sin\theta + b^2 + 2ab = 0,$$

and
$$\sin^2\theta_1 = \frac{b^2 + 2ab}{(b + a)^2}.$$

When b is small compared with a we may put

$$r_1 = 2a \sin \theta, \qquad r_2 = b/\sin \theta,$$

and then the integral with respect to r is carried out easily so that the above reduces to

$$\begin{aligned} 2\delta s \int_{\theta_1}^{\frac{\pi}{2}} \log \left(\frac{2a}{b} \sin^2 \theta \right) \sin \theta d\theta \\ &= 2\delta s \left[\cos \theta \left\{ 2 - \log \frac{2a \sin^2 \theta}{b} \right\} + 2 \log \tan \frac{\theta}{2} \right]_{\theta_1}^{\frac{\pi}{2}} \\ &= 2\delta s \left(\log \frac{8a}{b} - 2 \right) \text{ nearly.} \end{aligned}$$

Thus for the whole induction we get

$$a'_{12} = \frac{4\pi a}{c} \left(\log \frac{8a}{b} - 2 \right).$$

Since the magnetic force at any point whose distance from a curved wire is small compared with the radius of curvature, is nearly the same as if the wire had been straight, we can calculate the difference between the induction through the circle whose radius is $(a + b)$ and the circle a_2 by the formula

$$a'_{12} - a_{12} = 4\pi a \{\log b - \log r\}.$$

Hence we find the value of the induction between A and a to be

$$a_{12} = 4\pi a \left(\log \frac{8a}{r} - 2 \right)$$

approximately, provided r, the shortest distance between the circles, is small compared with a.

The coefficient of self-induction of a coil with n windings of wire, mean radius a and for which R is the geometrical mean distance of the transverse section of the coil from itself is

$$L = 4\pi n^2 a \left(\log \frac{8a}{R} - 2 \right).$$

441. Electromagnetic induction in conducting sheets and solid bodies*. We have already seen what an important part is played by metallic substances in the theory and practice of electrostatics and steady currents. We must now turn to discuss some aspects of the perhaps still more important part played by the same substances in electromagnetic theory. All technical electrical machinery is composed largely of metallic substances and the varying electromagnetic fields in their neighbourhood may thus be

* Maxwell, *Treatise*, II. ch. xii, *Proc. R. S.* 20, p. 160. Cf. also Larmor, *Phil. Mag.* (Jan. 1884), p. 4; G. H. Bryan, *Phil. Mag.* 38 (1894), p. 198; 45 (1898), p. 381; T. Levi Cività, *Rend. R. Acc. Lincei* (5), 11 (1902), pp. 163, 191, 228, *Nuovo Cimento* (5), 3 (1902), p. 442.

considerably modified by their presence. The considerations of the present section will be confined to comparatively slowly varying fields. The extent of this limitation will be discussed later, but we may now assert that when the motions are not too rapid we can neglect altogether the displacement currents in the aether and dielectrics, so that the conduction current is the only one to be reckoned with in the discussion. The electric current density $\mathbf{C}$ at any point is then given by

$$\mathbf{C} = \kappa \mathbf{E},$$

κ being the conductivity there and $\mathbf{E}$ the electric force in the field.

The fundamental equations of the theory are then

$$\frac{1}{c}\frac{d\mathbf{B}}{dt} = -\operatorname{curl}\mathbf{E},$$

$$\frac{\kappa\mathbf{E}}{c} = \operatorname{curl}\mathbf{H},$$

and if we restrict ourselves to homogeneous isotropic media for which

$$\mathbf{B} = \mu\mathbf{H},$$

we shall have

$$\operatorname{div}\mathbf{H} = \frac{1}{\mu}\operatorname{div}\mathbf{B} = 0,$$

at each point.

Now from the fundamental equations we deduce that

$$\begin{aligned} 4\pi\frac{\kappa\mu}{c^2}\frac{d\mathbf{H}}{dt} &= -4\pi\operatorname{curl}\frac{\kappa\mathbf{E}}{c} = -\operatorname{curl}\operatorname{curl}\mathbf{H} \\ &= \nabla^2\mathbf{H} - \operatorname{grad}\operatorname{div}\mathbf{H} \\ &= \nabla^2\mathbf{H}, \end{aligned}$$

so that $\mathbf{H}$ must at each point of the field satisfy the equation

$$\nabla^2\mathbf{H} - 4\pi\frac{\kappa\mu}{c}\frac{d\mathbf{H}}{dt} = 0.$$

Thus if we have any slowly varying magnetic system specified in a certain way we can determine the whole problem in the following manner. We solve this last fundamental equation for $\mathbf{H}$, choosing the appropriate solutions for the different regions of the field, and then fit them up across the boundary surfaces separating the different media. The conditions to be satisfied at a boundary are:

(1) The continuity of the normal component of the current or, what is the same thing, the continuity of the tangential component of $\mathbf{H}$.

(2) The continuity of the normal component of $\mathbf{B}$.

These conditions combined with the above equations determine the field of force completely.

442. Of course if there are any non-conducting media in the field a slight modification of the argument is possible, for then

$$\operatorname{curl} \mathbf{H} = 0,$$

so that the magnetic force is derivable in such regions from an appropriate potential ψ and

$$\nabla^2 \psi = 0,$$

as well as

$$\nabla^2 \mathbf{H} = 0.$$

The physical significance of these equations is now quite clear. The adjustment of the magnetic field in the dielectric media surrounding the conductors can be taken as practically instantaneous, so that the operative fields are sensibly statical, although, as we shall see later, they are in essence propagated. In the conductors however the field soaks in by diffusion; its penetration is counteracted by the mobility of the electrons, whose motion, by obeying the force, in so far annuls it by kinetic reaction; and it does not get very deep even when adjustment is delayed by the friction of the vast number of ions which it starts into motion, and which have to push their way through the crowd of material molecules; and the phenomena of surface currents thus arises.

It should thus appear that comparatively thin sheets of metal should act as good screens from electromagnetic action; and this is confirmed by the analysis of the following simple cases which are directly tractable as regards their mathematical form.

443. (*a*) The easiest and most famous example of the foregoing principles is afforded by the discussion of the effects of slow variation in the electromagnetic field in the neighbourhood and on one side of a plane conducting sheet indefinitely extended in all directions but whose thickness may be neglected. We shall take as the central plane of the sheet the coordinate plane xy and its thickness $\delta/2$ symmetrical on either side of this plane.

In the space on both sides of the sheet the magnetic force will be derived from a potential ψ_1 on the positive side and ψ_2 on the negative side. Just inside the sheet which will be presumed to be non-magnetic, the magnetic field agrees with that just outside on the same side but the tangential components change rapidly through the thickness of the sheet. The electric field is on the other hand perfectly continuous throughout the whole field, as there are no typical surface infinities to introduce discontinuity. To obtain the method of variation in the magnetic field we notice that at each point in the interior of the sheet we must have

$$\frac{4\pi\kappa \mathbf{E}_x}{c} = \frac{\partial \mathbf{H}_y}{\partial z} - \frac{\partial \mathbf{H}_z}{\partial y},$$

$$\frac{4\pi\kappa \mathbf{E}_y}{c} = \frac{\partial \mathbf{H}_z}{\partial x} - \frac{\partial \mathbf{H}_x}{\partial z}.$$

And then noticing that all functions vary steadily from one point of the sheet to another (in the x, y plane) we get on integration of these equations through the very small thickness of the sheet

$$\frac{4\pi\kappa\delta}{c}\mathbf{E}_x = \Big|\,\mathbf{H}_y\,\Big|_2^1,$$

$$\frac{4\pi\kappa\delta}{c}\mathbf{E}_y = \Big|\,-\mathbf{H}_x\,\Big|_2^1,$$

from which we easily deduce by differentiation

$$\frac{4\pi\kappa\delta}{c}(\text{curl}_z\,\mathbf{E}) = \left|\frac{\partial \mathbf{H}_x}{\partial x} + \frac{\partial \mathbf{H}_y}{\partial y}\right|_2^1,$$

but
$$\text{curl}_z\,\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{H}_z}{dt}$$

on either side of the sheet and also

$$\text{div}\,\mathbf{H} = 0,$$

at all points of the field : inserting these values we see that

$$\frac{4\pi\kappa\delta}{c^2}\frac{d\mathbf{H}_{z_1}}{dt} = \frac{4\pi\kappa\delta}{c^2}\frac{d\mathbf{H}_{z_2}}{dt} = \left|\frac{\partial \mathbf{H}_z}{\partial z}\right|_2^1.$$

These last relations really refer to the respective quantities just inside the sheet on either side : but as the field is continuous across a conducting surface we may just as well interpret them in terms of the same quantities just outside and then in terms of the magnetic potentials there. In other words this equation can be written in the form

$$\frac{4\pi\kappa\delta}{c^2}\frac{d}{dt}\left(\frac{\partial\psi_1}{\partial z}\right) = \frac{4\pi\kappa\delta}{c^2}\frac{d}{dt}\left(\frac{\partial\psi_z}{\partial z}\right) = \left|\frac{\partial^2\psi}{\partial z^2}\right|_2^1,$$

and provides us with surface condition connecting the potentials on either side across the sheet.

444. Let us now suppose that ψ_0 is the magnetic potential due to the external or inducing system and ψ_1', ψ_2' the magnetic potentials on the positive and negative sides respectively due to the currents induced in the sheet, so that we have

$$\psi_1 = \psi_0 + \psi_1',$$
$$\psi_2 = \psi_0 + \psi_2'.$$

Then ψ_0 with all its differential coefficients are continuous in crossing the sheet : moreover if the sheet is very thin we must have by symmetry

$$\psi_1'(x, y, z) \equiv -\psi_2'(x, y, -z),$$

so that
$$\frac{\partial^2\psi_1'}{\partial z^2} = -\frac{\partial^2\psi_2'}{\partial z^2} = \frac{1}{2}\left|\frac{\partial^2\psi'}{\partial z^2}\right|_2^1.$$

Thus if we write $k = \left(\frac{2\pi\kappa\delta}{c^2}\right)^{-1}$ the above surface condition may be written in the form

$$\frac{d}{dt}\left(\frac{\partial(\psi_0+\psi_1')}{\partial z}\right) = \frac{d}{dt}\left(\frac{\partial(\psi_0+\psi_2')}{\partial z}\right) = k\frac{\partial^2\psi_1'}{\partial z^2} = -k\frac{\partial^2\psi_2'}{\partial z^2},$$

or as the two equations

$$k\frac{\partial^2\psi_1'}{\partial z^2} - \frac{d}{dt}\left(\frac{\partial\psi_1'}{\partial z}\right) - \frac{d}{dt}\frac{\partial\psi_0}{\partial z} = 0,$$

and

$$k\frac{\partial^2\psi_2'}{\partial z^2} + \frac{d}{dt}\frac{\partial\psi_2'}{\partial z} + \frac{d}{dt}\frac{\partial\psi_0}{\partial z} = 0.$$

445. We suppose that there is no external magnetic system acting on the current sheet. We may then take $\psi_0 = 0$. The case then becomes that of a system of electric currents in the sheet left to themselves but acting on one another by their mutual induction, and at the same time losing their energy on account of the resistance of the sheet. The result is expressed by the equation

$$k\frac{\partial^2\psi_1'}{\partial z^2} = \frac{d}{dt}\frac{\partial\psi_1'}{\partial z},$$

and the analogous one on the other side of the sheet. On integration with respect to z this gives

$$k\frac{\partial\psi_1'}{\partial z} = \frac{d\psi_1}{dt},$$

the solution of which is $\psi_1' = \psi_0'(x, y, z + kt)$,

ψ_0' denoting the magnetic potential of the field of the currents at the time $t = 0$.

The differential equation thus solved has in reality reference only to the conditions at the boundary of the sheet, but the solution we have obtained satisfies all the conditions of the problem as well as the boundary conditions at $z = 0$ and will therefore be the proper general solution for all points of the field.

Hence the value of ψ_1' at any point on the positive side of the sheet whose coordinates are (x, y, z) at time t is equal to the original value of the potential at the point $(x, y, z + kt)$.

If therefore a system of currents is excited in a uniform plane sheet of infinite extent and then left to itself, its magnetic effect at any point on the positive side of the sheet will be the same as if the system of currents had been maintained constant in the sheet and the sheet moved away in the direction of a normal from its negative side with the constant velocity k. The diminution of the electromagnetic forces, which arises from a decay of the currents in the real case, is accurately represented by the diminution of the forces on account of the increasing distance in the imaginary case.

446. Now let us consider a slightly different type of circumstances. On integration of the equation for ψ_1' we find that

$$\psi_1' + \psi_0 = -\int k \frac{\partial \psi_2'}{\partial z}\, dt.$$

If now we suppose that at first $\psi_0 = \psi_2' = \psi_1' = 0$ and that a magnet or electromagnet is suddenly magnetised or brought from an infinite distance so as to change the value of ψ_0 suddenly from zero to ψ_0 then since the time integral on the right-hand side vanishes with the time, we must have at first at the surface of the sheet $\psi_1' + \psi_0 = 0.$

We have similarly at the sheet

$$\psi_2' + \psi_0 = 0.$$

Hence the system of currents excited in the sheet by the sudden introduction of the system to which ψ_0 belongs, is such that at the surface of the sheet it exactly neutralises the magnetic effect of the system.

At the surface of the sheet, therefore, and consequently at all points on the negative side of it, the initial system of currents produces an effect equal and opposite to that of the magnetic system on the positive side. We may express this by saying that the effect of the currents is equivalent to that of an image of the magnetic system, coinciding in position with that system, but opposite as regards the direction of its magnetisation and of its electric currents. Such an image is called a negative image.

The effect of the currents in the sheet at a point on the positive side of it is equivalent to that of a positive image of the magnetic system on the negative side of the sheet, the lines joining corresponding points of the two systems being bisected at right angles by the sheet.

The action at a point on either side of the sheet due to the currents in the sheet may therefore be regarded as due to an image of the magnetic system on the side of the sheet opposite to the point, this image being a positive or a negative image according as the point is on the positive or negative side of the sheet.

If the sheet is of infinite conductivity $k = 0$ and thus the image will represent the effect of the currents in the sheet at any time.

In the case of a real sheet the resistance k has some finite value. The image just described will therefore represent the effect of the currents only during the first instant after the sudden introduction of the magnetic system. The currents will immediately begin to decay and the effect of this decay will be accurately represented if we suppose the two images to move from their original positions, in the direction of normals drawn from the sheet, with the constant velocity k.

447. We are now in a position to investigate the system of currents induced in the sheet by any system M of magnets or electromagnets on the

one side (positive) of the sheet, the position and strength of which may vary in any manner.

Let ψ_0 be as before the magnetic potential from which the direct action of this system is to be deduced, then $\frac{d\psi_0}{dt}\delta t$ will be the potential function corresponding to the system represented by $\frac{dM}{dt}\delta t$. This quantity which is the increment of M in the time δt may be regarded as itself representing a magnetic system. If then we suppose that at the time t a positive image of the system $\frac{dM}{dt}\delta t$ is formed on the negative side of the sheet, the magnetic action at any point on the positive side of the sheet due to the currents in the sheet excited by the change in M during the first instant after the change, and the image will continue to be equivalent to the currents in the sheet, if as soon as it is formed, it begins to move in the negative direction of z with the constant velocity k. If we suppose that in every successive element of time an image of this kind is formed and that as soon as it is formed it begins to move away from the sheet with velocity k, we shall obtain the conception of a trail of images, the last of which is in process of formation, while all the rest are moving like a rigid body from the sheet with velocity k.

448. These general results are further illustrated by their application to the well-known case when the currents are excited by a magnetic pole placed at the fixed point $(x_0y_0z_0)$ on the positive side of the sheet and of strength $f(t)$, an arbitrary function of the time. Then

$$\psi_0 = \frac{f(t)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}},$$

and the potential on the negative side satisfies the equation

$$\frac{d}{dt}\left(\frac{\partial\psi_2'}{\partial z}\right) + k\frac{\partial^2\psi_2'}{\partial z^2} = -\frac{d}{dt}\frac{\partial\psi_2'}{\partial z},$$

so that
$$\left(\frac{d}{dt}+k\frac{\partial}{\partial z}\right)\psi_2' = -\frac{f'(t)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}},$$

and the solution of this may be written symbolically in the form

$$\psi_2' = -\int_{-\infty}^{t} e^{-k(t-\tau)\frac{\partial}{\partial z}} \frac{f'(\tau)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\,d\tau,$$

which from a well-known result in the differential calculus easily transforms to

$$\begin{aligned}\psi_2' &= -\int_{-\infty}^{t}\frac{f'(\tau)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0-kt+k\tau)^2}}\,d\tau \\ &= \frac{-f(t)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} \\ &\quad + \int_{-\infty}^{t} kf(\tau)\frac{\partial}{\partial z}\frac{d\tau}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0-kt+k\tau)^2}}.\end{aligned}$$

The first term represents the effect of a pole equal and opposite to the given pole at the point $(x_0 y_0 z_0)$, while the second term shows that the effect of the pole between the times τ and $\tau + d\tau$ is represented at the time t by a magnet of strength $kf(\tau)\, d\tau$ situated at the point $(x_0,\ y_0,\ z_0 + k(t - \tau))$.

The potential on the positive side is found from the relation

$$\psi_1'(x,\ y,\ z) = -\psi_2'(x,\ y,\ -z).$$

When the pole is in motion the results thus arrived at enable us to plot out the moving images contributed during every element of time $\delta\tau$ of the motion; but these may be obtained more directly as follows.

449. If instead of a single magnetic pole we are dealing with a magnetic distribution on the positive side of the sheet such that the volume density of magnetisation of the point $(x_0 y_0 z_0)$ is $F_0(x_0 y_0 z_0,\ t)$ then we must write $F_0 dv_0$ for $f(t)$ in the expressions for the potential and integrate over the whole field on the one side of the sheet. Thus

$$\psi_0 = \int \frac{F_0 dv_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}},$$

and

$$\psi_2' = -\psi_0 - \int_{-\infty}^{t} d\tau \int \frac{kF_0(\tau)(z - z_0 - \overline{kt - \tau})\, d\tau}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0 - k(t - \tau))^2}}.$$

If now the inducing system consist of a single pole of strength $f(t)$ which moves in any manner so that its coordinates at any time t are $\xi(t)$, $\eta(t)$, $\zeta(t)$ functions of t, we have

$$F_0 = 0,$$

except when $\quad x_0 = \xi(\tau), \quad y_0 = \eta(\tau), \quad z_0 = \zeta(\tau),$

and then $\quad F(x_0 y_0 z_0 \tau)\, dv_0 = f(\tau),$

so that
$$\psi_0 = \frac{f(t)}{\sqrt{(x - \xi(t))^2 + (y - \eta(t))^2 + (z - \zeta(t))^2}},$$

and

$$\psi_2' = -\psi_0 - \int_{-\infty}^{t} kf(\tau) \frac{\{z - \zeta(\tau) - k(t - \tau)\}\, d\tau}{\{(x - \xi(\tau))^2 + (y - \eta(\tau))^2 + (z - \zeta(\tau) - k(t - \tau))^2\}^{\frac{3}{2}}}.$$

This is the expression for the magnetic potential due to the induced currents on the negative side of the sheet, i.e. on the opposite side to the moving pole; and it is to be noticed that z must be taken to be negative on this side.

450. (*b*) As a second example of these principles illustrating further aspects of the fields we consider the currents induced in a thin uniform spherical metallic shell by varying magnetic fields outside the shell. As before, if the variations are not too rapid, we may neglect the displacement currents in the dielectrics so that the magnetic fields both inside and outside the shell are derived from potentials ψ_2 and ψ_1 respectively.

To obtain the boundary conditions connecting the potentials across the spherical shell it is best to interpret the general electromagnetic equations in terms of spherical polar coordinates (r, θ, ϕ) with the pole at the centre of the sphere. This is easily done by reference to the general theorem established in the introduction governing such rotational transformations and the resulting equations are of the type

$$\frac{4\pi r^2 \sin\theta}{c}\mathbf{C}_r = \frac{\partial}{\partial\phi}(r\mathbf{H}_\theta) - \frac{\partial}{\partial\theta}(r\sin\theta\mathbf{H}_\phi),$$

$$\frac{4\pi r^2 \sin\theta}{c}\mathbf{C}_\theta = \frac{\partial}{\partial r}(r\sin\theta\,\mathbf{H}_\phi) - \frac{\partial}{\partial\phi}(\mathbf{H}_r),$$

$$\frac{4\pi r^2}{c}\mathbf{C}_\phi = \frac{\partial}{\partial\theta}(\mathbf{H}_r) - \frac{\partial}{\partial r}(r\mathbf{H}_\theta).$$

In the present case there is no electric flux across the shell, $\mathbf{C}_r = 0$ and the other two components are proportional to the corresponding components of the electric force

$$\mathbf{C}_\theta = \kappa\mathbf{E}_\theta, \qquad \mathbf{C}_\phi = \kappa\mathbf{E}_\phi.$$

Substituting these values in the last two equations written down, and integrating across the small thickness of the shell, neglecting the integrals of all certainly continuous functions we find that*

$$\frac{4\pi\kappa\delta a}{c}\mathbf{E}_\theta = \Big|\,\mathbf{H}_\phi\,\Big|_1^2,$$

$$\frac{4\pi\kappa\delta a}{c}\mathbf{E}_\phi = \Big|-\mathbf{H}_\theta\,\Big|_1^2,$$

so that

$$\frac{4\pi\kappa\delta a}{c}\left\{\frac{\partial}{\partial\phi}(r\mathbf{E}_\theta) - \frac{\partial}{\partial\theta}(r\sin\theta\,\mathbf{E}_\phi)\right\} = \left|\left\{\frac{\partial}{\partial\phi}(r\mathbf{H}_\phi) + \frac{\partial}{\partial\theta}(r\sin\theta\mathbf{H}_\theta)\right\}\right|_1^2.$$

Then making use of the first fundamental equation of the second set,

$$-\frac{4\pi\kappa r^2\sin\theta}{c}\frac{d\mathbf{H}}{dt} = \frac{\partial}{\partial\phi}(r\mathbf{E}_\theta) - \frac{\partial}{\partial\theta}(r\sin\theta\mathbf{E}_\phi),$$

and the conditional equation for $\mathbf{H}$, viz.

$$\operatorname{div}\mathbf{H} = \frac{\partial}{\partial r}(\mathbf{H}_r) + \frac{\partial}{\partial\phi}(r\mathbf{H}_\phi) + \frac{\partial}{\partial\theta}(r\sin\theta\,\mathbf{H}_\theta) = 0,$$

we see at once that

$$+\frac{4\pi\kappa\delta a}{c^2}\frac{d}{dt}(r^2\mathbf{H}_r)_1 = +\frac{4\pi\kappa\delta}{c^2}\frac{d}{dt}(r^2\mathbf{H}_r)_2$$

$$= \left|\frac{\partial}{\partial r}(r^2\mathbf{H}_r)\right|_1^2,$$

the normal component H_r of the magnetic force being continuous through

* a is the radius of the shell.

the sheet. These equations may be interpreted in terms of the magnetic potentials and if we use as before

$$k = \left(\frac{2\pi\kappa\delta}{c^2}\right)^{-1}$$

we get
$$\frac{d}{dt}\left(r^2 \frac{\partial\psi_1}{\partial r}\right) = \frac{d}{dt}\left(r^2 \frac{\partial\psi_2}{\partial r}\right) = \frac{ka}{2}\left|\frac{\partial}{\partial r}\left(r^2 \frac{\partial\psi}{\partial r}\right)\right|_1^2.$$

Thus if we can find appropriate potentials ψ_1 and ψ_2 inside and outside the sphere and these must of course be regular solutions of the equation

$$\nabla^2\psi = 0,$$

which agree with the implied field at a distance and if these values satisfy the boundary conditions implied in the above relations, they represent the complete solution of the problem from which all the circumstances of the field can be deduced.

451. We may illustrate the problem by the case in which the applied field is practically uniform in the neighbourhood of the sphere but is oscillating with the period p: we may then assume as the applied potential

$$\psi_0 = -Hr\cos\theta\, e^{ipt}.$$

We then try tentative solutions for the total field resulting from the superposition on this field of the field of the currents which it induces in the shell (i) outside the sphere

$$\psi_1 = \psi_0 + a^3 \frac{A_1 \cos\theta}{r^2} e^{ipt},$$

and (ii) inside the sphere

$$\psi_2 = \psi_0 + A_2 r\cos\theta\, e^{ipt},$$

the radius of the sphere being a. These forms satisfy the general conditions of regularity and the outside field reduces to ψ_0 at a great distance. They satisfy the above boundary conditions if

$$ip(-H + A_2) = ip(-H - 2A_1) = ak(A_1 - A_2),$$

i.e. if
$$A_1 = -\tfrac{1}{2}A_2 = \frac{-H}{2\left(1 + \dfrac{3ka}{i2p}\right)}.$$

If we write
$$\alpha = \frac{3ka}{2p},$$

we have
$$A_2 = \frac{H}{1 - i\alpha} = \frac{1 + i\alpha}{1 + \alpha^2} H.$$

Thus taking real parts only, we find for a field of force of the present type oscillating in intensity according to the harmonic law

$$\psi_0 = -Hr\cos\theta\cos pt,$$

the intensity of the field inside the shell is determined by its potential

$$\psi_2 = \psi_0 + Hr\cos\theta \,.\, \frac{\cos pt - \alpha\sin pt}{1 + \alpha^2},$$

or using $\tan\chi = p$ the additional part may be written in the form

$$\psi_2 = Hr\cos\theta\cos\chi\cos(pt+\chi).$$

This solution represents a new and opposite field with the strength reduced by the factor $\cos\chi$, and a phase of variation retarded by the fraction $\frac{\chi}{2\pi}$ of a complete period superposed on the original field. This is of course the part of the field due to the induced currents.

The external potential in this case is got from the above relations and is determined by

$$\psi_1 = \psi_0 - \frac{Ha^3\cos\theta}{2r^2}\cos\chi\cos(pt+\chi),$$

and the part due to the induced currents is represented by the second term: the induced currents thus produce the same external effect as a simple magnet of moment

$$\tfrac{1}{2}Ha^3\cos\chi\cos(pt+\chi),$$

at the centre of the sphere with its axis pointing in the stable direction along the lines of force of the original field.

Now
$$\tan\chi = \frac{3ka}{2p} = \frac{3c^2a}{4\pi\kappa\delta p}.$$

Now for copper at 0° C.

$$\frac{\kappa}{c^2} = 1640,$$

and if we take the thickness of the shell to be 1 mm. and the radius 3 cm. and a frequency of about 2400 alternations per second we find that half the original field is suppressed inside the shell, and a new field of the same intensity as this half is added whose phase of oscillation is increased by one-sixth of a period. By decreasing the thickness the same effect will be produced by a corresponding slower alternation.

452. It is important to notice that the electric field associated with the magnetic field and which is determined by the three equations expressing Faraday's rule in these coordinates, viz.

$$-\frac{r^2\sin\theta}{c}\frac{d\mathbf{H}_r}{dt} = \frac{\partial}{\partial\phi}(r\mathbf{E}_\theta) - \frac{\partial}{\partial\theta}(r\sin\theta\mathbf{E}_\phi)$$

$$-\frac{r\sin\theta}{c}\frac{d\mathbf{H}_\theta}{dt} = \frac{\partial}{\partial r}(r\sin\theta\mathbf{E}_\phi) - \frac{\partial\mathbf{E}_r}{\partial\phi}$$

$$-\frac{r}{c}\frac{d\mathbf{H}_\phi}{dt} = \frac{\partial\mathbf{E}_r}{\partial\theta} - \frac{\partial}{\partial r}(r\mathbf{E}_\theta),$$

in which of course for the external field

$$\mathbf{H}_r = H\cos\theta\left\{\cos pt - \frac{a^3\cos\chi\cos(pt+\chi)}{r^3}\right\},$$

whilst $$\mathbf{H}_\theta = -H\sin\theta\left\{\cos pt - \frac{a^3\cos\chi\cos(pt+\chi)}{r^3}\right\},$$

and $$\mathbf{H}_\phi = 0,$$

is determined by $\mathbf{E}_r = \mathbf{E}_\theta = 0$ with however

$$\frac{\partial}{\partial\theta}(r\sin\theta\mathbf{E}_\phi) = -\frac{r^2\sin\theta}{c}\frac{d\mathbf{H}_r}{dt}$$

$$= +\frac{pr^2H}{c}\cos\theta\sin\theta\left(\sin pt - \frac{a^3\cos\chi\sin(pt+\chi)}{r^3}\right),$$

so that $$\mathbf{E}_\phi = \frac{1}{2}\frac{prH\sin\theta}{c}\left(\sin pt - \frac{a^3\cos\chi\sin(pt+\chi)}{r^3}\right).$$

The lines of electric force are circles round the direction of the lines of force in the original magnetic field. The current flux in the sheet itself will be in the same sense.

453. The method here adopted for the case of the thin shell is not directly applicable to the more general case when the thickness is not negligible but the circumstances of this more general case can be treated directly from the fundamental equations of the theory* without resorting to the simplifications introduced by the existence of magnetic potentials in the external field. We can illustrate the method by the consideration of a solid sphere in the same uniform field.

The field being uniform in the neighbourhood of the sphere and in the direction of the polar axis it is obvious from considerations of symmetry that the induced field must be symmetrical about the axis of polar coordinates so that none of the functions depend on the coordinate ϕ : it follows therefore from the fundamental equations which in the general case inside the sphere are of the type

$$\frac{4\pi\kappa r^2\sin\theta\mathbf{E}_r}{c} = \frac{\partial}{\partial\phi}(r\mathbf{H}_\theta) - \frac{\partial}{\partial\theta}(r\sin\theta\mathbf{H}_\phi),$$

$$\frac{4\pi\kappa r^2\sin\theta\mathbf{E}_\theta}{c} = \frac{\partial}{\partial r}(r\sin\theta\mathbf{H}_\phi) - \frac{\partial\mathbf{H}_r}{\partial\phi},$$

$$\frac{4\pi\kappa r\mathbf{E}_\phi}{c} = \frac{\partial\mathbf{H}_r}{\partial\theta} - \frac{\partial}{\partial r}(r\mathbf{H}_\theta),$$

and $$-\frac{r^2\sin\theta}{c}\frac{d\mathbf{H}_r}{dt} = \frac{\partial}{\partial\phi}(r\mathbf{E}_\theta) - \frac{\partial}{\partial\theta}(r\sin\theta\mathbf{E}_\phi)$$

$$-\frac{r\sin\theta}{c}\frac{d\mathbf{H}_\theta}{dt} = \frac{\partial}{\partial r}(r\sin\theta\mathbf{E}_\phi) - \frac{\partial\mathbf{E}_r}{\partial\phi}$$

$$-\frac{r}{c}\frac{d\mathbf{H}_\phi}{dt} = \frac{\partial\mathbf{E}_r}{\partial\theta} - \frac{\partial}{\partial r}(r\mathbf{E}_\theta),$$

* Cf. Larmor, *l.c.* p. 389; Lamb, *Proc. L. M. S.* (1884), p. 139; C. Niven, *Phil. Trans.* 172 (1882), p. 307; C. S. Whitehead, *Phil. Mag.* 48 (1899), p. 165; Hertz, *Dissertation*, Berlin (1880), *Ges. Werke*, I. p. 37.

that
$$\mathbf{E}_r = \mathbf{E}_\theta = \mathbf{H}_\phi = 0,$$
so that they reduce to
$$\frac{4\pi\sigma r \mathbf{E}_\phi}{c} = \frac{\partial \mathbf{H}_r}{\partial \theta} - \frac{\partial}{\partial r}(r\mathbf{H}_\theta)$$
$$-\frac{r^2 \sin\theta}{c}\frac{d\mathbf{H}_r}{dt} = -\frac{\partial}{\partial \theta}(r \sin\theta \mathbf{E}_\phi)$$
$$-\frac{r \sin\theta}{c}\frac{d\mathbf{H}_\theta}{dt} = \frac{\partial}{\partial r}(r\sin\theta\mathbf{E}_\phi).$$
If therefore we use
$$\Pi = r\sin\theta\mathbf{E}_\phi,$$
then
$$\frac{r^2\sin\theta}{c}\frac{d\mathbf{H}_r}{dt} = \frac{\partial\Pi}{\partial\theta}, \qquad -\frac{r\sin\theta}{c}\frac{d\mathbf{H}_\theta}{dt} = \frac{\partial\Pi}{\partial r},$$
so that from the first of the last three equations we get
$$\frac{4\pi\sigma}{c^2}\frac{d\Pi}{dt} = \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Pi}{\partial\theta}\right) + \frac{\partial^2\Pi}{\partial r^2},$$
or using $\mu \equiv \cos\theta$
$$\frac{\partial^2\Pi}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\Pi}{\partial\theta^2} = \frac{4\pi\sigma}{c^2}\frac{d\Pi}{dt}.$$
Outside the sphere the only difference is that $\sigma = 0$ so that the appropriate equation for Π is
$$\frac{\partial^2\Pi}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\Pi}{\partial\theta^2} = 0.$$

454. Now if the applied uniform field is varying in a simple harmonic manner with the period $\frac{2\pi}{p}$ it is easy to verify that in the neighbourhood of the sphere it is defined by the function
$$\Pi = \frac{ipr^2}{2}(1-\mu^2)He^{ipt}.$$
We therefore take the hint suggested by this form and try solutions for the internal field of the type
$$\frac{ipR_2}{2}(1-\mu^2)He^{ipt},$$
and for the external field
$$\frac{ip}{2}(r^2+R_1)(1-\mu^2)He^{ipt},$$
R_1 and R_2 being functions of r only and R_1 in addition tending to zero as r becomes infinite. On substitution of these forms in the above equations we find that R_1 satisfies the equation
$$\frac{d^2R_1}{dr^2} - \frac{2R_1}{r^2} = 0,$$

of which the appropriate solution is

$$R_1 = \frac{A_1}{r_1},$$

whereas R_2 satisfies the equation

$$\frac{d^2R_2}{dr^2} - \left(\frac{2}{r^2} + \frac{4\pi ip\sigma}{c^2}\right) R_2 = 0,$$

the appropriate solution of which is easily verified to be

$$R_2 = A_2\lambda r \frac{d}{d(\lambda r)}\left(\frac{\sin(\lambda r)}{\lambda r}\right)$$

where

$$-\lambda^2 = \frac{4\pi ip\sigma}{c^2}.$$

With these forms of solution all the implied conditions of the two fields are satisfied but it remains to connect them up across the boundary. The conditions at the boundary between the conducting sphere and the non-conducting free space surrounding it are simply that the current shall not become infinite at the surface, or what comes to the same thing, the tangential electric and magnetic forces are continuous across this surface. This requires that both Π and $\frac{\partial \Pi}{\partial r}$ are continuous at the surface of the sphere $(r = a)$: in other words

$$A_2\left(\lambda a \frac{d}{d(\lambda a)}\right)\left(\frac{\sin \lambda a}{\lambda a}\right) = -a^2 + \frac{A_1}{a},$$

$$A_2\left(\lambda a \frac{d}{d(\lambda a)}\right)^2\left(\frac{\sin \lambda a}{\lambda a}\right) = -a^2 - \frac{A_1}{a}.$$

455. Now as a general rule λa is very small so that we may approximate not only to these relations but also to the general forms of the functions for the internal field: we write in fact

$$\frac{d}{d(\lambda r)}\left(\frac{\sin \lambda r}{\lambda r}\right) = -\frac{\lambda r}{3} + \frac{\lambda^3 r^3}{30},$$

and to this approximation it is easily verified that

$$A_2 = \frac{3a^2}{\lambda^2 a^2\left(1 - \frac{\lambda^2 a^2}{6}\right)},$$

whilst

$$A_1 = -a^3 \cdot \frac{\lambda^2 a^2}{15},$$

and then also the internal field is determined of the type specified by

$$\Pi = \frac{ipr^2(1-\mu^2)}{2}\left\{1 - \frac{\lambda^2 r^2}{10} + \frac{\lambda^2 a^2}{6}\right\} He^{ipt},$$

whilst the external field is specified by

$$\Pi = \frac{ip(1-\mu^2)}{2}\left\{r^2 - \frac{a^3}{r} \cdot \frac{\lambda^2 a^2}{15}\right\} He^{ipt}.$$

Taking real parts only we see that to an applied uniform field determined by

$$\Pi = \frac{pr^2(1-\mu^2)}{2} H \cos pt,$$

the internal field is given by

$$\Pi = \frac{pr^2(1-\mu^2)}{2}\left\{\cos pt - \frac{2\pi p\sigma}{c^2}\left(\frac{a^2}{3} - \frac{r^2}{5}\right)\sin pt\right\} H,$$

whilst the external field has

$$\Pi = \frac{p(1-\mu^2)}{2}\left\{r^2 \cos pt - \frac{4\pi p\sigma a^5}{15 \,.\, rc^2}\sin pt\right\} H.$$

At external points the effect of the sphere is to introduce a new varying field with phase increased by a greater period, and which in other respects bears the same relation to the old as the electrostatic field produced by an insulated sphere of the same radius bears to the inducing field when the strength of the former is diminished in the ratio

$$\frac{2\pi p\sigma}{15c^2}.$$

The general form of the solution obtained for the internal field and the fundamental equation from which it is deduced suggests another way of looking at the problem. The induced distribution gives a field inside the sphere which is determined by the function

$$\Pi = -\frac{ip(1-\mu^2)}{2} He^{ipt}\left\{r^2 - A_2(\lambda r)\frac{d}{d(\lambda r)}\left(\frac{\sin(\lambda r)}{(\lambda r)}\right)\right\}.$$

The first term gives a field equal and opposite to the inducing field at each instant; the following term represents its decay. Since

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{r^2(1-\mu^2)}{2}He^{ipt}\right) = 0,$$

the opposite field can only be due to a surface distribution of currents. Thus if at each instant we suppose a system of currents to start in the superficial layer of the body which neutralises for internal space the effect of the outside changes, the actual state of the body is that produced by these currents soaking into it and decaying.

456. (*c*) The principles illustrated in the first section (*a*) have been applied by Prof. Larmor* to more general problems of steady motions of electrical systems. He starts by considering the case of an electric point charge q moving with velocity v in front of the plane sheet as above. This continuous motion of the charge may be replaced by a series of instantaneous jerks between the positions which it occupies after successive infinitesimal intervals of time δt. Each such displacement of its position is equivalent

* *Proc. L. M. S.* (2), 8 (1809), p. 1.

to the creation of a doublet of moment $qv\delta t$. The initial (magnetic) effect of the currents induced in the sheets is to annul the field of the doublet thus instantaneously created, as regards the region beyond the sheets: thus the action of the currents in the sheets is, on either side of it, equivalent initially to that of this doublet with sign changed, supposed to be situated on the other side. The currents thus impulsively produced in succession gradually die away by resistance (κ) in the sheet, and in so doing each of them exerts the same influence as if the equivalent doublet on the other side of the sheet moved steadily away with the velocity u. The fields of

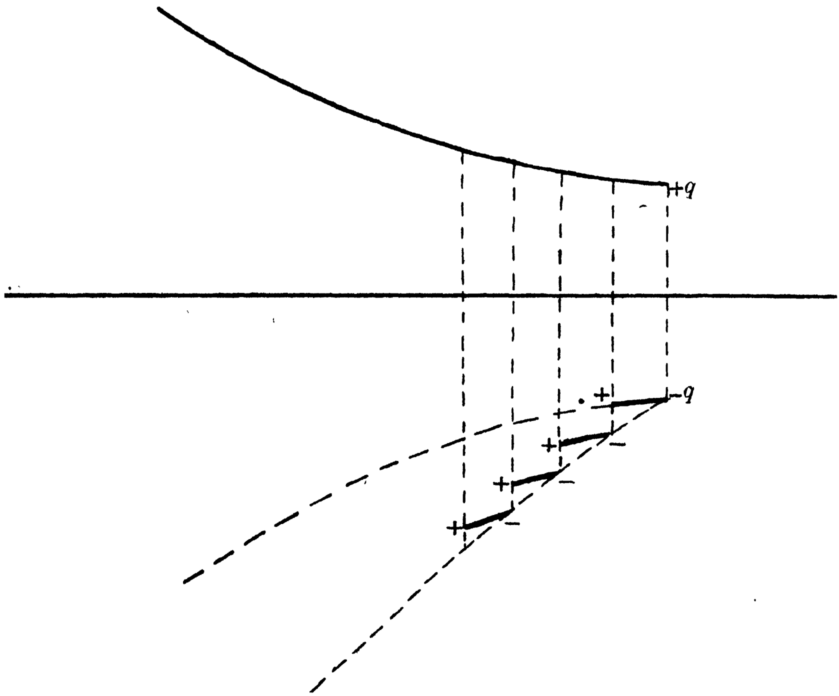


Fig. 76

force persisting from successive past intervals of time δt are thus expressed as due to the sources graphically represented in the diagram; in it the image system is inserted on the remote side of the sheet, the conjugate image for the near side being, of course, its reflection in the sheet, with the sign changed if it is magnetic, but not if it is an electric pole. Each step between one receding doublet image and the next is of length $u\delta t$. At the instant when the moving charge $+e$ has reached the point marked as the end of the path, the aggregate image system consists of the doublets repre-

sented by the short continuous lines, each of which is a persisting travelling reflection of the instantaneous effect of the jerk representing a previous element of the path of $+ q$.

It is the change, occurring per unit time, in the conformation of this image system of doublets that represents convection of electric charges and so produces magnetic effect. It is in fact only while complementary charges are being separated, so as to create a doublet, that a current element exists, of moment equal to charge multiplied by velocity: when the doublet has once been established and merely continues to exist the magnetic flux around it ceases, though there is a latent accumulation of such flux which persists without further electrodynamic effect and would be undone if the doublet were again absorbed by its poles moving into coincidence.

Thus the magnetic effect of the induced currents in the sheet is the same as that of the electric flux in the image system; and the aggregate flux in time δt amounts to the distribution of vertical doublets, each of moment $qu\delta t$, as marked by the $+$ and $-$ in the diagram, each receding from the sheet a distance $u\delta t$, and in addition the creation of the fresh isolated image $- q$ at the end of the series. It is the limiting configuration of this convection as δt vanishes with which we are concerned.

457. If the inducing charge q is at rest this image system amounts to nothing, as of course it must. If the inducing charge is moving directly towards the sheet with velocity v, the image system again assumes a simple form; for if each doublet $- qu\delta t$ is replaced by an equivalent but longer one

$$- qu/(u + v) \,.\, (u + v)\, \delta t,$$

the doublets will then form a continuous line in which adjacent poles cancel and will thus represent in the aggregate the removal in time δt of an image charge $- qu/(u + v)$ to infinite distance or rather to the distance appropriate to the duration of the motion. Thus of the instantaneous image $- q$ there remains in position, after subtracting the part thus removed, a residue $\dfrac{-qv}{u+v}$ moving towards the sheet with velocity v. This reduced geometrical image $\dfrac{-qv}{u+v}$ constitutes in the present case the entire image system, as regards magnetic field due to the disturbance excited in the sheet but as regards electric field the effect of the sheet is the same as that of the complete electrostatic image $- q$. For a receding charge the sign of v is of course changed.

This result may of course be generalised. When any rigid electric system is approaching directly with uniform velocity from a distance towards an infinite sheet of specific resistance κ and small thickness δ, the magnetic

effect on its own side of the sheet, due to the currents induced in the sheet, is the same as that of the moving optical image of the system with all charges altered in the ratio $-\left(1+\frac{\kappa\sigma}{u}\right)^{-1}$.

Obviously the argument here applied to a moving charge applies equally to a moving magnetic pole; but in that case the image must be positive instead of negative as here, in order to annul the normal component of the magnetic field at the sheet: thus the last result can be generalised so that the system may include magnets as well as electric charges. As circuital electric currents can be represented by magnetic polarity, the principle also extends so as to include such currents.

The representation by reduced geometrical images thus applies to any steady electromagnetic system whatever which is approaching directly to the conducting sheet or thin layer; and it is easy to extend it to a changing system.

458. Now revert to the case of a single electron $+q$ travelling in a curved path, as represented in the diagram. In the differential interval of time, δt, next before the time exhibited, each of the vertical doublets in the diagram will have receded a distance $u\delta t$, while a new image $-q$ will have been instituted opposite to the charge, or rather transferred there from the further end of the image system. To determine the magnetic field due to convection of these parallel doublets, we make use of the vector potential, which for a current element qu is parallel to it and equal to $\frac{qu}{r}$. The operation of moving an electric doublet of moment e along its own direction (that of z) with a velocity $-u$, thus produces a vector potential $u\frac{\partial}{\partial z}\left(\frac{e}{r}\right)$. In the present case $qu\delta t$ is the aggregate moment of these doublets, all normal to the sheet, which correspond to an element δt of the time of motion. Thus the aggregate vector potential is parallel to z and is thus

$$\mathbf{A}_z = u\int\frac{\partial}{\partial z}\left(\frac{qu}{r}\right)dt \quad \text{or} \quad u^2\int\frac{\partial}{\partial z}\left(\frac{q}{v'r}\right)ds.$$

It is thus u^2 multiplied by the component along z of static attraction (towards the sheet) of a line density equal to $\frac{e}{v'}$ distributed along the (elongated) image curve, where v' is the velocity along that curve corresponding to the motion of the inducing charge. The magnetic force intensity is thus parallel to the sheet and equal to $\left(\frac{\partial\mathbf{A}_z}{\partial y}, -\frac{\partial\mathbf{A}_z}{\partial x}, 0\right)$; in fact $\mathbf{A}_z$ is a stream function. The whole magnetic influence reflected from the conducting sheet is this magnetic field together with the electric and magnetic fields of an image

$-q$ existing and travelling in the position of the optical image of the inducing charge: as the latter is the instantaneous shielding image, the former part represents the trail due to imperfect conductance of the sheet, which prevents the currents from adapting themselves instantaneously into the shielding distribution.

If the inducing charge travels uniformly in an oblique straight line, the distribution of which A_2 is the Newtonian potential is a uniform straight linear one: thus A_2 is expressible in logarithmic form and the solution is at once completed in simple finite terms, which it is needless to express at length.

In the case of an inducing magnetic pole m, it is the actual poles of the image system that produce the magnetic field, in contrast with the convection which alone operates in the electric case. Thus in addition to the direct instantaneous image $+m$ (and its complement $-m$ at the other end of the trail) there is a trail of decaying previous disturbance having a magnetic potential

$$-u\frac{\partial}{\partial z}\int\frac{m}{v'r}\,ds',$$

wherein the integral is the potential of a line density $\frac{m}{v}$, distributed along the receding image path in the diagram.

Similar conclusions apply if the sheet itself is in motion, everything being determined by the relative motion of the inducing system with regard to it.

Thus the specification of the disturbance reflected from the infinite sheet for any convected system containing charges, magnets and currents is formally complete. The conducting sheet will cut off the direct action of the moving system from the other side, replacing it by a decaying trail represented by the receding images here investigated.

CHAPTER XI

ELECTRODYNAMICS OF LINEAR CURRENTS

459. The mutual mechanical interaction between a current and a magnetic system. We have so far merely dealt with the electromagnetic aspect of the interaction between a magnetic system and a linear current system. We now turn to an analysis of the mechanical action which is also involved, basing ourselves on the elementary principles of the previous chapter. There are as usual two ways of attacking the problem, the direct or synthetical method and the indirect analytical method based on ordinary mechanical considerations. We start with the first now and analyse the interaction between a single steady linear current J and a single magnetic pole m.

The potential of the field of the current at any point is

$$\psi = -\frac{J\Omega}{c} = -\frac{J}{c}\int_f \frac{\cos\theta\, df}{r^2};$$

the integral being taken over any barrier surface f bounded by the circuit, r denoting the distance of the point from the typical element df and θ the angle between the normal to df and r. This can be written in the form*,

$$\psi = \frac{J}{c}\int_f (\mathbf{n}_1\nabla)\frac{1}{r}\, df.$$

The force in the field is therefore the vector

$$-\nabla_P\psi = \nabla\psi = \frac{J}{c}\int_f (\mathbf{n}_1\nabla)\nabla\left(\frac{1}{r}\right) df,$$

which transforms by Stokes's theorem to the vector

$$\nabla\psi = \frac{J}{c}\int_s [d\mathbf{s},\ \nabla]\frac{1}{r},$$

the integral now being taken round the circuit s bounding the surface f.

This result admits of interpretation as the sum of effects due to each element of the current. If we take a vector $J\,d\mathbf{s}$ in the direction of the element ds of the current and another $\mathbf{r}$ from ds to the point in the field, the two form the sides of a triangle of area

$$J\,ds\,.\,r\sin(\widehat{r, ds}),$$

* In these expressions the operator ∇ refers to differentiation at the point on the surface or curve, these being equal but opposite in sign to the same differentiations ∇_p at the field point.

and thus the part of the force due to ds is

$$\frac{J\,ds\,.\,\sin \widehat{r,\,ds}}{r^2},$$

and is at right angles to both vectors and so as to turn positively round the tangent to the circuit when it is taken positively onwards*.

460. The action of the single magnetic pole on the circuit has a resultant equal and opposite to this force. The constituent forces may however be applied at the elements of the circuit; the reason being that the couples

$$\frac{J\,ds \sin \widehat{r\,ds}}{cr}$$

so introduced are in equilibrium†. Thus the action of a pole m on a current J may be calculated by combining the forces

$$-\frac{1}{c}[J\,d\mathbf{s},\ \nabla]\frac{m}{r}$$

at each element of the current.

This result admits of immediate generalisation: in fact for any magnetic system we see that the force on the current element in ds is

$$-\frac{1}{c}\Sigma\,[J\,d\mathbf{s},\ \nabla]\frac{m}{r}$$

$$= -\frac{1}{c}[J\,d\mathbf{s},\ \nabla]\,\Sigma\frac{m}{r},$$

the sum Σ being taken over all the elementary poles of the system; but

$$\mathbf{H} = -\nabla\Sigma\frac{m}{r}$$

is the magnetic force due to the whole magnetic system at the position of ds. Thus the force on the current in any field is

$$\frac{1}{c}[J\,d\mathbf{s},\ \mathbf{H}],$$

it is perpendicular to the plane of $\mathbf{H}$ and $d\mathbf{s}$ and its actual amount is

$$\frac{J\,.\,H}{c}\,ds \sin \widehat{H\,ds}.$$

There is however one important proviso to be placed on the above result. It is restricted to the case when there are no magnetisable substances present in the field; it is in fact only true in that case. The restriction is quite

* The law of Biot and Savarts, *Jour. de Savants*, Paris (1821), p. 221.

† The couple associated with any element of the circuit is completely represented in the vector sense by the chord of the projection of this element on a unit sphere round the field point, this point being the centre of projection. The circuit being closed the vector polygon of the couples is closed so that they are in equilibrium.

obvious as the argument adopted is based essentially on a distance action theory, starting from the potential function of the single current circuit $-\frac{J\Omega}{c}$, which is not valid if there are magnetic substances about anywhere in the field. The correct result for the most general case however will be shown to be

$$\frac{1}{c}[J\,d\mathbf{s}\,.\,\mathbf{B}],$$

the magnetic induction **B** replacing the magnetic force. This of course is the same as the formula above under the conditions mentioned.

461. We can now present another aspect of these forces in basing them on the energy principle. To do this we must first estimate the mutual potential energy of the current and the magnetic system.

The potential energy of a single magnetic pole of strength m in the magnetic field of the current is $m\psi$, where ψ is the magnetic potential (if such exists) of the field of the current

$$W = m\psi,$$

for if we displace the pole the work we do *on it* is equal to

$$-m\mathbf{H}_s ds = m\frac{\partial\psi}{\partial s}ds = m\delta\psi = \delta W.$$

The work done on the pole is equal to the energy gained by it.

It is important to notice again that this result is restricted in as far as the magnetic pole m must exist at a place in the magnetic field where there is a potential. Every magnetic field of electric current distributions has not a magnetic potential at each point of the field. Take for instance a finite volume distribution of electric flow (not a current in a linear conductor). We have then if **C** denotes the current density at any point of the field and **H** the magnetic force

$$\int_s \mathbf{H}_s ds = \frac{4\pi}{c}\int_f \mathbf{C}_n df,$$

the first integral being taken round any closed path s drawn in the field and the latter over a closing barrier across this path. Thus it is only at points not internal to the current distribution that it is possible to choose the path so that

$$\int_s \mathbf{H}_s ds = 0,$$

which is the condition that **H** should be derived from a potential function. A field of currents has therefore a potential only at points external to the current distribution. This is the general result.

462. With this restriction the potential energy of the pole due to the current is

$$W = m\psi$$
$$= -m\frac{J\Omega}{c},$$

provided again that there are no magnetisable substances anywhere in the field. But this is equal to*

$$W = -\frac{JN}{c},$$

where N denotes the number of tubes of force of the field of the pole m which pass through J, or its aperture in the positive direction corresponding to the circulation.

The result in this form can be immediately generalised to any number of poles for we can add up the tubes of force. This is easily seen for the number of tubes of force through any circuit s is

$$\int_f \mathbf{H}_n\, df,$$

taken over any barrier abutting on the circuit; but $\mathbf{H}_n$ is obtained by simple addition of the component intensities of the separate fields and so the number of tubes is additive. The total number of tubes is equal to the sum of the numbers belonging to each pole if existing alone. In the general case we have therefore

$$W = -\frac{JN}{c},$$

where N now denotes the number of lines of force due to the whole magnetic system which thread the circuit.

N is measured of course in the positive direction relative to the equivalent shell.

463. In the whole of the above argument it is assumed that it is possible to draw a barrier surface across the circuit entirely in free space at a distance from the magnetic media. The formula must however be generally true if only it is interpreted properly.

If we calculate N by means of the formula

$$N = \int_f \mathbf{H}_n\, df,$$

where $\mathbf{H}$ is the field strength in the total field the formula would still appear to be correct provided only that it has a definite meaning, as it represents the potential energy of the equivalent shell in the field considered as the work

* The positive direction through the sheet is the direction of increasing N and Ω but of decreasing ψ. It is the direction in which the unit pole would be urged by the field.

done in separating the sheet of positive magnetism from coincidence with the negative. Thus if the formula is to be true generally N must be the same on whatever barrier we count the tubes crossing it. Thus

$$N = \int_f \mathbf{H}_n df,$$

taken over any barrier surface must be the same. This means that

$$\operatorname{div} \mathbf{H} = 0,$$

or that $\mathbf{H}$ is a stream vector. This result is generally not true except when there is no magnetism about. We know however that if there are lumps of soft iron or other magnetisable substances present, the magnetic induction vector $\mathbf{B}$ satisfies this condition, viz.

$$\operatorname{div} \mathbf{B} = 0,$$

always; so that it appears probable that we should count the number of tubes of magnetic induction for N rather than the tubes of force. Of course the two are the same in free space. The formula would otherwise be meaningless.

464. We can easily show that this surmise is correct. In fact if we draw any one barrier surface f across the circuit which lies entirely in free space and does not therefore cut through any magnetic substances, then the formula is correct if applied to that surface by counting the tubes of magnetic force cutting across it; but if any other barrier surface f' is drawn we know that

$$N = \int_f \mathbf{H}_n df = \int_{f'} \mathbf{B}_n' df'$$

because the first integral is in fact the same as

$$\int_f \mathbf{B}_n df.$$

If it is not possible to draw one single surface wholly in free space we might remove a small layer of the magnetic substance over one of the surfaces so that it is practically in free space. The small quantity of matter thus removed would not materially affect the field of the rest*.

We must therefore in general count the induction tubes instead of the force tubes: in fact the formula, if not so interpreted, is not consistent with itself. We shall therefore always quote the formula with N interpreted as the number of tubes of induction.

465. The potential energy of the magnetic system in the field of the currents is thus

$$W = -\frac{JN}{c}.$$

* It is interesting to notice that the H_n in the thin slab of air thus made is practically the same as the component B_n of the magnetic induction in the medium.

But this potential energy is a mutual affair. It is in fact the mutual energy of the magnetic system and the current, the energy of each relative to the other. If it is a real potential it will determine not only the action of the current on the magnets but also the reaction of the magnets on the current; for if the current exerts forces on the magnets the magnets must, by reaction, exert forces on the currents. This principle of action and reaction, which in its simplest form is Newton's third law of motion, is in its generalised form a consequence of the existence of a potential energy function. This function contains the coordinates of both bodies, by varying the one we get the action and by varying the other the reaction.

We can thus determine the forces exerted by the magnetic system on the currents. All we have to do is to vary the corresponding coordinates in the energy function just obtained. The general principle is that due to any displacement of the current the work of the forces exerted on it by the magnets is

$$-\delta W;$$

the work done by these forces is equal to the energy exhausted; if the system does work without the help of an external agency its energy diminishes. The coefficients in this variation of the variations of each coordinate are respectively the forces in those coordinates. We have therefore merely to determine W in terms of suitable coordinates to determine the whole matter, the derivation of the forces being then merely a matter of differentiation.

466. The usual method of procedure is however to determine the forces exerted by the magnetic system on each element ds of the circuit carrying the current. This can be obtained by supposing that the variation in the configuration is produced by the element AB of the wire of length ds moving out into a near parallel position $A'B'$ at a vectorial distance $\delta\mathbf{s}$ from its original position. The change in W due to the displacement of this bit alone is

$$\begin{aligned}\delta W &= -\frac{J\delta N}{c}\\ &= -J\,(\text{number of tubes of induction through } ABB'A')/c.\end{aligned}$$

We must now estimate the number of tubes through this small area $ABA'B'$. It is obviously equal to the product of the component of the magnetic induction perpendicular to the area by the area, and this is

$$\delta N = ds\,.\,\delta s'\,\mathbf{B}_p \sin \widehat{ds\delta s'} = ([\delta\mathbf{s}'\,.\,d\mathbf{s}]\,\mathbf{B}).$$

But if F is this resultant force

$$\begin{aligned}(\mathbf{F}\delta\mathbf{s}') = -\delta W &= \frac{J}{c}\,(\mathbf{B}\,.\,[\delta\mathbf{s}'d\mathbf{s}])\\ &= \frac{J}{c}\,([d\mathbf{s}\,.\,\mathbf{B}]\,.\,\delta\mathbf{s}'),\end{aligned}$$

and thus the resultant force on the current element is

$$\mathbf{F} = \frac{J}{c}[d\mathbf{s} \,.\, \mathbf{B}],$$

its direction being perpendicular to the directions of **B** and $d\mathbf{s}$ and the actual magnitude

$$\frac{J\,ds}{c} \,.\, \mathbf{B} \,.\, \sin \widehat{B\,ds}.$$

This agrees with the former result deduced from elementary principles.

The whole force exerted by the magnetic system on the current is thus compounded of all these elementary forces applied at the corresponding elements of the circuit. Their composition is effected in the usual way.

467. If we take the displacement of each element of the circuit alone and separate from the rest the force here determined is the force acting on it. But this is the weak point in the argument although the result happens to be right. We cannot really have one element of the current without the rest. The analysis given implies, and is only correctly applicable to, continuous movements which keep the currents complete, i.e. which do not break them. One is not allowed to break the current circuit or to stretch the element *ds*. But any restriction of this kind excludes a certain type of force. There might in fact be a tension in the conductor which on the above assumption would do no work and would thus escape detection; δW could not contain a part depending on such a force and so the expression obtained above for the force on the element would be incomplete.

A complete investigation of this difficulty necessitates a more detailed knowledge of the structure of the current. We can however state at this point why it is that the result is correct. We have already been led to consider a current in a wire as being made up of moving electrons, or isolated charge elements. We shall prove later that if we have an electron with a charge e moving with a velocity v in a magnetic field it is subject to a mechanical force vectorially represented by

$$\frac{e}{c}[\mathbf{v}, \mathbf{B}].$$

By simple addition over all the electrons in the current element *ds* we therefore obtain at once that since $J\,d\mathbf{s} = \Sigma\, e\mathbf{v}$ the current element $J d\mathbf{s}$ is subjected to a force $\frac{1}{c}[J\,d\mathbf{s}\;\mathbf{B}]$, according to the result stated above. There will be no tension as the electrons are discrete and free entities.

The main upshot of all this is that instead of the artificial current element $J\,ds$, which involves all the other elements all round and cannot exist by itself, we have got a real movement of electricity, a real thing, out of which currents are built up. The new current is a perfectly independent entity and might be called the *rational* current element.

It would thus appear that the resultant reaction of the magnetic field on the current can be represented as composed of forces on each element of the current, that on the element ds being the vector

$$\mathbf{F} = \frac{J}{c}[d\mathbf{s} \,.\, \mathbf{B}].$$

The argument however is restricted to the case when no magnetisable substances are present at any point of the field. It is however obvious that the general result here obtained is true even if such substances were present, for the resultant force on the current element must be the same as if all the magnetism were rigid: it is only in the relations of the total magnetic field to the magnetism induced and its mechanical effect on the masses in which it is induced that the difference enters, and this is susceptible of treatment as in the chapter on magnetism. As far as slow movements of the magnetic masses are concerned the steady current may be regarded as producing a rigid magnetic field.

468. Kinetic or Potential Energy? We have now seen that a steady current in a linear conductor is surrounded by a steady magnetic field, the maintenance of which even in the presence of other stationary magnetic systems, requires the expenditure of no work: the work done by the applied electromotive force in driving the current is completely compensated by the heat developed in the circuit. We may therefore say that an electric current has a certain amount of energy in virtue of the existence of its field, and by the conservation principle the amount of this energy must be independent of the method adopted in setting up the field.

In the establishment of the current by the direct application of a finite electromotive force in the current circuit we find that the full current is not immediately produced, it rises gradually to its steady value. We may therefore well ask what the electromotive force is doing during this time that the opposing resistance is not able to balance it. It is increasing the current of course. But an ordinary force acting on a body in the direction of its motion increases its momentum and communicates kinetic energy to it, or the power of doing work on account of its motion. Thus if, as appears most natural, we assume that a current has motional energy, we may say that the unresisted part of the electromotive force has been employed in increasing the internal motional or kinetic energy of the current. This, of course, implies this definite hypothesis as to the nature of the current, and to this extent is going beyond our previous range of ideas. We shall however find it most convenient to regard it in this way.

But the internal motional energy of a current is identical with the energy of the magnetic field associated with the current, so that if we regard the energy of a current as of the kinetic type we ought also to regard the energy

in the magnetic field as kinetic energy. Moreover, as there is no essential difference between the magnetic field of a current and that of a magnet, we must also assume in this case that *all* magnetic energy is kinetic. This of course fits in with the Ampèrean view which regards magnetism as constituted of minute molecular current whirls, but it would require a revision of much of our previous work in so far as all quantities designated as potential energy of magnetic nature would have to be classed as kinetic energy and in consequence be reckoned with the reverse sign.

469. In general dynamics it is the Lagrangian function $L = T - W$ that determines everything, so that if a part of the energy of any system be counted as kinetic energy, that is, reckoned in T, it must have the opposite sign to what it would have if reckoned in W. At the bottom all potential energy is probably of kinetic origin but if counted as kinetic energy its sign must be reversed. Calling it potential energy merely means that we do not want to trouble ourselves about its actual constitution*.

The difference of sign in the energies does not of course affect the sign of the forces with which they are associated. In the most general case of steady systems the Lagrangian analysis shows that the internal forces are determined as the positive gradients of the function L so that for instance the force in the θ-coordinate is

$$\frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial W}{\partial \theta}.$$

Thus if the energy is reckoned as kinetic the force is determined as its positive gradient, whereas if the energy is potential the negative gradient has to be taken. The difference of sign in the gradient thus just balances the difference in the sign of the types of energy.

470. In spite of the slight confusion which may thus arise we shall follow Faraday's suggestion and treat all magnetic energy as of the kinetic type and we shall henceforth denote it by T. It must however be particularly emphasised that we have no definite proof that this energy is kinetic, it is merely a matter of convenient choice so to regard it.

In our previous work we have always regarded the energy of magnetism as potential energy and all our results were deduced on this basis. Thus if we quote in our future discussions any of our former expressions for magnetic energy care must be exercised to see that they are in all cases quoted with the reversed sign.

* The process of "ignoring" coordinates in analytical dynamics is practically equivalent to converting the energy in the coordinates from kinetic to potential. Cf. Larmor, "On the direct application of the Principle of Least Action to the dynamics of solid and fluid systems," *Proc. L. M. S.* xv. (1884).

The distinction here involved between the two types of energy is very important and must be strongly emphasised, especially in view of the fact that practically all the more prominent writers* on this subject have failed to appreciate it. As a result the treatment of the present subject offered in the usual text-books is hopelessly confused; in several cases even a perfectly legitimate analytical procedure which properly leads to opposite signs in the two cases has either been rejected as inconsistent or explained away by further argument.

471. The energy of the electromagnetic field. A system of linear conducting circuits carrying currents and existing apart from other permanent magnetic systems could be said to possess a certain amount of energy which would be equal to the total work required to establish the currents in any order. The total amount of this work is, in fact, the measure of the electromagnetic energy of the currents relative to the state in which the currents are all zero and is what we now want to determine. It would be transformed into other forms of energy if the currents were destroyed.

A distance action theory regards the electromagnetic conditions of the system as characteristic properties of the currents in the circuits and thus the currents are the most essential quantities required for the specification of the system. We therefore require a definition of the electromagnetic energy which makes it depend on the currents alone. This is readily obtained in the following manner.

The discussion refers to a system of linear conductors carrying currents $J_1, J_2, \dots J_n$, existing alone in a field, which may however be occupied by magnetisable substance and permanent magnets. Each of these currents has a magnetic field of its own and all the fields superposed form the total magnetic field of the system. Now let $N_1, N_2, \dots N_n$ be the numbers of lines of magnetic induction of the total magnetic field which thread the various current circuits respectively.

We first enquire as to the amount of work required from the driving batteries to increase each of the currents by a small differential amount. The work required to increase the current J_1 by the amount δJ_1 is equal to

$$\frac{1}{c} N_1' \delta J_1,$$

where N_1' is the number of tubes of induction of the total field which are ultimately enclosed by δJ_1; but this is exactly the same as the number enclosed by J_1 or $N_1' = N_1$; so that the work done is equal to

$$\frac{1}{c} N_1 \delta J_1.$$

* Notable exceptions are Maxwell and Larmor.

Thus we see that the total work necessary to increase the currents in the circuits by respective amounts $\delta J_1, \delta J_2, \dots \delta J_n$ is

$$\delta T = \frac{1}{c}(N_1 \delta J_1 + N_2 \delta J_2 + \dots + N_n \delta J_n).$$

This is the fundamental characteristic equation of energy of the system and must be integrated somehow.

472. The test of integrability, i.e. the criterion for the existence of the energy function T is that for all values of r and s

$$\frac{\partial N_r}{\partial J_s} = \frac{\partial N_s}{\partial J_r} = c\frac{\partial^2 T}{\partial J_r \partial J_s},$$

and we have then also

$$\frac{\partial T}{\partial J_r} = \frac{1}{c} N_r.$$

As a matter of fact we know that in the simplest cases the inductions $N_1, N_2, \dots N_n$ through the circuits must be linear functions of the currents in them, for if we double the currents, we double the field intensity at every point and therefore also the density of the tubes of induction*. Thus T is a quadratic function of these quantities and by Euler's theorem

$$2T = \frac{\partial T}{\partial J_1} . J_1 + \frac{\partial T}{\partial J_2} J_2 + \dots,$$

but

$$\frac{\partial T}{\partial J_1} = \frac{N_1}{c} \text{ etc.},$$

so that

$$T = \frac{1}{2c} \Sigma N . J.$$

If we now write $N_r = (a_{r1} J_1 + a_{r2} J_2 + a_{r3} J_3 + \dots),$

then we see from the above that

$$a_{rs} = a_{sr}, \quad r = 1, \dots, n, \quad s = 1, \dots, n,$$

and thus

$$T = \frac{1}{2c}(a_{11} J_1^2 + a_{22} J_2^2 + \dots + 2a_{12} J_1 J_2 + 2a_{23} J_2 J_3 + \dots),$$

and this is the total energy stored up in the form of electromagnetic energy in the system. It is the electrokinetic energy of the electrodynamic field, to use Maxwell's expression.

The coefficients $a_{11}, a_{22}, \dots a_{rr} \dots$ are the coefficients of *self-induction* of the respective circuits, and the coefficients $a_{12} \dots a_{rs}$ are the coefficients of *mutual induction*. They have already been introduced in a previous connection. They are numerical coefficients which are functions of the forms and relative positions of the circuits : a_{rr} is the number of tubes of induction

* A certain restriction on the law of induction of the magnetisable substances in the field is hereby implied. Permanent magnets are presumed to be absent.

that thread the rth circuit when unit current flows in that circuit and none in all the others, it is therefore a function of the one circuit only; a_{rs} is the number of tubes of induction due to the field of a unit current in the rth circuit that thread the sth.

If we interpret the energy as kinetic energy these coefficients appear as the inertia coefficients, they determine the electrical inertia of the circuits. The currents are then the generalised velocities in the Lagrangian sense. At least it looks like this and the impulse is to try it and see if it gives the right results. We shall give a fuller discussion of the subject later.

473. Following Maxwell we have however continually attempted to interpret all our results in the language of a medium-action theory. On such a theory the motion involved in the electric current is not necessarily confined to the conductors but may take place in the whole of the space surrounding them, so that the current itself is merely the manifestation of the varying condition in the medium occupying that space. The electromagnetic and mechanical relations of the field of a system of steady currents should therefore be expressible in terms of the medium between the circuits. But for all practical purposes the only change which has taken place in the surrounding field during the starting of the currents is that involved in the statement that a magnetic field has been established. We should therefore be able to express all the mechanical relations of the theory in terms of the magnetic field vectors instead of the currents as above.

We have obtained the total energy of a system of linear currents expressed generally in the form

$$T = \frac{1}{c} \Sigma \int_0^J N \delta J$$

$$= \frac{1}{c} \Sigma \int_0^J \delta J \int_f \mathbf{B}_n \, df,$$

where Σ refers to the separate barrier surfaces f closing the various current circuits. But if ϕ is the magnetic potential of the field at the typical stage of its establishment

$$4\pi \frac{\delta J}{c} = \int_s \delta \mathbf{H}_s \, . \, ds = \delta\phi_{f_1} - \delta\phi_{f_2},$$

where s denotes any closed path which encircles once the first current only and $\delta\phi_{f_1}$, $\delta\phi_{f_2}$ denote the values of $\delta\phi$ at two infinitely near points on either side of the barrier f, the first being obtained on starting along s where it cuts through f and the second on arriving back at f. Thus

$$T = \Sigma \frac{1}{4\pi} \iint_0^\phi \mathbf{B}_n (\delta\phi_{f_1} - \delta\phi_{f_2}) \, df,$$

or if we reckon the two sides of the barrier as different sides of the same closed surface we have that

$$4\pi T = \Sigma \int_f \int_0^\phi \delta\phi B_n df.$$

Since each of the currents has a barrier surface the whole space is a singly connected region with the two sides of each barrier and a surface at infinity as boundary. We may therefore apply Green's theorem to the region and convert the surface integral into the volume integral

$$4\pi T = - \iint_0^\phi (\mathbf{B}, \nabla)\, \delta\phi \, dv,$$

the infinite surface contributing nothing in all regular cases. Since div $\mathbf{B} = 0$ and div $\delta\phi = - \delta\mathbf{H}$ this is simply

$$4\pi T = - \int dv \int_0^\phi (\mathbf{B}, \nabla)\, \delta\phi$$

$$= + \int dv \int_0^H (\mathbf{B}\delta\mathbf{H}),$$

and is thus expressed completely in terms of the field vectors $\mathbf{B}$ and $\mathbf{H}$.

The total electromagnetic energy of the currents may thus be regarded as belonging to the medium occupying the field between, being distributed through that medium with a density

$$\int_0^H (\mathbf{B} d\mathbf{H}).$$

This is the result necessitated by Maxwell's theory and is precisely the same as that deduced from the special assumption in the more general case in Chapter XIV.

474. If no magnetic substances are present in the field the density of the energy which now must be associated with the aether, is simply

$$\frac{1}{4\pi} \int_0^B \mathbf{B} d\mathbf{B} = \frac{1}{8\pi} \mathbf{B}^2,$$

but if there are magnetic media about this result is modified.

We include for generality a distribution of rigid magnetism throughout the field of intensity $\mathbf{I}_0$ at any point and if we imagine this to be built up with the current system an additional amount of work is done equal to*

$$+ \int dv \int_0^{I_0} (\mathbf{H} d\mathbf{I}_0),$$

so that the total work done is

$$T = \frac{1}{4\pi} \int dv \int_0^H (\mathbf{B} d\mathbf{H}) + \int dv \int_0^{I_0} (\mathbf{H} d\mathbf{I}_0).$$

* It is now kinetic energy.

But if there are magnetisable media in the field and if at any stage of the process of building up the system the intensity of induced polarity is **I** then

$$\mathbf{B} = \mathbf{H} + 4\pi (\mathbf{I} + \mathbf{I}_0)$$

so that

$$\begin{aligned} T &= \int dv \left[\frac{1}{4\pi}\int (\mathbf{B},\ d\mathbf{B} - 4\pi d\mathbf{I} - 4\pi d\mathbf{I}_0) + \int_0^{I_0} (\mathbf{H} d\mathbf{I}_0)\right] \\ &= \int dv \left[\frac{1}{4\pi}\int_0^B (\mathbf{B} d\mathbf{B}) - \int_0 (\mathbf{H} + 4\pi\mathbf{I} + 4\pi\mathbf{I}_0,\ d\mathbf{I} + d\mathbf{I}_0) + \int_0^{I_0} (\mathbf{H} d\mathbf{I}_0)\right] \\ &= \int dv \left[\frac{1}{8\pi}\int d\,\{\mathbf{B}^2 - 16\pi^2 (\mathbf{I} + \mathbf{I}_0)^2\} - \int_0^I (\mathbf{H} d\mathbf{I})\right] \\ &= \frac{1}{8\pi}\int \{\mathbf{B}^2 - 16\pi^2 (\mathbf{I} + \mathbf{I}_0)^2\}\, dv - \int dv \int_0^I (\mathbf{H} d\mathbf{I}). \end{aligned}$$

The last term in this expression represents with sign changed the increase of the intrinsic potential energy of the polarisations in the magnetic media. As it stands it therefore represents the increase, consequent on the induction of the polarisation, of the intrinsic elastic or motional energy of the magnetic media, regarded now as kinetic energy: it is stored up in the media as energy of an effectively non-magnetic nature and is mechanically unavailable. The first term therefore represents the total energy of purely magnetic nature and of kinetic type stored up in the system; on a tentative theory we could regard this energy as distributed throughout the field with the density

$$\frac{1}{8\pi}\{\mathbf{B}^2 - 16\pi^2 (\mathbf{I} + \mathbf{I}_0)^2\}$$

at any place. Of this energy the part

$$-2\pi (\mathbf{I} + \mathbf{I}_0)^2$$

represents intrinsic energy in the magnetic media of a purely local or constitutive nature, depending as it does only on the conditions in the medium at any point. The remainder, being a distribution with density

$$\frac{1}{8\pi}\mathbf{B}^2,$$

is then to be associated solely with the magnetic conditions in the aether.

These results are perfectly consistent with those deduced on the previous statical basis, if allowance be made for the difference of sign in all the terms arising from the different and more general conception as to the type of energy with which we are now dealing*.

* This point must again be strongly emphasised, as it is the cause of all the confusion in the subject. It is the origin of the inconsistency mentioned in connection with the usual treatment of the energy in static fields, for that treatment gives the same sign to a quantity which in the statical case is treated as potential energy and in the dynamical case as kinetic energy. Most authors seem to find satisfaction in this apparent agreement of sign and the real discrepancy has

475. Of the total energy of purely magnetic nature stored up in the field the part

$$\int dv \int_0^H (\mathbf{I}_0 d\mathbf{H})$$

is concerned mainly with the rigidly magnetised masses, being in fact the kinetic force function of their mechanical interactions and of the interactions of the currents and induced magnetism with them. Of the remainder the part

$$\int dv \int_0^H (\mathbf{I} d\mathbf{H})$$

is concerned in a similar manner with the induced polarity, being the kinetic force functions of the mechanical actions on the induced magnets.

The remainder or

$$\frac{1}{8\pi}\int [\mathbf{B}^2 - 16\pi^2 (\mathbf{I} + \mathbf{I}_0)^2]\, dv - \int dv \int_0^H (\mathbf{I} + \mathbf{I}_0,\ d\mathbf{H})$$

is therefore concerned solely with the currents and is the force function of their mechanical interactions and the interactions of the magnets with them.

Since $$\mathbf{B} = \mathbf{H} + 4\pi (\mathbf{I} + \mathbf{I}_0),$$

this part reduces to

$$\frac{1}{4\pi}\int dv \int_0^B (\mathbf{H} d\mathbf{B}),$$

and may therefore be regarded as distributed throughout the field with the density

$$\frac{1}{4\pi}\int_0^B (\mathbf{H} d\mathbf{B})$$

at each place. If as is generally the case there are no rigid magnets about in the field this latter part of the total energy is the only part that is mechanically available, provided it is not prevented from so being by frictional forces tending to degrade it. The part of the energy corresponding to the induced polarisations and arising from their reaction with the currents is in some

rarely been noticed. The same inconsistency is also involved in the assumption of similarity between the electrostatic and magneto-static fields, which is frequently adopted as a reason for the analogy between the electric potential energy and the magnetic kinetic energy. It also enables Jeans and Richardson (*Electron Theory of Matter*, p. 103) to talk of "proofs" of the kinetic nature of magnetic energy.

Cf. J. J. Thomson, *Elements of Electricity and Magnetism*, pp. 267, 363; J. H. Jeans, *Electricity and Magnetism*, pp. 403, 432, 433; M. Abraham, *Theorie der Elektrizität*, I. pp. 218, 253; Cohn, *Das elektromagnetische Feld*, pp. 202, 281, 298; Schaeffer, *Maxwellsche Theorie*, p. 61; Classen, *Theorie der Elektrizität und des Magnetismus*, II. p. 48.

Maxwell's own treatment (*Treatise*, II. ch. xi.) is perfectly clear and consistent. Cohn suggests the present development of it but finds a difficulty in the opposite signs for the potential and kinetic energies! Cf. also Livens, "On the mechanical relations of the energy of magnetisation," *Proc. R. S.* 93 (A), p. 20 (1916).

respects mechanically available, but only in so far as the presence of the magnetic masses increases the available energy associated with the currents giving rise to the field.

A similar transformation to that employed above in the reverse manner shows that

$$\frac{1}{4\pi}\int dv \int_0^B (\mathbf{H}d\mathbf{B}) = \frac{1}{c}\Sigma\int^N JdN,$$

where Σ denotes a sum relative to the linear circuits. This exhibits the energy in its relation to the currents in a more vivid form.

476. If the induction of the polarisation **I** follows a linear isotropic law so that

$$\mathbf{I} = \mu'\mathbf{H},$$

and $$\mathbf{B} = \mu\mathbf{H} + 4\pi\mathbf{I}_0,$$

where $$\mu = 1 + 4\pi\mu',$$

then it is easy to verify in a manner already elaborated in detail for the statical case that the total energy in the magnetic field, including both the energy of the aethereal field and the intrinsic energy of the material polarisations, can in the most general case be expressed in the form

$$\frac{1}{8\pi}\int\frac{\mathbf{B}^2 - 16\pi^2\mathbf{I}_0{}^2}{\mu}dv,$$

or in the mechanically equivalent form

$$\frac{1}{8\pi}\int\frac{\mathbf{B}^2}{\mu}dv.$$

As there is now no leakage by hysteresis this expression for the energy of the field also determines the mechanically available energy of the system.

Thus in this special case the energy of the system in both senses may be regarded as distributed throughout the field with the density

$$\frac{\mathbf{B}^2}{8\pi\mu}.$$

In the parts of the field where there is no rigid magnetism we have

$$\mathbf{B} = \mu\mathbf{H},$$

so that the energy density there is also expressed by

$$\frac{\mu\mathbf{H}^2}{8\pi}.$$

These expressions now represent kinetic energy so that there is no discrepancy in their having the opposite sign to the similar expressions for the potential energy in the statical theory.

477. The most definite and consistent way to treat magnetism and its energy is to consider it as consisting in molecular current whirls*; so that in magnetic media we have the ordinary finite currents, combined with the molecular currents so numerous and irregularly orientated that we can only average them up into so much polarisation per unit volume of the space they occupy. If there were no such molecular currents, the magnetic force H in the aether would in steady fields be derived from a potential cyclic only with regard to the definite number of the ordinary circuits. But when magnetism is present this potential is cyclic also with regard to the indefinitely great number of molecular circuits. The line integral of magnetic force round each circuit is $\frac{1}{c}(\Sigma J + \Sigma J')$, where $\Sigma J'$ refers to the practically continuous distribution of magnetic molecular currents that the circuit threads. This latter vanishes when these currents are not orientated with some kind of regularity. If we extend the integral from a single line to an average across a filament or tube of uniform cross section df', with that line as axis, by multiplication by df', we obtain readily the formula

$$df' \int \mathbf{H}_s ds = 4\pi \frac{df'}{c} \Sigma J + 4\pi \int \mathbf{I}_s ds\, df',$$

in which $\mathbf{I}dv$ represents the magnetisation in volume dv. Thus after transposition of the last term and removal of the factor $\delta f'$ after the average has now been taken we obtain

$$\int (\mathbf{H} - 4\pi\mathbf{I} \,.\, \delta\mathbf{s}) = \frac{4\pi}{c} \Sigma J.$$

In other words the new vector $(\mathbf{H} - 4\pi\mathbf{I})$ is derived from a potential cyclic in the usual manner with regard to the ordinary current circuits *alone*.

It thus appears that $\mathbf{H}$ must now represent the induction vector of our ordinary theory and so will hereinafter be denoted by $\mathbf{B}$ as usual. The new vector which has a potential cyclic with respect to the finite currents only, represents the 'force' and will hereafter be denoted by $\mathbf{H}$, whose significance is now changed. The induction $\mathbf{B}$ has not necessarily a potential but is by constitution of the free aether always circuital; that is it satisfies the condition of streaming flow

$$\operatorname{div} \mathbf{B} = 0.$$

The expression for the energy now includes terms

$$\Sigma \int_0^T N dJ,$$

for the ordinary currents J; this transforms as usual into

$$\frac{1}{2} \Sigma \iint d\psi \mathbf{B}_n df$$

* Cf. Larmor, *Proc. R. S.* 71 (1903); "On the mechanical and thermal relations of the energy of magnetisation."

over both faces of each barrier, which by Green's theorem is equal to

$$\frac{1}{4\pi}\int dv \int_0^H (\mathbf{B}d\mathbf{H}) \quad\text{.................................(i)}$$

extended throughout all space. But there are also terms

$$\Sigma \int_0^{J'} N'dJ',$$

for the molecular currents; now taking N' to be the cross section of the circuit multiplied by the component of the average induction normal to its plane, and remembering that J' multiplied by this cross section is the magnetic moment of this molecular current, it appears that $dJ'N'$ is equal to the magnetic induction multiplied by the component of the magnetic moment in its direction and therefore $\Sigma \int_0^{J'} N'dJ'$ is equal to

$$\int dv \int_0^I (\mathbf{B}d\mathbf{I}).$$

Thus the magnetic circuits add to the energy the amount

$$\int_0^I (\mathbf{H}d\mathbf{I}) \quad\text{.....................................(ii)},$$

together with

$$2\pi\int \mathbf{I}^2 dv \quad\text{.....................................(iii)}.$$

Thus the total electromagnetic energy in the field is

$$\frac{1}{8\pi}\int \mathbf{B}^2 dv,$$

a result which differs from our previous estimate by the purely local part represented by (iii): of course when it is remembered that in our previous theories the local forces and their associated energies were always presumed to be negligible as regards their observable mechanical effects and were therefore left entirely out of account, it is not surprising that this difference occurs. In fact using the same arguments as there employed we should neglect this local part and the results would then agree perfectly.

It may however be noticed that although this discussion provides us with the correct allotment of the total energy in reference to its association with the different parts of the system it does not indicate that the part of it

$$\int dv \int_0^I (\mathbf{H}d\mathbf{I})$$

arises mostly at the expense of the internal store of kinetic energy in the medium, being in fact merely the part of this energy which is temporarily classed as magnetic, and is not effectively available, because the magnetism cannot be annulled quickly enough to develope any considerable available energy by induction.

478. Equilibrium theories in dynamics. Before proceeding further with the applications of our ordinary dynamical analogies to these electromagnetic phenomena it will be as well to say a few words about the limitations to the theory which are necessarily implied by our analysis.

In the study of the electrodynamic relations of the field of a system of currents we always attempt to interpret everything in terms of the current strengths and the positions of the circuits, these being the palpable coordinates of the system which are directly accessible to measurement. This of course implies, not only that there is such a definite quantity as a current in each circuit (the same all round that circuit) but also that the relations of the surrounding electromagnetic fields can always be interpreted correctly in terms of these currents. In the work so far we have treated the currents as constant and the circuits as fixed and therefore the implied conditions are evidently satisfied; but what if the current strengths or the positions of the circuits are varying in any manner?

In the usual treatment of ordinary dynamics of rigid bodies we express the energy functions, and dynamical relations generally, in terms solely of the palpable coordinates of the body and the velocities in them. This however implies certain restrictions on the nature of the body and its motion which are in practice never actually realised. In fact we know that when we hit a body so as to start it off suddenly, the actual observed motion is set up by a most complicated process. Firstly a wave of compression is sent off through the body from the point of application of the blow, and the further parts of the body take up their motion only when this wave reaches them and imparts the necessary momentum and energy. The uniform motion of the body is not therefore attained until this wave disturbance has been smoothed out over the whole body; in most cases of practical importance however the time occupied by this process is so exceedingly small that we can neglect it altogether and regard the observed conditions as instantaneously established, and this is implied in the term 'rigid body.' Of course there is a slight dissipation of the energy of the wave disturbance, and it is not all transformed into the energy of the observable motion, but some goes into heat; but this quantity is usually so small that we are fully justified in treating it as non-existent or, again, the body as rigid. Of course there is the theory at the other end where we investigate these waves of compression, but the two cases are extremes and as a rule we need not mix the one with the other.

479. This is the usual sense of the term 'equilibrium theory' as used in dynamical and related discussions. The state of the motion actually observed adjusts itself so quickly that at each instant it is practically in equilibrium under the conditions pertaining at that instant, and the process of the establishment of this condition at each instant can be ignored.

A simple mechanical analogy will perhaps help to elucidate the matter. Consider the extension of a spring by a given weight. If we add the weight gradually bit by bit the spring is at each instant practically in an equilibrium condition. If however we hang the whole weight on suddenly (i.e. in general terms, suddenly produce a finite change of the steady conditions) the spring would first extend twice as far as in the above case and would then oscillate up and down before settling into the same equilibrium position. In this example if the forces are applied suddenly the phenomena of elastic inertia come in and the problem is complicated; but if they are applied gradually the conditions at each instant are statical.

There is just the same point in the usual electrodynamic theories. We have already seen how Maxwell was induced to his theoretical explanation of all observed electrodynamic actions as being transmitted through and by that hypothetical medium, the aether, which occupies the whole of space; an essential point being that the actions are transmitted by this medium with a finite velocity. It therefore follows that any disturbance produced in the conditions at any part of an electromagnetic field will smooth itself over the whole field by sending out an electromagnetic wave through the field in a manner exactly analogous to that described above. If however the variations arbitrarily produced in the conditions at any point of an electromagnetic field are slow enough, we can consider the adjustment of the corresponding new conditions throughout the whole of the field to take place so quickly as to be almost instantaneous, so that the whole field is at each instant practically in the equilibrium condition under the circumstances pertaining at that instant.

Thus if we confine ourselves to the discussion of phenomena the period of any change of which is large compared with the time taken by radiation to get across the system and adjust any new conditions we may adopt an equilibrium theory and use the results obtained for absolutely stationary states as applicable at each instant to the slowly varying motion (quasi-stationary state). That is, in the present case if the changes in the values of the current strengths or the positions of the circuits are very small and insignificant in the time taken to adjust an equilibrium condition throughout the field, then we may treat the currents in each circuit as definite quantities, i.e. the same all round the circuit, and the field at each instant can be determined in terms of their instantaneous values.

480. The velocity of adjustment of electrical conditions in air turns out to be 3×10^{10} cms. per second, so that for any ordinary sized system the condition that changes in its configuration should be small in the time occupied by radiation in getting across the system hardly restricts the application of the results obtained except in the case of very rapid oscillations, of the type in fact used to start electric waves.

In such cases we must proceed as in dynamics, to investigate the process of the adjustment of electrical conditions by this wave propagation. This however belongs to another chapter in the general theory and will not concern us at present.

Thus with the restrictions implied by the assumption of an equilibrium theory we can discuss the relations of our current system in terms of the currents in the circuits and the ordinary geometrical coordinates. The formula obtained for the electromagnetic energy, in either form, for the steady state will then apply similarly as representing the instantaneous value of that quantity in the quasi-stationary system; and the mechanical relations already deduced for the steady conditions from elementary principles will still be available for slowly varying ones.

481. Deduction of Faraday's law by the energy principle. We have already spoken in Chapter IX of induction currents caused by variations in the magnetic field surrounding conductors and Faraday's law defining them more precisely was quoted as an experimentally proved fact.

The existence of these currents and their mathematical relations can however easily be deduced by energy considerations from the results obtained regarding the electromagnetic actions discovered by Oersted; Helmholtz* and Kelvin† were the first to indicate the possibility of this deduction, ten years after the actual discovery of the currents by Faraday.

Helmholtz takes the case of a conducting circuit of resistance R, in which an electromotive force E, arising from voltaic action exists. The current in this circuit at any instant is J. Suppose now that this current by means of its magnetic field is moving a magnet about so that it is doing work on it‡. The work done on the magnet may then be calculated by considering the motion of the magnet as taking place by infinitely small displacements so that at each instant the ponderomotive forces are determined by the field and the current strength at that instant. It then follows that the work of these forces during one of the small displacements may be reckoned as $\int(\mathbf{I}\delta\mathbf{H})\,dv$, where $\mathbf{I}$ is the polarisation intensity at the typical point of the matter; $\delta\mathbf{H}$ the increase in the magnetic force at this point due to its displacement in the field, and the integral is taken over the whole field. Thus the work of the electromagnetic actions during the small instant δt may be reckoned as

$$\delta t \int \left(\mathbf{I}\frac{d\mathbf{H}}{dt}\right) dv.$$

But during this time the following additional changes have taken place; (i) an amount of heat has been generated in the circuit equal to $J^2R\delta t$;

* *Über die Erhaltung der Kraft* (1847).

† *Trans. Brit. Ass.* (1848); *Phil. Mag.* Dec. (1851).

‡ The motions however all being so slow that an equilibrium theory can be adopted.

(ii) an amount of energy equal to the work of the electromotive forces in the circuit in driving the current, viz. $EJ\delta t$, has been added to the system; (iii) the internal electromagnetic energy of the current has been altered by

$$\delta t \frac{d}{dt}\int \frac{dv}{8\pi}\{\mathbf{B}^2 - 16\pi^2\mathbf{I}^2\}.$$

The principle of energy thus requires that

$$JE\delta t = J^2R\delta t + \delta t\int \frac{dv}{4\pi}\left\{\left(\mathbf{B}\frac{d\mathbf{B}}{dt}\right) - 16\pi^2\left(\mathbf{I}\frac{d\mathbf{I}}{dt}\right) - 4\pi\left(\mathbf{I}\frac{d\mathbf{H}}{dt}\right)\right\}.$$

We have of course assumed that the magnetism of the magnet moved about is rigid, i.e. the magnet must not be capable of absorbing or storing internal energy from the electromagnetic field. There are difficulties about this assumption but it is sufficient for the present theoretical purposes. We thus see that

$$JR = E - \frac{1}{4\pi}\int\left(\mathbf{H}\frac{d\mathbf{B}}{dt}\right)dv.$$

The integral in this last equation can as usual be reduced to the form which expresses it as

$$\frac{1}{c}J\frac{dN}{dt},$$

where N represents the number of lines of induction through the circuit due to the total field comprised of the field of the magnets superposed on that of the current. Thus we have

$$JR = E - \frac{1}{c}\frac{dN}{dt}.$$

In other words in addition to the electromotive force of the battery there is an additional electromotive force on the circuit equal to $-\frac{1}{c}\frac{dN}{dt}$. Thus whenever the total number of lines of induction enclosed by the circuit changes there is an induced electromotive force created in the circuit and the amount of it is proportional to the rate of diminution of the total induction through this circuit. This is precisely Faraday's rule.

Another important point illustrated by the present example is that of the induction of a current on itself, self-induction as it is called. Whenever the current in a conductor is varying there is always a corresponding variation in the flux of magnetic induction in its own field through its circuit, which gives rise to an electromotive force in the circuit tending to prevent the variation. Induction in a circuit thus acts as a sort of electric inertia

482. A single circuit with capacity as well as induction*. We shall now discuss another important example of these matters and one which introduces us to further new principles. We have so far always regarded our

* Kelvin, "On transient electric currents," *Phil. Mag.* [4], 5 (1853), p. 393.

currents as conduction currents flowing in complete circuits: we shall now consider the case of the current discharge of an ordinary parallel plate condenser of high capacity when its plates are connected by a wire conductor. As soon as we connect the plates there is a rush of electricity round the wire and we want to investigate the nature of the current so produced. Although the theory of Maxwell which we shall eventually adopt states that the current is even in this case closed by an aethereal displacement current in the space between the condenser plates, we shall not make much direct use of the idea because we can avoid the difficulty here by imagining that the plates of the condenser are so very near together that we may practically regard the circuit as a complete one, in that the small distance between the plates is too small to make any difference to such quantities as the resistance and self-induction coefficients, which may in any case be considered as experimentally determinate.

In Ampère's time the question was tested experimentally to see if it was an ordinary current circuit, the method being to make the connecting wire into a solenoid and to examine whether an iron needle stuck in it during the discharge became magnetised. This was found to be the case but there was a certain irregularity in the phenomena in that the magnetism induced along the needle was sometimes in one direction and sometimes in the other. This irregularity was quite a mystery until Helmholtz suggested that the discharge was an oscillation. There was in the circuit a considerable amount of inertia and so the rush in one direction always overdid itself and got past the equilibrium position and thus had to swing backwards and forwards until killed by damping. In such a case it would of course be largely a matter of chance whether the first, second, third or any subsequent swing gave the preponderating magnetisation to the needle. Kelvin developed the idea more precisely a few years later.

483. We shall adopt an equilibrium theory, of course, so that at each instant the current will practically have settled down so that it is the same across every cross section of the wire and corresponds to the equilibrium value if all the other conditions pertaining at that instant could be maintained invariable. We shall thus be able to define our theory in terms of the current J in the circuit. The above simple experiment shows that such a current has associated with it a magnetic field in which there will be a certain amount of energy measured by

$$T = \frac{1}{2}\frac{aJ^2}{c},$$

c being the coefficient of self-induction of the circuit. This of course again implies that the magnetic field at each instant is that steady field corresponding to the current at that instant.

We have now a condenser in the circuit and if its capacity b is large the electrostatic energy in the system is no longer negligible. In fact if Q is the charge at any instant on the condenser plates ($+Q$ on one and $-Q$ on the other) the electrostatic potential energy W of the system is practically

$$W = \frac{1}{2}\frac{Q^2}{b},$$

which is that of the condenser. The potential energy of the whole system is practically confined to the condenser, the other parts being of small capacity and carrying also a small charge.

The current at any instant along the discharging wire may be expressed as the rate of diminution of charge on the condenser or

$$J = -\frac{dQ}{dt},$$

so that the kinetic energy in the circuit is

$$T = \frac{1}{2}\frac{a}{c}\left(\frac{dQ}{dt}\right)^2.$$

If also there is a resistance k in the circuit there is a dissipation of energy at a rate

$$kJ^2$$

per unit time. This can be introduced into the general dynamical scheme by using the dissipation function which for this case is

$$F = \tfrac{1}{2}kJ^2.$$

On the mechanical analogue of these things the inverse capacity $\dfrac{1}{b}$ appears as a coefficient of elasticity, the self-induction a as a modulus of electric inertia, and the resistance k as a coefficient of electric resistance.

484. As there is but one variable in the system, viz. Q, it is sufficient to apply the generalised energy principle to solve the problem; it gives

$$\frac{d}{dt}(T + W) = -2F,$$

the total amount by which the internal energy of the system falls is equal to the energy dissipated, which reappears in the form of heat; this is Joule's result. Thus

$$\frac{d}{dt}\left[\frac{1}{2}\frac{a}{c}\left(\frac{dQ}{dt}\right)^2 + \frac{1}{2b}Q^2\right] = -k\left(\frac{dQ}{dt}\right)^2.$$

The equilibrium theory assumption now asserts that both a and b are constant, a statical distribution being attained at each instant, so that the equation reduces to

$$\frac{d^2Q}{dt^2} + \frac{ck}{a}\frac{dQ}{dt} + \frac{c}{ab}Q = 0,$$

the solution of which determines the complete circumstances of the affair. This equation, first obtained by Kelvin, is easily solved, for if

$$Q = Q_0 e^{pt}$$

is a solution then
$$p^2 + \frac{kc}{a} p + \frac{c}{ab} = 0,$$

or
$$p = -\frac{kc}{2a} \pm \sqrt{-\left(\frac{c}{ab} - \frac{k^2c^2}{4a^2}\right)},$$

say p_1 and p_2, so that the complete solution of the equation is of a type

$$Q = Q_1 e^{p_1 t} + Q_2 e^{p_2 t}.$$

There are two distinct types of solution of an equation of this kind. If

$$q^2 = +\left(\frac{c}{ab} - \frac{k^2c^2}{4a^2}\right)$$

is positive, i.e. if $k < \sqrt{\frac{4a}{bc}}$, imaginary values are obtained for p and the solution is an oscillatory one. In this case we can in fact write the general solution in the form

$$Q = Q_0 e^{-\frac{kct}{2a}} \sin\left[qt + kc\right],$$

the integration constants Q_0 and χ being obtained from the initial conditions.

485. If the resistance R is practically negligible or the conduction nearly perfect, the solution reduces to

$$Q = Q_0 \sin\left(\frac{\sqrt{c}\, t}{\sqrt{ab}} + \chi\right)$$

and represents a permanent oscillation of period

$$2\pi \sqrt{\frac{ab}{c}}.$$

Even if k is sensible its effect on the period is practically always negligible; in fact in the general case

$$q = \sqrt{\frac{c}{ab} - \frac{k^2c^2}{4a^2}} = \frac{\sqrt{c}}{\sqrt{ab}}\left(1 - \frac{k^2cb}{8a}\right) \text{ approx.},$$

so that the period of the oscillation is increased in the ratio

$$1 + \frac{k^2cb}{8a} : 1,$$

the effect of k thus being of the second order and therefore negligible unless k is very big. This is a general result in dynamics, when the dissipation is comparatively small its effect on the period is of the second order of smallness; the main effect of resistance is in damping the amplitude, which gradually decreases to zero.

Sometimes however the resistance is so very considerable that it actually destroys the periodic nature of the motion altogether. This is the case when $k > \sqrt{\frac{2a}{bc}}$, when the solution is of the type

$$Q = Q_0 e^{-\frac{Rt}{2a} - qt},$$

so that the discharge falls off in one direction only dying down to zero. A swinging pendulum in a liquid settles down at once if the viscosity is very big.

Helmholtz's suspicions are therefore entirely corroborated. The theoretical possibility of these oscillatory discharges in a condenser was however first recognised by Kelvin. The formula obtained above for the period of the undamped discharge is in fact known after him as Thomson's formula. In order that the period may be large we must have a or b or both very big. We might for instance increase a by bringing iron into the field: practically however this method leads us into further difficulties because iron behaves so erratically that the simplicity of the above solution is spoilt. The iron cannot respond quick enough to such rapid oscillations.

486. To illustrate the various points more vividly let us take a special case given by Abraham*. The capacity b is that of a spherical condenser with radii 10 cm. and 10·2 cm. with a dielectric medium of constant 5. In the electromagnetic units adopted this gives a capacity

$$b = \frac{2550}{9 \times 10^{20}}.$$

This condenser is discharged through a circuit of resistance 1 ohm and coefficient of self-induction $a = c \,.\, 10^7$ cm. In absolute units $k = 10^9$ so that in this case

$$\frac{k^2cb}{8a} = \tfrac{1}{3} \,.\, 10^{-7} \text{ approx.};$$

the influence of the resistance on the frequency is therefore extremely slight and we can use Thomson's formula. This gives for the period

$$2\pi \sqrt{\frac{ab}{c}} = \tfrac{1}{3} \,.\, 10^{-4} \text{ sec. approx.},$$

and the corresponding wave length is

$$\lambda = 10^6 \text{ cm}.$$

The damping constant is

$$\frac{kc}{2a} = 50,$$

so that in ·02 of a second the amplitude has been decreased by $\frac{1}{e} = \frac{1}{2{\cdot}718}$;

* *Theorie der Elektrizität*, I. p. 292.

but during this time about 600 oscillations have occurred so that the rate of reduction of the amplitude from period to period is slow.

These results are of the order of those first experimentally determined by Feddersen*, who examined the spark of the discharge by a revolving mirror. In such a case the equilibrium theory certainly does apply because the dimensions of the whole system are very small compared with the wave length of the oscillation, and the state of affairs is at each instant practically the smoothed out equilibrium one. The actual distinguishing characteristics of Maxwell's theory are therefore not appreciable in such cases so that it was necessary to get beyond the limits where this analysis applies by creating much faster electrical oscillations.

487. From Thomson's formula it follows that we can theoretically reduce the wave length by decreasing the capacity or self-induction. This was accomplished experimentally by H. Hertz†, who was the first to demonstrate the existence of electric waves of sufficiently short wave length. The original form of exciter adopted by Hertz was in the form of that shown diagrammatically in the figure, this form having very little capacity and induction. When the current in the wire is concentrated into a small cross section, when the strands, as it were, of the current are close together the induction in the circuit is very large, but if they are spread out a bit the inductance is smaller. A similar argument applies to the electrostatic phenomena and shows that the energy is large if the charges are concentrated together, but small if they are spread out over large areas far apart. This is the distinction between the form of circuit adopted by Hertz and that discussed above. In our case the elastic spring is almost entirely confined to the space between the condenser plates and the inertia is in the actual circuit wire connecting them so that both the electrokinetic and electrostatic energies are very large. In the Hertzian form things are much more spread out. The theory above however does not apply to a case of this kind because the quasi-stationary conditions are no longer satisfied. It was in fact this electric oscillation with tremendously short period that sent out the electric waves detected by Hertz. In such cases an appreciable part of the energy of the system is dissipated in the form of radiation: this loss is not accounted for by the present theory where the

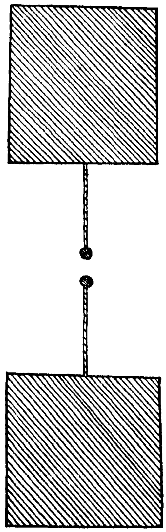


Fig. 77

* *Pogg. Ann.* 103, p. 69.

† *Untersuchung über die Ausbreitung der elektrischen Kraft.* English translation by D. E. Jones (London, 1900). Cf. below, ch. xii.

only dissipation is in the currents driving against the friction and converting their energy into heat according to Joule's law.

On the Maxwellian theory the oscillation of the concentrated electric field between the plates of the condenser represents a violent disturbance in the aether, which if quick enough should send out electric waves of disturbance through the aether: such waves carry energy away with them. An analogy is provided by the vibrations of a tuning fork in air. If the vibrations are slow the fork does not send out sound waves, because the air has time to get round the moving prong and settle down at each instant: the motion of the fork is thus independent of the air because such slow alternations cannot get hold of the elasticity in the air. If however the vibration is rapid the air has no time to get out of the way and there is a compression caused which relieves itself away in a sound wave.

488. The general electrodynamical theory of currents*. The developments in the previous paragraph illustrate in the various simple cases the application of ideas of ordinary dynamics to the solution of electrodynamic problems concerning the action and interaction of linear current systems, the results obtained being entirely consistent with the experimental facts. We have not however got beyond applications of the energy principle and this is not sufficient for the general discussion of any system with more than one degree of freedom. It is therefore very desirable that we should have, if possible, a general principle from which all the results, statical or dynamical, can be deduced for the most general system. We shall now show that we can adopt the more general results of analytical mechanics into a scheme which will enable us to deduce from the expressions for the kinetic and potential energies already obtained all the equations of a system of linear currents with any number of degrees of freedom. By applying Lagrange's dynamical equations to the more general problem expressed in a definite manner we are again led to results which are in entire agreement with the experimental facts.

Following Maxwell we assume that in the field of any system of electric currents there is some motion of which we cannot take direct cognisance. The kinetic energy of this motion is that energy which we have already obtained as the electromagnetic energy in the field; Maxwell calls it the electrokinetic energy. For a system of stationary or quasi-stationary currents it appears as a quadratic function† of the current strengths in the various circuits, the coefficients in which depend merely on the forms and relative positions of the various circuits.

489. Whatever mechanical analogy we adopt in the application of dynamics to electric currents we are evidently always involved in the class

* Cf. Maxwell, *Treatise*, II. chs. v–vii.

† If there is no iron about.

of motions which Helmholtz described as cyclic. The parameters which determine the instantaneous position of such a system are of two kinds: (i) the parameters of the first kind which are of the general type of coordinates in mechanics. These parameters occur in general in the expression of the kinetic energy with their differentials with respect to the time. To this group belong, in the present instance, those geometrical parameters which determine the position of the circuits: (ii) the parameters of the second group on the other hand, which are the coordinates of the cyclic motion, do not themselves appear in the expression for the kinetic energy, only their time rates of change being involved. If the kinetic energy is known to be correctly expressed in terms of the coordinates and velocities explicitly we have therefore no difficulty in separating the coordinates into their respective classes. But in systems where the internal connections are only partially known, a difficulty may occur, in as far as in obtaining the expression for the kinetic energy, it may have been convenient or even necessary to introduce the generalised momenta in the cyclic motions, in order to obtain a usable expression. For example in the hydrodynamical analogue, in determining the forcives between cores in problems of cyclic motion, the circulations in terms of which the energy is usually expressed, must be treated as generalised momenta. It is therefore necessary and essential to have a clear view of the circumstances which determine whether the various quantities which enter into the specification of the energy are to be classed as velocities or momenta. The basis of the distinction between these two classes of quantities is of course fundamental; it is to be found in the way in which they occur in the Hamiltonian analysis of the dynamical problem. The essential property of a velocity is that it is a perfect differential coefficient with respect to the time; any function involving rate of change of configuration, which enjoys this property, so that its time integral is a function of position only may be taken to be a velocity; provided we, if need be, contemplate also a corresponding force. On the other hand, any such function of the rate of change of configuration, even though it be a perfect differential with respect to the time, must be treated as a momentum, if it is known to remain constant with time while no external forces are applied to it; for if it were a velocity, linked up with other velocities, its constancy in the free motion could not usually fit in with the analytical theory.

490. In the theory of cyclic fluid motion, the circulations being constant, must thus be taken as momenta, and, when the energy is expressed in terms of them, it must be modified before the forcives can be derived from it in the manner of Lagrange and Hamilton. In the theory of electrodynamics, on the other hand, the electric currents are not unalterable with the time, even if no applied electromotive forces are applied, and as they are the differential coefficients with respect to the time of definite physical quantities, the charges

of electricity, they may be taken as the velocities, provided we recognise the play of the corresponding (electromotive) forces.

In the electrodynamics of complete circuits however there is no reason, in that theory taken by itself, why the functions defined hereinafter as the electrokinetic momenta should not be taken as the velocities instead, if so desired, for they satisfy all the above conditions, though of course the corresponding forcives would be of quite different types from the usual ones. This remark is in illustration of the fact that the distinction between momenta and velocities is to a certain extent one of convenience. We shall however adopt the method indicated on account of the enormous difficulties underlying this suggested alternative.

491. The system we shall treat will consist of n-conducting circuits carrying currents $J_1, J_2, \ldots J_n$. We have therefore n-cyclic coordinates in which the velocities are $J_1, J_2, \ldots$, in addition to a certain number of ordinary geometrical coordinates $\theta_1, \theta_2, \ldots \theta_m$ which determine the relative configuration of the system. The actual cyclic coordinates may be taken as the integrals of the currents with respect to the time reckoned from a definite instant, i.e. the quantities $Q_1, Q_2, \ldots Q_n$ of electricity which since that instant have crossed any cross section of the respective conductors

$$J_1 = \frac{dQ_1}{dt}, \qquad J_2 = \frac{dQ_2}{dt}, \ldots.$$

The generalised force components corresponding to the cyclic-coordinates are the electromotive forces which work on the currents flowing. The work function of the forces applied in these coordinates would thus be

$$\delta W_Q = \sum_{r=1}^{n} E_r \delta Q_r,$$

if we assume impressed electromotive forces $E_1, E_2, \ldots E_n$ in each circuit respectively, since the work of the electromotive force E in any virtual increase of the coordinate Q defined as above is $E\delta Q$.

We have also to include the virtual work of the applied mechanical forces if there are any. In the general Lagrangian method the generalised force component Θ which corresponds to the coordinate θ is the coefficient in the work done when that coordinate is alone altered. Thus

$$\delta W_\theta = \Sigma \Theta \delta\theta.$$

If the forces are applied from without δW would represent energy added to the system. If they are merely forces exerted by one part of the system on another or against the external system the work in them would come from the energy of the system and must therefore be taken as $-\delta W_\theta$.

492. We have to take into account the resistances to the flow of the currents. These may be introduced either by including them in the generalised impressed or external force components corresponding to the

cyclic coordinates, or by the more general method involving the introduction of a dissipation function. If the resistances in the circuits are R_1, R_2, ... then on the first method the impressed electromotive forces in the separate circuits would have been respectively diminished by $R_1 J_1$, $R_2 J_2$, ... $R_n J_n$, in order to obtain the resultant force components. The more general method consists in the introduction of the function

$$F = \tfrac{1}{2}(R_1 J_1^2 + R_2 J_2^2 + \dots),$$

into the general dynamical scheme.

In order to obtain complete generality we shall assume that each circuit has included in it a condenser, or an appreciable capacity for storing energy. It is only in this case that the potential electric energy is at all comparable with the magnetic kinetic energy. If the original charges in these condensers were Q_{0_1}, Q_{0_2}, ... then the general potential energy function for the system would be

$$W = \tfrac{1}{2}\Sigma b_{rr}(Q_{0_r} - Q_r)^2 + \Sigma b_{rs}(Q_{s_r} - Q_r)(Q_{0_s} - Q_s),$$

and the product terms would be negligible in most cases if the condenser in each circuit is of such a form as to concentrate its field sufficiently to prevent mutual influence with the others.

We shall leave out of account any so-called permanent magnets or magnetisable substances in which the magnetisation is not capable of following the field without hysteretic loss. 'Permanent' magnets are in reality far from permanent and are indeed very erratic things; their properties are very indefinite and a theory including them becomes largely an empirical subject. We may thus regard the coefficients of induction of the circuits to be dependent merely on the geometrical configurations in the circuits.

493. The electrokinetic energy of the system is, as before, given by

$$2T = \Sigma a_{rr}\left(\frac{dQ_r}{dt}\right)^2 + 2\Sigma a_{rs}\frac{dQ_r}{dt}\cdot\frac{dQ_s}{dt}.$$

This is kinetic energy of some kind; we do not as before need to know of what kind. We only want its amount in suitable terms to enable us to apply general dynamical methods; this is the great advantage of the present line of attack.

We must now also include the kinetic energy of the movement of the material conductors because the material conductors involved may possess very considerable masses. The positions and general configurations of these masses are as before specified by the generalised coordinates θ_1, θ_2, ... θ_m and the kinetic energy corresponding to them will be denoted by T_1 so that the total kinetic energy is given by

$$T + T_1;$$

T_1 is of course a quadratic function of $\dot{\theta}_1$, $\dot{\theta}_2$, ... $\dot{\theta}_m$ in which the coefficients are functions of $\theta_1 \dots \theta_m$.

For absolute generality we should include in the complete expression for the kinetic energy terms involving such things as $(\dot{Q}_r \dot{\theta}_s)$ but this would be getting beyond our theory. In all realisable cases and certainly in all those cases where an equilibrium theory is applicable the electric changes adjust themselves so quickly compared with the slow motions of ordinary matter that the general electromagnetic system is at each moment sensibly in an equilibrium condition; so that there is practically no interaction between the kinetic energies of the electromotive and material systems such as would arise from mixed terms in the energy function involving both their velocities —a fact verified experimentally by Maxwell. The expression for T thus represents completely the energy of the system as far as electromotive disturbances are concerned, whether the system is in motion or not. It is therefore sufficient for the determination of the electrical conditions. The other part T_1 is solely the ordinary kinetic energy of motion of the conductors and is alone necessary for the determination of the mechanical relations of the electrodynamic field.

494. We can now proceed to apply any of the usual methods of obtaining the equations of the motion in the various coordinates, electrical and geometrical. The most general method involves a use of the principle of least action which represents the most general principles of dynamics in their most condensed form. If the system is governed by dynamical laws at all, we have merely to obtain the energies in their most compact form and then to substitute them in this principle. Lagrange's principle of least action is in fact the infallible method to apply to all dynamical systems and is the one which avoids the investigation every time of the conditions peculiar to each case. We shall however find it more convenient for the present case, in which we have actually determined the form of the functions, to take the slightly less general form of the principle which is contained by the expression that the motion is determined by the ordinary Lagrangian equations in dynamics.

If we apply the general dynamical equations to the electrical coordinates first we find that there is an equation for each circuit of a type

$$\frac{d}{dt}\frac{\partial}{\partial \dot{Q}_r}(T + T_1) - \frac{\partial (T + T_1)}{\partial Q_r} + \frac{\partial W}{\partial Q_r} - \frac{\partial F}{\partial \dot{Q}_r} = E_r,$$

but since T_1 is a function of the θ's and $\dot{\theta}$'s and does not contain either Q_r or $\dot{Q}_r$ this reduces to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}_r}\right) - \frac{\partial T}{\partial Q_r} + \frac{\partial W}{\partial Q_r} - \frac{\partial F}{\partial \dot{Q}_r} = E_r.$$

There are as many of these equations as there are circuits and so they are sufficient to determine the electrical motions in the circuits.

495. On the analogy with the ordinary Lagrangian equations the term

$$E_r - \frac{\partial W}{\partial Q_r} + \frac{\partial F}{\partial \dot{Q}_r},$$

is the applied force acting in the coordinate. The term on the other side

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}_r}\right) - \frac{\partial T}{\partial Q_r},$$

which, since the actual coordinates $Q_1, Q_2 \ldots Q_n$ corresponding to the cyclic velocities $J_1, J_2, \ldots$ do not explicitly occur in any of the functions involved so that for each circuit

$$\frac{\partial T}{\partial Q_r} = 0,$$

reduces to
$$\frac{d}{dt}\frac{\partial T}{\partial J_r},$$

represents what Kelvin called the 'kinetic reaction' of the system, taken in D'Alembert's sense. In the electrical case the applied electromotive force is balanced by the kinetic reaction of the changing current.

But

$$\frac{\partial T}{\partial J_r} = a_{r1}J_1 + a_{r2}J_2 + \ldots a_{rn}J_n,$$

and differential coefficients of T with respect to the cyclic velocities are the momenta in the respective coordinates; we call them the cyclic momenta. We see at once that they are identical for each circuit with the magnetic induction flux through the circuit; we can therefore describe them, after Maxwell, as the electrokinetic momenta of the circuits. The electrokinetic reactions to the variation of these momenta are determined as usual by

$$-\frac{d}{dt}\left(\frac{\partial T}{\partial J_r}\right) = -\frac{d}{dt}(a_{r1}J_r + \ldots),$$

so that they correspond exactly to the induced electromotive forces in the circuits. We thus deduce that Faraday's law is quite consistent with the general dynamical hypothesis, even in the most general case when the current circuits are in motion.

496. We have above limited ourselves to the application of the dynamical analysis to the cyclic coordinates, thereby determining the electrical motions only. We must now apply the same method to find the equations corresponding to the geometrical coordinates which determine the configurations of the conducting circuits. For each geometrical coordinate θ_r we have an equation of type

$$\frac{d}{dt}\frac{\partial (T + T_1)}{\partial \dot{\theta}_r} - \frac{\partial (T + T_1)}{\partial \theta_r} + \frac{\partial W}{\partial \theta_r} - \frac{\partial F}{\partial \dot{\theta}_r} = \Theta.$$

which since $\frac{\partial T}{\partial \dot\theta_r} = 0$ can be put in the form

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot\theta_r}\right) - \frac{\partial T_1}{\partial \theta_r} = \Theta - \frac{\partial W}{\partial \theta_r} + \frac{\partial T}{\partial \theta_r} + \frac{\partial F}{\partial \dot\theta_r}.$$

The two terms on the right represent with signs changed the kinetic reaction of the circuits to ordinary motions. The terms on the right

$$-\frac{\partial W}{\partial \theta_r} + \frac{\partial T}{\partial \theta_r},$$

represent the mechanical force exerted by the system in the coordinate θ_r. The second part arises from the electrokinetic energy of the system and the first from the electrostatic part. We call the former the electrodynamic forces and the others the electrostatic ones.

We thus see that the electrodynamic forces of stationary currents always tend to increase the electrokinetic energy T. We then speak of $-T$ in this sense as the *electrodynamic potential* of the system; it plays the part of the force function of the electrodynamic forces. This is Neumann's result*.

Prof. Larmor† criticises the above argument on the ground that the mechanism that links the mechanical and electrodynamical systems together is too complicated to be treated otherwise than statically. Such a procedure is however quite sufficient for our purposes because the mechanical changes in the conductors, as already explained, have usually a purely statical aspect compared with the extremely rapid electric disturbances. Larmor puts the argument in the following manner. A small displacement of the system increases T by δT; this increase must come from some source; if we suppose for the moment, to avoid complications, that there is no dissipation, we see that this energy must come from the energy of the material system. During the displacement the electromotive system is at each instant sensibly in an equilibrium condition so that somehow, by means of unknown connecting actions, the displacement alters the mechanical energy by $-\delta T$ and of this, considered as potential energy, the mechanical forces are the result. The expression T given above with its sign changed thus appears as the potential energy of the mechanical electrodynamic forces acting between the material conductors which carry the currents. This is the result as deduced above.

497. The general conclusions thus arrived at are in perfect agreement with the actual facts of the phenomena. Consider for example the case of two rigid conductors, the one fixed and the other moveable about a fixed point, and suppose the currents maintained constant or that there is no

* "Ueber ein allgemeines Prinzip der mathematischen Theorie der induzierten Ströme," *Berlin. Abhdg.* (1848).

† *Phil. Trans.* A. 185 (1894), p. 761.

resistance in the circuits to cause dissipation. The electrokinetic energy of the system then alters by an amount which is determined solely by the alteration of $a_{12}J_1J_2$; but this is proportional to $a_{12}J_1$, the number of tubes of induction in the field of the first conductor which passes through the second. The second moveable conductor will thus always tend to turn so as to enclose as many as possible of the lines of induction in the field of the first current.

The deductions are also true for non-rigid or flexible circuits, wherein the coefficients a_{11}, a_{22}, ... of self-induction are variable.

The success of these investigations points to the conclusion that the laws of electrodynamics can be all deduced from the general equations of mechanics, and a conclusion of this kind is of immense importance for our future speculations. Whatever view we may hold of the actions underlying such physical phenomena we must nevertheless admit the importance of the discovery that the motions of ponderable bodies and electrodynamic phenomena are subject to the same laws. We need not necessarily regard this close connection between mechanics and electrodynamics as providing any evidence of a mechanical basis for electrical phenomena. We might just as well say that it favours the view that mechanical laws have an electrodynamic basis. This latter point of view is in fact characteristic of the general trend of modern physical speculations.

498. On the solution of the equations for circuits with capacity and induction. We can now proceed to discuss the numerical solution of the equations representing the conditions in a set of n-circuits with capacity and induction. We have then the kinetic and potential energies and the dissipation function in the form

$$2T = \Sigma a_{rr}\left(\frac{dQ_r}{dt}\right)^2 + 2\Sigma a_{rs}\frac{dQ_r}{dt}\cdot\frac{dQ_s}{dt},$$

$$2W = f(Q_1, Q_2, \ldots Q_n),$$

$$2F = \Sigma R_r\left(\frac{dQ_r}{dt}\right)^2,$$

and in addition there are the impressed electromotive forces $E_1, E_2, \ldots E_n$ in the respective circuits.

The general equations are then of the type

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}_r}\right) + \frac{\partial W}{\partial Q_r} - \frac{\partial F}{\partial \dot{Q}_r} = E_r,$$

from which one or two important conclusions can be directly inferred. The complete solution of the equations in all their generality will be examined at a later stage in the discussion.

If the solution shows that the currents are periodic forced vibrations with a frequency n then all the quantities may be treated as dependent on the time by the factor e^{int} and thus we can have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}_r}\right) = in \frac{\partial T}{\partial \dot{Q}_r},$$

and when n increases indefinitely we must have

$$\frac{\partial T}{\partial \dot{Q}_r} = 0$$

in each circuit. Thus in the case of enormously rapid vibrations of the currents, their distribution in the various conductors is independent of the resistances and is determined by the fact that the kinetic energy (and not the dissipation) function is a minimum.

A similar remark applies when the question under consideration is one of initial impulse effects.

This explains why it is that when a rapidly alternating current is sent along a wire, the current really only travels in the outer layer of the wire; the mean distance between the various filaments in the current being thereby increased and their mutual inductance and the kinetic energy of the field decreased to their minimum values.

499. When $W = 0$ it is convenient to express everything in terms of the currents $J_r = \frac{dQ_r}{dt}$. Thus in the problem of steady electric flow when all the quantities E_r representing impressed electromotive forces are constant, the currents are determined directly by the linear equations

$$\frac{\partial F}{\partial J_r} = E_r,$$

which express the condition that the function

$$(F - \Sigma E_r J_r)$$

is a minimum. If all the E's are zero this is Joule's law of minimum dissipation for steady currents. The above is the more generalised form including impressed electromotive forces.

500. The general problem dealing with forced vibrations in circuits with potential energy can easily be reduced to the simpler case when $W = 0$ if attention is directed to the following point.

In the most general case in which

$$W = \Sigma \tfrac{1}{2} b_{rr} Q_r^2 + \Sigma b_{rs} Q_r Q_s,$$

the equations for each circuit are of the form

$$a_{11}\dot{J}_1 + a_{12}\dot{J}_2 + \dots a_{1n}\dot{J}_n + b_{11}Q_1 + b_{12}Q_2 + \dots b_{1n}Q_n + \frac{\partial F}{\partial J_1} = E_1,$$

but when vibrations proportional to say $\cos pt$ are in progress in the circuit

$$\frac{dJ_r}{dt} = \frac{d^2Q_r}{dt^2} = -p^2Q_r,$$

and thus the equation above reduces to

$$\left(a_{11} - \frac{b_{11}}{p^2}\right)\dot{J}_1 + \left(a_{12} - \frac{b_{12}}{p^2}\right)\dot{J}_2 + \ldots + \frac{\partial F}{\partial J_1} = E_1,$$

and then it is of precisely the type obtained without a potential energy function.

This remark allows us to simplify our equations by omitting the potential energy part altogether. When the solution is obtained we may at any time generalise it to include these cases, by the introduction of $a_{rs} - \frac{b_{rs}}{p^2}$ in place of any induction coefficient a_{rs}. In following this course we must be prepared to admit negative values of these coefficients a_{rs}.

We can now illustrate these general principles by the exact solution of the equations in a few special cases*.

501. A single current circuit with resistance and induction only: here we have

$$2T = a\dot{Q}^2,$$
$$2F = R\dot{Q}^2,$$
$$W = 0,$$

and the equation of motion assumes the form

$$\frac{d}{dt}(a\dot{Q}) + R\dot{Q} = E,$$

where E is the impressed force. We use J for the current strength in the circuit and then

$$a\dot{J} + RJ = E,$$

and the integration of this equation determines completely the time variation of the current intensity J. Two cases present themselves.

(1) E constant. In this case we have, since a and R are constants,

$$J = \frac{E}{R} - Ce^{-\frac{Rt}{a}}.$$

The integration constant C is determined from the initial conditions. If at the time $t = 0$ the circuit is closed and the constant electromotive force applied we shall have $J = 0$ initially and thus

$$C = \frac{E}{R}.$$

* These examples are given by Maxwell and Rayleigh, "The Theory of Sound," *Phil. Mag.* [5], 21 (1886), p. 369; *Proc. R. S.* 48 (1891), p. 203. Cf. also J. J. Thomson, *Recent Researches in Electricity and Magnetism* (Oxford, 1893).

or
$$J = \frac{E}{R}\left(1 - e^{-\frac{Rt}{a}}\right),$$
the constant $\frac{R}{a}$, called the 'time constant,' determines the rate of increase of the current up to its steady value $J = \frac{E}{R}$ which corresponds to Ohm's law.

(2) E periodic: say
$$E = E_0 \sin pt,$$
we have then
$$a\dot{J} + RJ = E_0 \sin pt;$$
the solution of this equation will, as is well known, be the same as the imaginary part of the solution of the simpler equation
$$a\dot{J} + RJ = E_0 e^{ipt}.$$
We try a solution of the form
$$J = J_0 e^{ipt},$$
which will satisfy if
$$J_0 (aip + R) = E_0,$$
i.e. if
$$J_0 = \frac{E_0}{R + aip}.$$

The solution of our first equation is then the imaginary part of the function
$$\frac{E_0 e^{ipt}}{R + aip} = \frac{E_0 (R - aip)\, e^{ipt}}{(R^2 + a^2p^2)},$$
or of
$$\frac{E_0 e^{(ipt-\theta)}}{\sqrt{R^2 + a^2p^2}},$$
where
$$\tan\theta = \frac{ap}{R}.$$

We have therefore
$$J = \frac{E_0}{\sqrt{R^2 + a^2p^2}} \sin(pt - \theta).$$

The complete solution is therefore
$$J = \frac{E_0}{\sqrt{R^2 + a^2p^2}} \sin(pt - \theta) + Ce^{-\frac{Rt}{a}},$$
the integration constant C being again determined by the initial conditions; if these are again such that $J = 0$ when $t = 0$ then
$$C = \frac{E_0 \sin\theta}{\sqrt{R^2 + a^2p^2}}.$$
Thus
$$J = \frac{E_0}{\sqrt{R^2 + a^2p^2}}\left(\sin(pt - \theta) + e^{-\frac{Rt}{a}} \sin\theta\right).$$
The second part of the solution is however unimportant except for a consideration of the initial establishment of the steady oscillating current finally

established. If we consider the conditions started at some remote past time the solution is represented correctly by the one term

$$J = \frac{E_0 \sin (pt - \theta)}{\sqrt{R^2 + a^2p^2}}.$$

The phase of the current therefore lags behind the phase of the impressed force by an amount $\theta = \tan^{-1} \frac{ap}{R}$, which amounts to a quarter phase when p is infinitely large.

We also see that the ratio of maximum current to maximum impressed force is

$$\frac{1}{\sqrt{R^2 + a^2p^2}},$$

this ratio in the steady case being

$$\frac{1}{R}.$$

Thus if p is very much bigger than $\frac{R}{a}$ the current is greatly reduced by the self-induction. The quantity $\sqrt{R^2 + a^2p^2}$ is called the 'impedance' in the circuit for the given periodic disturbance.

When we introduce an appreciable capacity into the circuit (perhaps a Leyden jar) the current is again increased. This case is easily solved on the above lines.

502. Two detached circuits without capacity influencing one another only by induction.

We shall examine the effect on the second circuit of the instantaneous establishment and subsequent maintenance of a current J_1 in the first circuit. At the first moment the question is one of the function T only, where

$$2T = a_{11}J_1^2 + 2a_{12}J_1J_2 + a_{22}J_2^2,$$

and the solution is to be obtained by making T a minimum under the condition that J_1 has the given value. Thus initially

$$J_{20} = -\frac{a_{12}}{a_{22}} J_1,$$

and accordingly after a time t

$$J_2 = -\frac{a_{12}}{a_{22}} J_1 e^{-\frac{R_2 t}{a_{22}}},$$

R_2 is the resistance of the circuit. The whole induced current as measured by a ballistic galvanometer is

$$\int_0^\infty J_2 dt = -\frac{a_{12}}{R_2} J_1,$$

in which a_{22} does not appear.

The current in the secondary circuit due to the cessation of a previously established steady current J_1 in the first circuit is the opposite of the above.

The general equations for two detached circuits as above may be obtained in the usual manner from the above form of T and

$$2F = R_1 J_1^2 + R_2 J_2^2.$$

Thus
$$a_{11}\dot{J}_1 + a_{12}\dot{J}_2 + R_1 J_1 = E_1,$$
$$a_{12}\dot{J}_1 + a_{22}\dot{J}_2 + R_2 J_2 = E_2.$$

If a harmonic electromotive force

$$E_1 = Ee^{ipt},$$

act in the first circuit, and the second circuit be free from imposed forces ($E_2 = 0$) we have on elimination of J_2

$$J_1\left[ip\left(a_{11} - p^2\frac{a_{12}^2 a_{22}}{p^2 a_{22}^2 + R_2^2}\right) + R_1 + \frac{p^2 a_{12}^2 R_2}{p^2 a_{22}^2 + R_2^2}\right] = Ee^{ipt},$$

showing that the reaction of the secondary circuit upon the first is to *reduce* the inductance by

$$\frac{p^2 a_{12}^2 a_{22}}{p^2 a_{22}^2 + R_2^2}$$

and to *increase* the resistance by

$$\frac{p^2 a_{12}^2 R_2}{p^2 a_{22}^2 + R_2^2}.$$

503. Let us now consider two circuits in parallel.

Firstly it is not necessary to include the influence of the leads outside the points of bifurcation; for provided there be no mutual induction between these parts and the remainder, their inductance and resistance enter into the result by simple addition.

Under the sole operation of resistance the total current J would divide itself between the two conductors (of resistances R_1 and R_2) in the parts

$$\frac{R_2}{R_1 + R_2} J \quad \text{and} \quad \frac{R_1}{R_1 + R_2} J,$$

and we may conveniently so choose the second coordinate that the currents in the two conductors are in general

$$\frac{R_2}{R_1 + R_2} J + J', \qquad \frac{R_1}{R_1 + R_2} J - J',$$

J still representing the total current in the leads. The dissipation function found by multiplying the squares of the above currents by $\frac{1}{2}R_1$ and $\frac{1}{2}R_2$ is

$$F = \frac{1}{2}\frac{R_1 R_2}{R_1 + R_2} J^2 + \tfrac{1}{2}(R_1 + R_2) J'^2.$$

also a_{11}, a_{12}, a_{22} being the induction coefficients of the two branches

$$T = \frac{1}{2}\frac{a_{11}R_2^2 + 2a_{12}R_1R_2 + a_{22}R_1^2}{(R_1+R_2)^2}J^2$$
$$+ \frac{(a_{11}-a_{12})R_2 + (a_{12}-a_{22})R_1}{R_1+R_2}JJ' + \tfrac{1}{2}(a_{11} - 2a_{12} + a_{22})J'^2.$$

Thus in the notation

$$\alpha_{11} = \frac{a_{11}R_2^2 + 2a_{12}R_1R_2 + a_{22}R_1^2}{(R_1+R_2)^2}$$
$$\alpha_{12} = \frac{(a_{11}-a_{12})R_2 + (a_{12}-a_{22})R_1}{R_1+R_2},$$
$$\alpha_{22} = a_{11} - 2a_{12} + a_{22}, \qquad \rho_1 = \frac{R_1R_2}{R_1+R_2}, \qquad \rho_2 = R_1 + R_2,$$

we have
$$2F = \rho_1 J^2 + \rho_2 J'^2.$$
$$2T = \alpha_{11}J^2 + 2\alpha_{12}JJ' + \alpha_{22}J'^2,$$

and we are then in the same case as previously discussed. We thus see at once that the effective resistance R and the effective inductance a of the combination* are obtained as

$$R = \rho_1 + \frac{p^2\alpha_{12}^2\rho_2}{p^2\alpha_{22}^2 + \rho_2^2},$$
$$a = \alpha_{11} - \frac{p^2\alpha_{12}\cdot\alpha_{22}}{p^2\alpha_{22}^2 + \rho_2^2}.$$

By substitution and reduction we find that

$$R = \frac{R_1R_2(R_1+R_2) + p^2[R_1(a_{12}-a_{22})^2 + R_2(a_{11}-a_{12})^2]}{(R_1+R_2)^2 + p^2(a_{11} - 2a_{12} + a_{22})^2},$$
$$a = \frac{a_{11}R_2^2 + 2a_{12}R_1R_2 + a_{22}R_1^2 + p^2(a_{11}a_{22} - a_{12}^2)(a_{11} - 2a_{12} + a_{22})}{(R_1+R_2)^2 + p^2(a_{11} - 2a_{12} + a_{22})^2},$$

in which $(a_{11} - 2a_{12} + a_{22})$ and $(a_{11}a_{22} - a_{12}^2)$ are both positive by virtue of the nature of T.

504. As p increases from zero, we see that R continually increases and that a continually decreases.

When p is small

$$R = \frac{R_1R_2}{R_1+R_2}, \qquad a = \frac{a_{11}R_2^2 + 2a_{12}R_1R_2 + a_{22}R_1^2}{(R_1+R_2)^2},$$

in this case the distribution of the main current between the conductors is determined by the resistances.

On the other hand when p is very great

$$R = \frac{R_1(a_{12}-a_{22})^2 + R_2(a_{11}-a_{12})^2}{(a_{11} - 2a_{12} + a_{22})^2},$$
$$a = \frac{a_{11}a_{22} - a_{12}^2}{a_{11} - 2a_{12} + a_{22}},$$

* Defined as the effective constants in the solution for the total current J.

in this case the distribution of the currents is independent of the resistances, being determined as usual so that the ratio of the currents in the two conductors is

$$\frac{a_{22} - a_{12}}{a_{11} - a_{12}}.$$

When the conductors exert no mutual induction the formulae are simpler: $a_{12} = 0$ and so

$$R = \frac{R_1 R_2 (R_1 + R_2) + p^2 (R_1 a_{21}{}^2 + R_2 a_{11}{}^2)}{(R_1 + R_2)^2 + p^2 (a_{11} + a_{22})^2},$$

$$a = \frac{R_2{}^2 a_{11} + R_1{}^2 a_{22} + p^2 a_{11} a_{22} (a_{11} + a_{22})}{(R_1 + R_2)^2 + p^2 (a_{11} + a_{22})^2}.$$

505. The more general form of these results applicable to the case of n current circuits in parallel can be similarly treated. Let J_0 be the current in the leads, $J_1, J_2, \ldots J_n$ the currents in the wires; we may assume, for simplicity, that there is no mutual induction between the wires and the leads. Let a_{rr} be the self-induction and k_r the resistance of the wire through which the current is J_r, a_{rs} the coefficient of mutual induction between this wire and the wire through which the current is J_s. Let a_0 be the self-induction, r_0 the resistance of the leads, ϕ_0 the electromotive force in the external circuit; we shall suppose that this varies as e^{ipt}. The current through the leads and those through the wires in parallel are connected by the relation

$$J_0 - (J_1 + J_2 + \ldots + J_n) = 0,$$

so that these variables are not all independent. Thus in forming the general equations we may treat these currents as independent if we introduce an undetermined multiplier in the usual manner in connection with this relation: we then get

$$\begin{aligned}
&(a_0 ip + k_0) J_0 && + \lambda = \phi_0,\\
&(a_{11} ip + k_1) J_1 + a_{12} ip J_2 + \ldots && - \lambda = 0,\\
&(a_{12} ip) J_1 + (a_{22} ip + k_2) J_2 + \ldots && - \lambda = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots && \\
&a_{1n} ip J_1 + a_{2n} ip J_2 + \ldots && - \lambda = 0,
\end{aligned}$$

solving the last n-equations linearly we find

$$\frac{J_1}{A_{11} + A_{12} + \ldots A_{1n}} = \frac{J_2}{A_{21} + A_{22} + \ldots A_{2n}} = \ldots = \frac{\lambda}{\Delta},$$

where

$$\Delta \equiv \begin{vmatrix} a_{11} ip + k_1, & a_{12} ip, & \ldots & a_{1n} ip \\ a_{21} ip, & \ldots & & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{1n} ip, & a_{2n} ip, & \ldots & a_{n22} ip + k_n \end{vmatrix},$$

and A_{pq} denotes the minor of Δ corresponding to the constituent in which a_{pq} occurs. Since

$$J_0 = J_1 + J_2 + \ldots,$$

we have from the above equations

$$\frac{J_0}{A_{11} + A_{22} + \ldots + 2A_{12} + 2A_{23} + \ldots} = \frac{\lambda}{\Delta},$$

substituting this value of λ in the first equation for the currents we find

$$\left(a_0 ip + k_0 + \frac{\Delta}{S}\right) J_0 = \phi_0,$$

where $$S \equiv A_{11} + A_{22} + \ldots + 2A_{12} + \ldots,$$

whence the self-inductance and impedance of the leads can be deduced; their expressions are however in general very complicated, but they take as usual comparatively simple forms when ip is either very large or very small.

506. When ip is very large

$$\frac{\Delta}{S} = ip\frac{D}{S'} + \frac{k_1 (A_{11}' + A_{12}' + \ldots A_{1n}')^2 + k_2 (A_{12}' + \ldots A_{2n}')^2 + \ldots}{S'^2},$$

where

$$D = \begin{vmatrix} a_{11}, & a_{12}, & \ldots & a_{1n} \\ a_{21}, & a_{22}, & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

and A_{pq}' is the minor of D corresponding to the constituent a_{pq}, while

$$S' = A_{11}' + A_{22}' + \ldots + 2A_{12}' + \ldots.$$

Thus the self-inductance of the wires in parallel is in this case

$$\frac{D}{S'},$$

while the impedance is

$$\{k_1 (A_{11}' + \ldots A_{1n}')^2 + k_2 (A_{21}' + \ldots A_{2n}')^2 + \ldots\}/S'^2.$$

When ip is very small

$$\frac{\Delta}{S} = ip\frac{\left(\frac{a_{11}}{k_1^2} + \frac{a_{22}}{k_2^2} + \ldots + \frac{2a_{12}}{k_1 k_2} + \ldots\right)}{\left(\frac{1}{k_1} + \frac{1}{k_2} + \ldots \frac{1}{k_n}\right)^2} + \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \ldots \frac{1}{k_n}}.$$

So that in this case the self-induction of the wires in parallel is

$$\frac{\frac{a_{11}}{k_1^2} + \frac{a_{22}}{k_2^2} + \ldots + \frac{2a_{12}}{k_1 k_2} + \ldots}{\left(\frac{1}{k_1} + \frac{1}{k_2} + \ldots \frac{1}{k_n}\right)^2},$$

and the resistance is

$$\frac{1}{\dfrac{1}{k_1}+\dfrac{1}{k_2}+\dots}.$$

When there is no induction between the wires in parallel, a_{12}, a_{22}, ... all vanish; hence when ip is very large the self-induction is

$$\frac{1}{\dfrac{1}{a_{11}}+\dfrac{1}{a_{22}}+\dots\dfrac{1}{a_{nn}}},$$

and the impedance

$$\frac{\dfrac{k_1}{a_{11}{}^2}+\dfrac{k_2}{a_{22}{}^2}+\dots}{\left(\dfrac{1}{a_{11}}+\dots\right)^2}.$$

These are the general results.

507. We shall now briefly consider the general case of any number of circuits: the investigation will apply whether the circuits are arranged so as to form separate circuits or whether some or all of them are metallically connected so as to form a network of conductors.

Let $J_1, J_2, \dots J_n$ be the variables required to fix the distribution of currents through the circuits; let T, the kinetic energy due to these currents, be expressed by the equation

$$T = \tfrac{1}{2}(a_{11}J_1{}^2 + a_{22}J_2{}^2 + \dots + 2a_{12}J_1J_2 + \dots),$$

while the dissipation function F is given by

$$F = \tfrac{1}{2}(k_{11}J_1{}^2 + k_{22}J_2{}^2 + \dots + 2k_{12}J_1J_2 + \dots).$$

Let us suppose that there are no external forces of types J_2, J_3, ... and that ϕ_1, the external force of type J_1, is proportional to e^{ipt}. The equations giving the currents are

$$\begin{aligned}(a_{11}ip + k_{11})J_1 + (a_{12}ip + k_{12})J_2 + \dots &= \phi_1,\\ (a_{12}ip + k_2)J_1 + \dots\qquad\qquad &= 0,\\ \dots\dots\dots\dots\dots\dots\qquad\qquad &= 0.\end{aligned}$$

From the last $(n-1)$ of these equations we have

$$\frac{J_1}{B_{11}} = \frac{J_2}{B_{12}} = \frac{J_3}{B_{13}} = \dots,$$

where B_{pq} denotes the minor of the determinant

$$\begin{vmatrix} a_{11}ip + k_{11}, & a_{12}ip + k_{12}, & \dots \\ a_{12}ip + k_{12}, & \dots\dots\dots & \\ \dots\dots\dots & \dots\dots\dots & \dots \end{vmatrix},$$

corresponding to the constituent $a_{pq}ip + r_{pq}$; we shall denote the determinant by Δ.

Substituting the values of J_2, J_3, ... in the first equation, we have

$$(a_{11}ip + r_{11})\,J_1 + \frac{1}{B_{11}}\{(a_{12}ip + k_{12})\,B_{12} + \ldots\}\,J_1 = X_1,$$

which may be written in the form

$$\frac{\Delta}{B_{11}}J_1 = \phi_1.$$

If $\dfrac{\Delta}{B_{11}}$ be written in the form $Lip + R$, where L and R are real quantities, then L is the effective self-induction of the circuit and R the impedance.

508. We have also

$$\frac{\Delta}{B_{12}}J_2 = \phi_1.$$

If an electromotive force ϕ_2 of the same period as ϕ_1 acted on the second circuit, then the current J_1 induced in the first circuit would be given by

$$\frac{\Delta}{B_{12}}J_1 = \phi_2.$$

Comparing these results we get Lord Rayleigh's theorem, that when a periodic electromotive force F acts on a circuit A the current induced in another circuit B is the same in amplitude and phase as the current induced in A when an electromotive force equal in amplitude and phase to F acts on the circuit B.

When there are only two circuits in the field

$$\frac{\Delta}{B_{11}} = a_{11}ip + k_{11} - \frac{(a_{12}ip + k_{12})^2}{a_{11}ip + k_{22}},$$

if the circuits are not in metallic connection $k_{12} = 0$ and we have

$$\frac{\Delta}{B_{11}} = \left(a_{11} - \frac{p^2a_{22}a_{12}^2}{a_{22}^2p^2 + k_{22}^2}\right)ip + k_{11} + \frac{p^2k_{22}a_{12}^2}{a_{22}^2p^2 + k_{22}^2}.$$

Thus the presence of the second circuit diminishes the self-induction of the first by

$$\frac{p^2a_{22}a_{12}^2}{a_{22}^2p^2 + k_{22}^2},$$

while it increases the impedance by

$$\frac{p^2k_{22}a_{12}^2}{a_{22}^2p^2 + k_{22}^2}.$$

509. On the propagation of waves along a cable. We now turn to a final illustration of the principles of this chapter, which analyses from a simple point of view a problem to be subsequently discussed in greater detail. The subject hardly belongs to the title of the present chapter but it is convenient to include it in the general discussion at the present stage.

The problem concerns a very long cylindrical metallic conductor surrounded by a coaxial cylindrical metal sheath (somewhat like a cable but in this case the outer conductor is water). If we create an electrical disturbance in the space between the metals at one end of the conductor, say by discharging a condenser into it, the complementary plates being connected to the inner and outer metallic coatings, then an electric disturbance in the form of a pulse will run along the conductor, in the sense that a disturbance of the otherwise statical electrical conditions will appear to run along the conductor much as a wave, representing a disturbance of the statical conditions of the liquid surface, runs along a water trough when started at one end. The general case discussed is one that really involves the application for a short time of a periodic disturbance at one end so that a group of electric waves runs along the conductor with a definite velocity of propagation and wave length.

If we send a disturbance along the inner conductor these will be, as we have seen, a complementary disturbance in the outer conductor and the two are connected across with one another by the electrical field in the dielectric shell between. The apparent current in the inner conductor at any point produced by the instantaneous disturbance of the field is thus accompanied down the cable by the opposite and equal complementary current in the outer conductor. Thus the energy in the wave or disturbance is locally distributed and does not spread much, the outer conductor preventing this spreading. This means that we can calculate the induction and capacity of the cable as so much per unit length, because they arise merely as local effects between neighbouring elements of charge and current. We shall thus assume that our cable has a self-induction a per unit length at the point distant s from the one end where the disturbance is started, a capacity b per unit length and also a resistance k. There may also be a certain dissipation by leakage across between the conducting core and the outer sheath, but we shall for the present neglect any effect of this kind.

510. The current in the inner conductor will be measured by the amount of charge crossing the typical section per unit time, and may be denoted by

$$\frac{dQ}{dt},$$

so that the kinetic energy of the current at this place will be of amount

$$\frac{1}{2}\cdot\frac{a}{c}\left(\frac{dQ}{dt}\right)^2$$

per unit length, or in all

$$T=\int_{s=0}^{s=s_1}\frac{1}{2}\frac{a}{c}\left(\frac{dQ}{dt}\right)^2 ds,$$

integrated along the cable of length s_1.

The charge on an element of length of the cable will be then equal to the difference of the charges which have crossed the sections bounding this length and will therefore be

$$\frac{dQ}{ds}\,ds,$$

and since the capacity of this element is bds the potential energy of this charge element is

$$\frac{1}{2b}\left(\frac{dQ}{ds}\right)^2 ds,$$

or in all the potential energy will be

$$W = \int_{s=0}^{s_1} \frac{1}{2b}\left(\frac{dQ}{ds}\right)^2 ds.$$

There is then the dissipation function which is similarly seen to be

$$F = \int_0^{s_1} \frac{k}{2}\left(\frac{dQ}{dt}\right)^2 ds.$$

The coefficients a and b may change along the cable, that is they may be functions of s, but for the present analysis to be valid their rate of change must be slight in a length comparable with the wave length of the disturbance; the properties of such cables are not to change perceptibly in a length equal to that of the shortest wave transmitted along it. This implies that the cables must be uniform in a length large compared with the dimensions of their cross section (200 to 300 times is practically sufficient). The values of a, b and k will of course depend on the distribution of the current in the cross section. It appears that if the alternations are slow, as in telegraphy, the current practically goes full bore, or is uniformly distributed over the cross section, but if the period is very small the current is confined to a very thin layer at the surface of the conductors.

511. There is only one electric variable and we may use any of the usual methods of obtaining the equation of motion. The general method involves a use of the principle of least action, which, for a system with no dissipation and for which the potential and kinetic energies are W and T respectively, is expressible in the form

$$\delta \int_{t_1}^{t_2} (T - W)\,dt = 0.$$

If however there is a dissipative function for the system it must be introduced as follows. If the generalised coordinates of the system are θ_1, θ_2, ... and the corresponding velocities are $\dot{\theta}_1$, $\dot{\theta}_2$, ... then the equation of action is modified by adding

$$\int_{t_1}^{t_2} \left(\frac{\partial F}{\partial \dot{\theta}_1}\,\delta\theta + \dots\right) dt,$$

so that generally it assumes the form

$$\int_{t_1}^{t_2}\left[\delta(T-W)+\frac{\partial F}{\partial\dot{\theta}_1}\delta\theta_1+\frac{\partial F}{\partial\dot{\theta}_2}\delta\theta_2+\ldots\right]dt.$$

In our case this equation is

$$\delta\int_{t_1}^{t_2}dt\int_s\left[\frac{a}{2c}\left(\frac{dQ}{dt}\right)^2-\frac{1}{2b}\left(\frac{dQ}{ds}\right)^2\right]ds+\int_{t_1}^{t_2}dt\int_s k\frac{dQ}{dt}\delta Q\,ds=0.$$

The variation of the first integral with respect to the single independent variable Q is

$$\int_{t_1}^{t_2}dt\int_s\left[\frac{a}{c}\frac{dQ}{dt}\frac{d}{dt}(\delta Q)-\frac{1}{b}\frac{dQ}{ds}\cdot\frac{d}{ds}(\delta Q)\right]ds,$$

wherein we have the two different variations $\frac{d}{dt}(\delta Q)$ and $\frac{d}{ds}(\delta Q)$, which are not however independent. We must therefore get rid of them by integration by parts: this part of the variation is then equal to

$$\left|\int_s\frac{a}{c}\frac{dQ}{dt}\delta Q\,ds\right|_{t_1}^{t_2}-\int_{t_1}^{t_2}dt\int_s\frac{d}{dt}\left(\frac{a}{c}\frac{dQ}{dt}\right)\delta Q\,ds$$
$$-\left|\int_{t_1}^{t_2}\frac{1}{b}\frac{dQ}{ds}\cdot\delta Q\,dt\right|_{s_1}^{s_2}-\int_{t_1}^{t_2}dt\int_s\frac{d}{ds}\left(\frac{1}{b}\frac{dQ}{ds}\right)\delta Q\,ds,$$

and the total variation is thus equal to

$$\left|\int_s\frac{a}{c}\frac{dQ}{dt}\delta Q\,ds\right|_{t_1}^{t_2}-\left|\int_{t_1}^{t_2}\frac{1}{b}\frac{dQ}{ds}\delta Q\,dt\right|_{s_1}^{s_2}$$
$$-\int_{t_1}^{t_2}dt\int_0^{s_1}\left[\frac{d}{dt}\left(\frac{a}{c}\frac{dQ}{dt}\right)-\frac{d}{ds}\left(\frac{1}{b}\frac{dQ}{ds}\right)+k\frac{dQ}{dt}\right]\delta Q\,ds,$$

and this has now to vanish whatever δQ may be. The arbitrary nature of δQ not only ensures that the terms at the limits vanish independently but that at every point of the conductor and at any time

$$\frac{d}{dt}\left(\frac{a}{c}\frac{dQ}{dt}\right)-\frac{d}{ds}\left(\frac{1}{b}\frac{dQ}{ds}\right)+k\frac{dQ}{dt}=0.$$

This is the differential equation which tells us how the disturbance travels. The terms at the limits give us the conditions at the ends of the cable and for the initial and final displacements and need not further trouble us.

512. If the induction and capacity are constant along the cable the general equation becomes

$$\frac{a}{c}\frac{d^2Q}{dt^2}-\frac{1}{b}\frac{d^2Q}{ds^2}+k\frac{dQ}{dt}=0,$$

or

$$\frac{d^2Q}{dt^2}+\frac{kc}{a}\frac{dQ}{dt}=\frac{c}{ab}\frac{d^2Q}{ds^2}.$$

This method of deriving this equation is that consistent with a strict dynamical theory of the subject. The argument can however be interpreted in purely

electrical language and the same equations are obtained. Our method of deduction from the principle of least action is however independent of any special argument at all.

The solution of this equation is obvious. It is identical in form with the well known equation of propagation of damped waves along a string. The general type of solution is obtained as

$$Q = Q_0 e^{i(nt+ms)},$$

where n is real, $\frac{2\pi}{n}$ being the period of the wave, and m and n are connected by the equation

$$m^2 = \frac{ab}{c}\left(n^2 + \frac{iknc}{a}\right),$$

so that m is partly real and partly imaginary; say

$$m = f + ig,$$

so that
$$Q = Q_0 e^{-gs} e^{i(pt+fs)}.$$

The current is thus damped out as it goes along the wire, by the frictional resistance.

513. If the resistance is small, or at least if the period is such that $\frac{kc}{a}$ is very small then the imaginary part of m is negligible, and the wave will travel along without any appreciable decrease in its amplitude. This condition is satisfied if n is large enough, that is if the oscillations are very fast. We have already seen the significance of this statement: the inertia of the current is then so large that the ordinary resistance, depending on p only, is comparatively ineffective. In such a case the equation assumes the simpler form

$$\frac{d^2Q}{dt^2} = \frac{c}{ab}\frac{d^2Q}{ds^2},$$

of which the general solution is

$$Q = f_1(s + c_1 t) + f_2(s - c_1 t),$$

where
$$c_1^2 = \frac{c}{ab}.$$

The electric disturbance then travels along the cable in a simple permanent wave form with velocity

$$c_1 = \sqrt{\frac{c}{ab}}.$$

514. If the resistance and the dissipation depending on it are not too big it is possible to approximate to the general form of solution of the equation

$$\frac{\partial^2 Q}{\partial s^2} - kb\frac{\partial Q}{\partial t} = \frac{1}{c_1^2}\frac{\partial^2 Q}{\partial t^2},$$

but the most general case requires more careful treatment. The most elegant solution is obtained by a method due to Riemann*. We first transform the equation by writing

$$Q = qe^{-\lambda_1 c_1 t},$$

then, if $2\lambda_1 = kbc_1$ the equation for q reduces to

$$\frac{\partial^2 q}{\partial s^2} - \frac{1}{c_1^2}\frac{\partial^2 q}{\partial t^2} + \lambda_1^2 q = 0.$$

We next notice that†

$$q = I_0(z), \quad z \equiv \lambda_1 \sqrt{c_1^2(t-t_1)^2 - (s-s_1)^2}$$

is a solution of this equation which we can temporarily denote by q'. It follows then that

$$q'\left(\frac{\partial^2 q}{\partial s^2} - \frac{1}{c_1^2}\frac{\partial^2 q}{\partial t^2}\right) = q\left(\frac{\partial^2 q'}{\partial s^2} - \frac{1}{c_1^2}\frac{\partial^2 q'}{\partial t^2}\right),$$

or

$$\frac{\partial}{\partial s}\left(q'\frac{\partial q}{\partial s} - q\frac{\partial q'}{\partial s}\right) = \frac{1}{c_1^2}\frac{\partial}{\partial t}\left(q'\frac{\partial q}{\partial t} - q\frac{\partial q'}{\partial t}\right),$$

so that the integral

$$\int\left[\left(q'\frac{\partial q}{ds} - q\frac{\partial q'}{\partial s}\right)dt + \left(q'\frac{\partial q}{\partial t} - q\frac{\partial q'}{\partial t}\right)\frac{ds}{c_1^2}\right]$$

taken round the boundary of any region in the (s, t) plane, in which q, q' are continuous functions with continuous derivatives, must vanish. Such a region is that bounded by the axis of s and the lines

$$s - c_1 t = s_1 - c_1 t_1, \qquad s + c_1 t = s_1 + c_1 t_1$$

and on the two latter lines $q' = 1$ and $ds = \pm c_1 dt$. We conclude therefore by taking the integral in its three parts that

$$2q_{s_1, t_1} = q_{s_1 - c_1 t_1, 0} + q_{s_1 + c_1 t_1, 0} + \frac{1}{c_1}\int_{s_1 - c_1 t_1}^{s_1 + c_1 t_1}\left[q'\frac{\partial q}{\partial t} - q\frac{\partial q'}{\partial t}\right]ds.$$

This formula determines q as a function of the time t_1 and position s_1 on the cable in terms of the values of the function at the time $t_1 = 0$ between the points $s_1 + c_1 t_1$ and $s_1 - c_1 t_1$. If the time $t_1 = 0$ is the initial instant of starting the signal and the conditions then are specified at all points by

$$Q = q = f(s), \qquad \frac{1}{c_1}\frac{\partial q}{\partial t} = g(s),$$

it follows that at the s_1-point at time t the function q is determined by

$$2q = f(s_1 - c_1 t) + f(s_1 + c_1 t) + \int_{s_1 - c_1 t}^{s_1 + c_1 t}\left[g(s) + \frac{\lambda_1 c_1 t}{z} f(s)\frac{d}{dz}\right] I_0(z)\, ds,$$

where now $$z = \lambda_1 \sqrt{c_1^2 t^2 - (s - s_1)^2}.$$

* Riemann-Weber, *Die partielle Differentialgleichungen der mathematischen Physik*, Bd. II, (4th Ed.), p. 322.

† $I_0(z)$ is the zero Bessel function with imaginary argument.

515. To obtain some insight into the nature of the solution thus obtained let us take the particular case given by Heaviside where

$$f(s) = g(s) = Q_0 \text{ for } -\infty < s < 0$$
$$f(s) = g(s) = 0 \quad \text{for } 0 < s < +\infty$$

so that
$$2Qe^{\lambda_1 c_1 t} = Q_0 + Q_0 \int_{s_1 - c_1 t}^{0} \left[I_0(z) + \frac{\lambda_1 c_1 t}{z} \frac{dI_0}{dz} \right] ds,$$

provided $s - c_1 t < 0$; for all other values $Q = 0$. This means that the wave disturbance extending backwards originally from the origin expands itself in the positive direction of the cable with velocity c_1. The front of the wave is at all times distinctly marked but the amplitude of the disturbance in it is gradually being reduced on account of the resistance.

The case of an impulse of finite length initially can be obtained by a combination of such solutions slightly displaced relative to one another. It appears in the general case that any definite wave figure travels along the cable with a general distinctness as regards its terminations fore and aft, but the resistance causes dissipation and distortion of the signal as it proceeds. In addition to this the wave leaves behind a general trail of disturbance of small but finite amplitude which itself gradually dies away by dissipation.

516. It may however happen that the resistance is so very large or the period of the disturbance so very long that the damping term in the equation is of considerable importance and perhaps more important than the other part. The affair does not then travel in a wave at all because it is damped out very quickly. The wave characteristic is thus eliminated by the resistance; in such a case the electric inertia in the circuit is comparatively inoperative and the elasticity works against the resistance; the circumstances are those of the diffusion of a charge along the cable and no wave motion exists. A sharp well-defined disturbance sent in at one end of the cable would then arrive at the other end in a very weak and distracted form. This is of course not desirable in signalling, where a sharp signal should turn out sharp and distinct at the other end.

CHAPTER XII

ELECTROMAGNETIC OSCILLATIONS AND WAVES

517. The general problem with electromagnetic waves. In the previous chapter we have confined our attention entirely to stationary or quasi-stationary electromagnetic systems, i.e. systems in which the time of variation is small compared with the time taken by radiation to cross the system and we have in consequence found it unnecessary to consider the process by which the varying conditions established in one part of the field are smoothed out over the whole of the field by radiation. We shall now turn to the other side of the matter and make a special investigation of the radiation processes by which a given state of affairs is transferred from one part of the field to another. We have already had cause to investigate in a previous chapter a possible source of a very rapidly oscillating field (viz. that associated with a condenser discharging through an induction) in which we can no longer neglect the time taken to smooth out the field, and as this case has an important theoretical as well as practical bearing, we shall examine it more closely by more general methods.

The general problem is the investigation of the conditions in any electromagnetic field consequent on a rapid alteration of the conditions in any one part of it; perhaps by discharging one conductor in the field by connecting it through an induction to another conductor in the same field. Complete generality will be obtained by the disposition of dielectric and other conducting bodies throughout the field, the whole being then included in one scheme.

518. The generalised scheme adopted by Maxwell for the treatment of these cases has already been set out in detail, the underlying idea being to try and explain the electromagnetic phenomena by means of some action, mechanical or otherwise, transmitted from one body to the other by means of a supposed medium, the aether, occupying the space between them; the mode of action of this medium being completely specified by the two fundamental circuital relations of the theory.

The essential point of this scheme involves the assumption of a quasi-current in the aether of density

$$\frac{1}{4\pi}\frac{d\mathbf{E}}{dt},$$

which is equal to the time rate of change of the aethereal displacement. This current in addition to the real displacement current in the dielectrics of density*

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt}\frac{(\epsilon - 1)\,\mathbf{E}}{4\pi},$$

being sufficient to secure that all currents flow in complete cycles. We can thus adopt the two circuital relations of electrodynamics as descriptive of the general state of affairs. In their differential form they are

$$\frac{4\pi\mathbf{C}}{c} = \operatorname{curl}\mathbf{H},$$

$$-\frac{1}{c}\frac{d\mathbf{B}}{dt} = \operatorname{curl}\mathbf{E}.$$

These equations represent the simplest conception of a general electromagnetic theory of these things. They could not of course be right unless in addition

$$\operatorname{div}\mathbf{C} = 0,$$

and also

$$\operatorname{div}\mathbf{B} = 0,$$

the first being secured by Maxwell's hypothesis and the second indicating that it is **B** and not **H** that must be used to count the flux of induction.

519. These two dynamical equations expressing exact physical principles are independent of the constitution of the substances in which the action takes place. As however they involve four vectors they are not sufficient for a complete scheme and we must again introduce the constitutive relations depending on the nature of the media occupying the field. We have already had these relations: the first one expressing the total current as a function of the electric force is of the form

$$\mathbf{C} = \sigma\mathbf{E} + \frac{1}{4\pi}\frac{d}{dt}(\epsilon\mathbf{E}),$$

the first term, representing the conduction current, expressing an exact relation as far as experiment can follow it, but the second expresses the best we can do in our theory; it represents however a fairly good approximation to the facts. The second relation between the magnetic induction and magnetic force also assumes the form

$$\mathbf{B} = \mu\mathbf{H},$$

and in the simplest cases μ is constant. This relation is however not so exact as the above.

* Throughout this chapter where not otherwise specified we shall assume that linear isotropic relations hold between the electric and magnetic forces and the induced polarisations respectively.

These four relations represent Maxwell's complete scheme: adopting the latter we can write the first two in the form

$$\operatorname{curl} \mathbf{H} = \frac{4\pi\sigma\mathbf{E}}{c} + \frac{\epsilon}{c}\frac{d\mathbf{E}}{dt},$$

$$\operatorname{curl} \mathbf{E} = -\frac{\mu}{c}\frac{d\mathbf{H}}{dt}.$$

We shall now limit the complete generality of our scheme by the assumption that there are no magnetic bodies present so that we can take $\mu = 1$ everywhere.

520. We deduce at once that

$$\frac{4\pi\sigma}{c}\frac{d\mathbf{E}}{dt} + \frac{\epsilon}{c}\frac{d^2\mathbf{E}}{dt^2} = \operatorname{curl}\frac{d\mathbf{H}}{dt}$$

$$= -c \operatorname{curl} \operatorname{curl} \mathbf{E}$$

$$= c\nabla^2\mathbf{E} - c \operatorname{grad} \operatorname{div} \mathbf{E}.$$

But we have

$$4\pi\rho = \operatorname{div} \epsilon\mathbf{E},$$

and yet

$$0 = \operatorname{div}\left(\frac{\epsilon\dot{\mathbf{E}}}{4\pi} + \mathbf{C}_1\right)$$

$$= \frac{d\rho}{dt} + \frac{1}{\sigma}\operatorname{div} \mathbf{E},$$

or

$$\frac{d\rho}{dt} + \frac{1}{\epsilon\sigma}\operatorname{div}(\epsilon\mathbf{E}) = 0,$$

or again

$$\frac{d\rho}{dt} + \frac{4\pi\rho}{\epsilon\sigma} = 0,$$

or

$$\rho = \rho_0 e^{-\frac{4\pi t}{\epsilon\sigma}}.$$

This equation shows that the changes in ρ are independent of the external electromagnetic influence, so that even if there is an initial electrical volume charge distribution it will decrease very rapidly except in the very improbable case when $\epsilon\sigma$ is a very large quantity. We may thus consider that $\rho = 0$ always, for we may consider the origin of time to be chosen when $\rho = 0$. The equations then become

$$\frac{4\pi}{\sigma}\frac{d\mathbf{E}}{dt} + \epsilon\frac{d^2\mathbf{E}}{dt^2} = c^2\nabla^2\mathbf{E},$$

since now

$$\operatorname{div}\frac{\epsilon\mathbf{E}}{4\pi} = 0.$$

521. In general we can neglect the displacement current in the metallic conductors in comparison with the conduction currents. Our equation thus reduces to the form

$$\nabla^2\mathbf{E} = \frac{4\pi\sigma}{c^2}\cdot\frac{d\mathbf{E}}{dt},$$

which exhibits the propagation of the electromagnetic disturbances into the conducting substances as a simple process of diffusion.

In the dielectric parts of the field $\sigma = 0$ and the equation becomes

$$\nabla^2 \mathbf{E} = \frac{\epsilon}{c^2} \frac{d^2\mathbf{E}}{dt^2},$$

which shows that the electric field in the dielectric can be propagated as a simple wave motion in the medium with a velocity

$$\frac{c}{\sqrt{\epsilon}}.$$

A similar discussion easily shows that the magnetic force is propagated in an exactly similar manner, the equations satisfied by it in the separate media being

(i) in the conductors

$$\nabla^2 \mathbf{H} = \frac{4\pi\sigma}{c^2} \frac{d\mathbf{H}}{dt},$$

(ii) in the dielectric

$$\nabla^2 \mathbf{H} = \frac{\epsilon}{c^2} \frac{d^2\mathbf{H}}{dt^2}.$$

These equations represent the characteristic differential equations of the theory. In attacking any problem where there are different regions (conducting or dielectric) we of course have to solve the different equations for each region and the corresponding solutions have then to be fitted together or connected by the appropriate continuity conditions at the boundary. Before proceeding we must therefore obtain the boundary conditions.

522. We first notice that any discontinuities in crossing the surface at any point clearly arise from the distribution over a surface element δf surrounding that point, for the disturbances propagated from the more distant parts are virtually the same at points on the two sides of the surface whose distance apart is infinitesimal compared with the linear dimensions of δf. Moreover if this is the case the discontinuities will be the same as in the corresponding static or stationary condition of that part of the boundary δf, because the field from it produces almost instantaneously its effect at an infinitely near point. We may therefore conclude at once that, in the general case,

(i) the tangential electric force must be continuous unless a double sheet distribution exists on the surface. This case is excluded.

(ii) the normal magnetic induction is also continuous: this follows also as a consequence of the general circuital property of that vector.

(iii) the total normal electric current component is also continuous.

The first and second of these conditions are however not independent for we know that

$$-\frac{1}{c}\frac{d\mathbf{B}}{dt} = \text{curl}\,\mathbf{E},$$

so that if the tangential components of **E** are continuous the normal component of **B** must also be continuous.

Also since

$$\frac{4\pi\mathbf{C}}{c} = \text{curl}\,\mathbf{H},$$

we see from the third relation that unless there are surface current sheets at the surface the tangential components of the magnetic force must also be continuous.

There are thus in all two independent boundary conditions which have to be satisfied.

The previous general equations with these boundary conditions provide us with a complete scheme of equations for all electromagnetic wave problems. Before however proceeding to the consideration of particular problems we will apply these results to the general case discussed above with the additional assumption that all the conductors in the field are perfect.

523. In this case we have $\sigma = 0$ for all the conductors and therefore inside them

$$\nabla^2\mathbf{E} = 0,$$

and also

$$\nabla^2\mathbf{H} = 0,$$

which combined with the general results

$$\text{div}\,\mathbf{E} = 0 \quad \text{and} \quad \text{div}\,\mathbf{H} = 0,$$

show that inside the conductors

$$\mathbf{E} = \mathbf{H} = 0;$$

there is no field inside the conductors. In external space on the other hand we have still

$$\nabla^2\mathbf{E} = \frac{\epsilon}{c^2}\frac{d^2\mathbf{E}}{dt^2},$$

and also

$$\nabla^2\mathbf{H} = \frac{\epsilon}{c^2}\frac{d^2\mathbf{H}}{dt^2},$$

but if the alternations of the field are not too fast the time variations on the right-hand side containing the very small factor $\frac{\epsilon}{c^2}$ is negligible and the equations can then be written in the simpler form

$$\nabla^2\mathbf{E} = 0,$$

and

$$\nabla^2\mathbf{H} = 0.$$

We know moreover from the boundary conditions that the electric force just outside the conductors is normal to their surface but the magnetic force is tangential even in the general case.

524. The physical explanation of these solutions is now obvious. The electromagnetic field exists only in the dielectric between the conductors. The lines of electric force go across from one conductor to another and the magnetic ones are round about. The positive and negative charges on the surfaces of the conductors are the terminations of the tubes of electric force in the intermediate field. The real propagation of the effects thus takes place in the dielectric medium, a given field being propagated through that medium with a velocity depending only on the medium. Wherever the electric force arrives at a conductor it pulls the electric charges on the conductors (the electrons) about until the statical force due to their rearranged distribution counterbalances the electric force in the field on any one of them, i.e. until there is no resultant electric force inside the conductors. The conductors are full of charges (positive and negative) more or less free which slightly adjust themselves, concentrating on the surface so as to get the necessary field in the interior of the conductors which cancels that of the oncoming wave. If the conduction is perfect the redistribution of charge at each instant takes place instantaneously and thus the field in the conductor right up to its surface is annulled; the charge on the conductor creating the induced electric force is entirely on its surface. If the conduction were not so good, the electrons would not be so free and the electric field would at each instant penetrate into the conductors a little way before being annulled by the reaction of the field due to the electrons which it pulls about. If the conduction is good the cancelling takes place instantaneously at the surface.

This is the general idea of the phenomena. Before electrons were discovered one was however not able to put the matter so definitely. As early as 1884 however Poynting, Hertz and Heaviside emphasised the point that where a current is used to transmit the power the energy travels in the dielectric round about, which is the real elastic thing, the conductors only acting as guides to prevent the disturbance spreading.

525. There is a rough mechanical analogy in the propagation of waves in an elastic medium with holes in it. In this case the transverse waves or waves of shear travel along through the elastic material adjusting itself by material deformation of the surfaces of the holes so that there is no elasticity inside the holes. In the electrical case the dielectric is the elastic medium through which the field is propagated by wave motion; this field (or the elasticity in it) is annulled at the surfaces of the conductors (the holes) by the pulling about of the mobile electrons on their surface. The conductors thus appear as places where there is no elasticity, where the electrical elasticity of the aether is annulled by the mobile electrons.

The property of perfect conductors thus appears to be merely a negative one, viz. that of cancelling the elasticity of the aether. In imperfect conductors there is a damping action and the elasticity is only partially annulled.

We have thus the general idea that the whole affair of electric current phenomena is actually in the field outside the conductors, the current merely providing a convenient mode of describing the changes taking place. All the energy is to be found in the dielectric medium surrounding the conductors; on this view, the heat developed in the conductors is energy which has soaked in as it were from the store in surrounding field. The old method of describing the phenomena was to say that the energy was that of the moving charges; we know now that the energy of any charge, however small, is in the electromagnetic field which we have learnt to associate with it: thus ultimately all electric energy is in the aether. The electric charges (or electrons) which are the nuclei with which the fields are associated merely provide the means of communication of the energy from the aether to the matter.

526. The fundamental equation of wave propagation. We have so far tacitly assumed that any quantity which is determined mathematically by certain scalar or vector component quantities which satisfy an equation of the type

$$\nabla^2\phi = \frac{1}{c^2}\frac{d^2\phi}{dt^2} + \frac{2\lambda}{c}\frac{d\phi}{dt}$$

is essentially propagated by a wave motion throughout the field. That this is so follows from our knowledge of such phenomena as, for example, accompany the propagation of sound through any elastic medium; but it may be inferred directly as a mathematical consequence of the implied condition involved in the characteristic equation.

The general problem in the present aspect of the theory is to determine how any electromagnetic disturbance is propagated across space filled with dielectric and conducting masses in any specified configuration, and to see how the conditions at any one point of this field are affected by those occurring at any other. The complexity of the conditions involved naturally excludes the determination of a simple solution for the general problem and we must therefore be content with the examination of simpler problems with restricted circumstances.

We first examine the general case of the propagation of effects from a specified type of disturbance located in a finite region of an infinite homogeneous isotropic dielectric with zero conductivity. The disturbance will for simplicity be assumed to be of a continuous character and to have been in operation for an indefinite period previous to the instant at which the field is examined*.

* More general cases are examined by Love, *Proc. L. M. S.* (2), vol. I. (1903), p. 37.

Suppose we enclose the origin of the disturbance by any closed surface f: then at all points in the region outside this surface the field vectors will satisfy an equation of the type

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}.$$

The function ϕ satisfying this equation must necessarily be regular at all points outside f even at an infinite distance and it therefore follows from the general problem analysed in the introduction (§§ 25–29) that the appropriate form for the function at time t at the typical field point outside f is

$$\phi = \frac{1}{4\pi}\int_f \frac{df}{r}\left[\frac{\partial\phi}{\partial n}\right] + \int_f df \frac{d}{dn}\left(\frac{[\phi]}{r}\right)$$

where each integral is taken over the surface f bounding the origin of the disturbance; r is the distance of the element of this surface from the point of the external field where the conditions are examined and square brackets as usual indicate that the functions affected are to be taken for the time

$$\left(t - \frac{r}{c}\right).$$

527. Now let us see what this formula means. The conditions at any point in the dielectric medium outside the surface f depend only on the conditions of the field on the surface itself, so that any alteration of condition in the disturbing system inside f affects the external field only through the medium of the field on the arbitrary separating surface. Moreover the conditions existing on any element df of this surface at a given instant are not effective at any external point distant r from it until after the time r/c. This suggests the view that the conditions originated at any point in the field travel out from that point into the surrounding field, traversing each part of the intervening field in turn and proceeding from point to point with the velocity c.

This is the essence of a radiation theory and is exactly analogous to the phenomenon with which we are familiar in the theory of sound, and although it will appear that the type of radiation is essentially different from that met with in all such material phenomena, it is convenient to talk of electromagnetic waves and radiation in the same sense as we talk of waves of sound.

528. The analytical formula under review has an important physical significance which it is worth while examining in detail. The potential propagated from a point source variable with the time and of strength $f(t)$ is with the same characteristic equation

$$\frac{1}{r}f\left(t - \frac{r}{c}\right).$$

The potential propagated from a doublet consisting of simple sources $f(t)$ and $-f(t)$ separated by an interval δn is consequently

$$\delta n \frac{d}{dn}\left\{\frac{1}{r} f\left(t - \frac{r}{c}\right)\right\}.$$

Thus for a doublet of strength $F(t)$, the equivalent of $f(t)\,\delta n$, it is

$$\frac{d}{dn}\left\{\frac{F\left(t - \frac{r}{c}\right)}{r}\right\}$$

in which the function F comes under the differentiation.

The formula quoted above for ϕ thus implies that each differential element of the surface f acts, as regards the point P inside it as a complex radiating element consisting of a simple source of strength

$$\frac{\partial \phi}{\partial n}\, df,$$

and a normal doublet of strength

$$\phi\, df.$$

The wave disturbance originated by these elements travels out into the space inside f as a simple spherical wave propagation. The disturbance at P is thus just the same as if the surface itself acted as a sort of secondary radiator and this is the essence of Huyghens' well-known principle in physical optics. The above mode of deduction of the formula, not free from analytical difficulties, can hardly however be said to throw much light on the character of the simple principle which is thus demonstrated. In this connection however the following discussion due to Prof. Larmor* is of special interest as indicating the exact amount of precision in the specification of the secondary disturbance thereby introduced. Reference may also be made back to the discussions of Green's theorem and the equivalent stratum.

529. Consider a potential specified throughout all space as follows. It is a function ϕ, single valued and continuous as to itself and its first gradient, and satisfying $\nabla^2\phi = 0$, in all the space outside a boundary f, and as a consequence diminishing towards infinity according to the law r^{-1} or higher inverse power: it is zero everywhere inside the boundary. What is the distribution of attracting mass to which this belongs? This distribution is as usual in Green's manner determined by the singularities and discontinuities of the potential function. It consists of a surface density σ over f and a double sheet τ over f also; where

$$\sigma = -\frac{1}{4\pi}\frac{\partial \phi}{\partial n}, \qquad \tau = \frac{1}{4\pi}\phi,$$

δn is an element of the outward normal. For it follows by the usual procedure that if ϕ' is the potential of this distribution then $(\phi - \phi')$ is a potential

* *Proc. L.M.S.* (2), vol. I. (1903), p. 1.

function which has no singularities, or discontinuities throughout all space and is therefore identically null. Exprèssed analytically the potential at a point in space is

$$-\frac{1}{4\pi}\int_f \frac{1}{r}\frac{\partial\phi}{\partial n}\,df + \frac{1}{4\pi}\int \phi\,\frac{d}{dn}\left(\frac{1}{r}\right) df,$$

in other words the formula gives the value of a potential function ϕ in the free space outside the surface in terms of the values which it and its gradient assume on the surface, it constitutes in fact the analytical continuation of the function outward from the surface, while inside the surface the value of the expression is everywhere null. In the case of a closed surface as well as that of an open sheet either side may be called the outside for the present purpose. This continuation of the function is necessarily unique and determinate, but the form of the integral expressing it is far from being so. We may in fact generalise the formula immediately in Green's manner. Consider a function ϕ which is the potential throughout space of any assigned distribution of mass. Draw any surface dividing space into two regions A and B each of which contains part of the mass, these parts being represented by M_A and M_B. What distribution of masses and of surface densities and normal doublets on the surface f is required to produce a potential equal to ϕ in the region A and equal to zero in the region B? Clearly M_A together with

$$\sigma = -\frac{1}{4\pi}\frac{\partial\phi}{\partial n}, \qquad \tau = \frac{1}{4\pi}\phi.$$

What distribution is required to make the potential zero in the region A and ϕ in the region B? Clearly M_B with the same distribution on the surface but with the sign changed if δn is measured in the same way. Thus a distribution of surface density and normal doublets is found which exactly cancels the effect of M_A on the other side of the dividing sheet f; moreover an infinite number of such distributions can be found, for in determining it M_B is entirely arbitrary.

530. The same procedure can now be extended to a scalar potential propagated in time, i.e. which satisfies a characteristic equation involving the time as a variable.

Consider first the simplest case of a velocity potential ϕ satisfying

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}.$$

It is necessary to ascertain what distribution of sources on a surface f will create given discontinuity in the values of ϕ, and of its normal gradient, in crossing the surface, it being clear that such discontinuities in ϕ and $\partial\phi/\partial n$ constitute the most general type, involving only first differential coefficients of ϕ, that can exist.

We have already seen that the velocity potential propagated from a doublet consisting of simple sources $f(t)$ and $-f(t)$ separated by an interval δn is

$$\delta n \frac{d}{dn}\left\{\frac{1}{r}f\left(t-\frac{r}{c}\right)\right\};$$

and that for a doublet of strength $F(t)$, the equivalent of $f(t)\,\delta n$, it is

$$\frac{d}{dn}\left[\frac{F\left(t-\frac{r}{c}\right)}{r}\right],$$

in which the function F comes under the differentiation. Within a region of such small extent that the functions $f(t)$ and $F(t)$ do not sensibly change in the time required for the disturbance to pass across it, these potentials are of types $\frac{f(t)}{r}$ and $F(t)\frac{d}{dn}\left(\frac{1}{r}\right)$, so far as they relate to sources inside the region of which $f(t)$ and $F(t)$ are the strengths at this interval of time; for this modification only neglects lower inverse powers of r than those retained. Thus for such a region enclosing an element δf of the surface f, and at times for which the functions $f(t)$ and $F(t)$ do not change there abruptly, the potentials, subject to exceptions to be presently encountered in the case of double sheets, take, throughout any time of the order above specified, the form of simple gravitational potentials, the circumstances of propagation not sensibly inferring.

We are therefore invited to follow the procedure of Coulomb and Laplace for the ordinary potential and investigate the discontinuities arising from a surface distribution of simple sources and one of doublets orientated normally to the surface. The discontinuities, on crossing the surface at any point, clearly arise from the distribution over a surface element δf surrounding that point; for the disturbances propagated from the more distant sources are virtually the same at points on the two sides of the surface, whose distance apart is infinitesimal compared with the linear dimensions of δf, so that, as regards their effect, no discontinuities can arise.

531. Taking first then, the case of a simple surface density $\sigma(t)$ spread over δf, which we may take to be uniform all over it at each instant, its effect is to transmit towards both sides a train of plane waves with fronts parallel to δf, which remain plane until the distance n to which they have travelled becomes comparable with the linear dimensions of δf. For them the value of $d\phi/dn$ at a distance n at time t is $2\pi\sigma\left(t-\frac{n}{c}\right)$, but with different sign on the two sides; such a surface distribution $\sigma(t)$ of simple sources thus accounts for a discontinuity in $d\phi/dn$ of amount $4\pi\sigma(t)$, but introduces no discontinuity in ϕ itself.

This result now assists us to analyse the circumstances of a sheet of normal doublets of strength $\tau(t)$ per unit area, for we can replace it by two simple parallel sheets of densities $\sigma_1(t)$ and $-\sigma_1(t)$ at an infinitesimal distance δn apart, such that $\sigma_1(t)\,\delta n = \tau_1(t)$. At a point at a distance n from the sheet of density $+\sigma_1(t)$ the values of $\frac{d\phi}{dn}$ at time t arising from these two sheets are, as above, $\pm 2\pi\sigma_1\left(t_1 - \frac{n}{c}\right)$ and $\mp 2\pi\sigma_1\left(t - \frac{n+\delta n}{c}\right)$, in which signs are to be determined by the sides of the respective component sheets on which this point lies. If the point is not between the sheets, the signs are opposite and the sum for both is $-2\pi\frac{\delta n}{c}\frac{d}{dt}\sigma_1\left(t - \frac{n}{c}\right)$, which is the value of $\frac{d\phi}{dn}$ due to the element $\tau\delta f$; but it has the same sign on both sides of the double sheet: so that in crossing the double sheet there is no discontinuity in the value of $\frac{\partial\phi}{\partial n}$, though the element $\tau\delta f$ of the double sheet contributes $\frac{4\pi}{c}\frac{d\tau}{dt}\delta f$ to that quantity on each side. But between the sheets the value of $\frac{d\phi}{dn}$ arising from them is of a higher order of magnitude, being a sum instead of a difference, and is $4\pi\sigma_1\left(t - \frac{n}{c}\right)$, or simply $4\pi\sigma_1(t)$, when σ_1 is not discontinuous in the time; and this value integrated across the interval δn gives a discontinuity in ϕ itself, on crossing the double sheet, of amount $4\pi\delta n\sigma_1(t)$; that is $4\pi\tau(t)$.

Collecting these results we see that a discontinuity in ϕ over a surface f of amount $\chi(x, y, z, t)$ and a discontinuity in $\frac{\partial\phi}{\partial n}$ equal to $\psi(x, y, z, t)$ over the same surface are accounted for respectively by a double sheet on the surface of strength τ equal to $\frac{1}{4\pi}\chi$ and a single sheet of density σ equal to $\frac{1}{4\pi}\psi$.

532. We are thus in a position to proceed exactly as in the first instance. Consider any system of sources, and let ϕ, a function of (x, y, z, t) be their potential function in infinite free space. Assign any surface f_1 dividing space into two regions A and B and let m_A and m_B stand for the sources as divided between the two regions. What distribution of sources would give rise to a potential equal to ϕ in region A and equal to zero in region B? Clearly the sources m_A, together with a distribution $(\sigma_1\tau)$ over f given by

$$4\pi\sigma = -\frac{\partial\phi}{\partial n}, \qquad 4\pi\tau = \phi;$$

for if ϕ' is the potential arising from this distribution and Φ is a function equal to ϕ in region A and to zero in region B, then $\Phi - \phi'$ will be a potential having no singularities or discontinuities throughout infinite space, and must therefore be null by simple physical intuition, or analytically by the usual

type of theorem of determinacy based on the energy of the relative disturbance being of necessity essentially positive. As the total effect within the region B is zero, we can say thus that ϕ_A the part of it arising from the sources m_A outside the region is given at time t by

$$-4\pi\phi_A = -\int_f \frac{df}{r}\left[\frac{\partial\phi}{\partial n}\right] + \int_f df \frac{d}{dn}\left[\frac{[\phi]}{r}\right],$$

where $[\phi]$ and $\left[\frac{\partial\phi}{\partial n}\right]$ are the values of these quantities for the element δf at time $t - \frac{r}{c}$ and in forming $\frac{d}{dn}\frac{[\phi]}{r}$, the variation of $[\phi]$ with regard to the coordinates is calculated only in as far as it involves them implicitly as a function of $\left(t - \frac{r}{c}\right)$*. This formula expresses the vibration potential due to sources m_A within the surface f, throughout the region B outside, as determined by the values which it and its gradient assume on that surface. It is so to speak an analytical continuation beyond the surface of a function satisfying the aforesaid characteristic differential equation. Such a continuation must be unique, and it is determined by the value assumed by ϕ alone on the surface: as therefore $\frac{d\phi}{dn}$ is determined by a knowledge of ϕ over the surface, the data for the formula here given are redundant; if arbitrarily assigned they will usually be self-contradictory and the formula thus nugatory. Moreover the formula determines ϕ_A in terms of the surface distribution of ϕ, equal to $\phi_A + \phi_B$, where ϕ_B is due to an entirely arbitrary distribution of sources within the region B to which the formula relates. Thus the quantities integrated in it are very widely arbitrary and the element of the integral corresponding to δf in no sense represents any influence actually propagated from that part of the surface. The formula is purely analytical and in no degree a mathematical formulation of the principle of Hughens, relating to propagation of actual disturbance. In fact if m_B vanishes the formula represents a distribution of surface disturbances, which does not radiate at all into the region A.

533. In the more general case when the uniform medium possesses conducting qualities the above analysis is no longer applicable. The general character of the solution in this case can however be demonstrated by another method which is also suited to the simpler problem.

In this case the characteristic potential equation assumes the form

$$c^2\nabla^2\phi = \frac{\partial^2\phi}{\partial t^2} + 2\lambda c\frac{\partial\phi}{\partial t}.$$

* This point was overlooked by Prof. Larmor in the original paper.

If we transform this equation to a spherical polar coordinate system, then multiply it by $r^2 d\omega$, where $d\omega$ is the element of solid angle at the polar origin, and finally integrate over the unit sphere it reduces immediately to the form

$$c^2 \frac{\partial^2 \Phi}{\partial r^2} = \frac{\partial^2 \Phi}{\partial t^2} + 2\lambda c \frac{\partial \Phi}{\partial t},$$

where
$$\Phi = \frac{r}{4\pi} \int \phi \, d\omega.$$

Let us now examine the propagation of conditions from an initial disturbance specified by

$$\phi = f(x, y, z), \qquad \frac{1}{c}\frac{\partial \phi}{\partial t} = g(x, y, z)$$

at the time $t = 0$; the propagation is assumed to take place in a uniform isotropic medium possessing dielectric and conducting properties corresponding to the equation chosen.

We first transform the functions f and g to the same spherical polar coordinates and then use

$$F(r) = \frac{r}{4\pi} \int f \, d\omega, \qquad G(r) = \frac{r}{4\pi} \int g \, d\omega$$

so that $F(r)$ and $G(r)$ are respectively the initial values of Φ and $\frac{\partial \Phi}{\partial t}$.

We are of course concerned only with the positive values of r so we can choose the functions F and G for negative values as we please. We choose them so that

$$F(-r) = -F(r), \qquad G(r) = -G(-r).$$

We then have, by applying the formula obtained at the end of Chapter XI, the general solution for Φ in the form

$$2\Phi e^{\lambda ct} = F(r + ct) + F(r - ct) + \int_{r-ct}^{r+ct} \left[G(s) + \lambda F(s) + \frac{\lambda ct}{z} F(s) \frac{d}{dz} \right] I_0(z) \, ds$$

where again I_0 is the Bessel function of zero order and imaginary argument and

$$z = \lambda \sqrt{c^2 t^2 - (s - r)^2}.$$

We conclude that the average conditions propagated from the disturbance at any point in the field travel outwards radially from that point in exactly the same way as a signal travels along a telegraph cable. In other words if there is no friction the propagation is like that of a simple undamped wave form with the velocity c, but if there is appreciable conductivity rapid distortion and dissipation occur to destroy these simple propagation effects.

The value of the more general function ϕ can be easily obtained from the

value found above for Φ by determining the limiting value of the ratio Φ/r as r tends to zero. The result is that*

$$\phi = e^{-\lambda ct}\left[F'(ct) + G(ct) + \frac{\lambda ct}{2} F(ct)\right.$$
$$\left. + \lambda \int_0^{ct} \left\{G(s) + \lambda ct\, F(s) \frac{1}{z}\frac{d}{dz}\right\} \frac{1}{z}\frac{dI_0}{dz}\, s ds\right\}$$

where now

$$z = \lambda \sqrt{c^2t^2 - s^2}.$$

This formula determines completely the way in which the conditions at any one point in the field depend on those at the other points, and is in complete accord with our physical conception of these things.

534. The electromagnetic theory of light†. We have thus far been attempting to generalise an explanation of electromagnetic phenomena by ascribing it to some mechanical action transmitted from one body to another by means of a medium, the aether, filling all space; the dynamical or analytical theory of the activity of this medium being expressed in the representation of the mode in which electrodynamic action is propagated across free space outlined in the previous paragraph.

Starting from the general scheme, we showed that it is an essential consequence of the fundamental concept of displacement currents introduced by Maxwell that electrodynamic disturbances are propagated through the medium as a wave motion through an ordinary elastic solid with a velocity equal to $\frac{c}{\sqrt{\epsilon}}$: we shall soon prove that these electromagnetic waves are necessarily transverse waves of the type with which we are familiar in optics.

When Maxwell formulated his electrical theory any such waves of purely electrical origin were entirely unknown, so that the introduction of the concept of displacement currents was then a pure hypothesis, unsupported by any experimental evidence.

It was soon found however that the theoretical behaviour of these electromagnetic waves was governed by exactly the same laws as had been found through many years of combined theoretical and practical investigation to apply in the corresponding phenomena in physical optics, which was the one great branch of physical science, for the description of the phenomena in which it was found essentially necessary to presume the existence of transverse waves. The analogy thus suggested between the two sets of phenomena is moreover more than a mere qualitative one; as Maxwell soon found, it is quantitative as well and to an extent that led him to formulate the

* Cf. Riemann-Weber, *Die partielle Differentialgleichungen, etc.* II. pp. 299–312 (4th Ed. 1901).

† Cf. Maxwell, *Treatise*, I. Ch. XX.

opinion that the waves in light are in fact identical in type with the electromagnetic waves which his theory predicts. This is the now famous electromagnetic theory of light, about the correctness of which there are now hardly any doubts.

The velocity of electromagnetic waves in a vacuum is equal to a constant c introduced originally as a physical constant in the fundamental equations; and the value of this constant which can be determined by purely electrical measurements turns out to be

$$c = 3.10^{10} \text{ cms./sec.},$$

which is identical within the very close limits of experimental error with the velocity of light in vacuo. This identity was known previous to Maxwell's time.

535. In any other medium than a pure vacuum the velocity of radiation is

$$c_1 = \frac{c}{\sqrt{\epsilon}},$$

where ϵ is the specific inductive capacity of the medium. Now in optics the ratio

$$\frac{c}{c_1} = \mu = \sqrt{\epsilon},$$

is defined as the index of refraction of the medium under consideration relative to a vacuum; and thus if Maxwell's surmise is correct we must expect the square of the index of refraction of a medium to be equal to its dielectric constant, properly defined. It was usually inferred that the proper value of the dielectric constant to be used with this relation was the simple statical one, and it was then found that except in the case of gases and a few other substances which show but little or no dispersion, the relation was by no means verified in actual experience. For instance water has an index of refraction of about 1·33 and a dielectric constant of 81. But it must be remembered that the dielectric constant introduced as a physical constant in the relation between the complex electric displacement **D** and the electric force **E** producing it, viz.

$$\mathbf{D} = \frac{\epsilon}{4\pi} \mathbf{E},$$

can only be a definite constant in the statical theory when it depends only on the simple internal statical forces tending to annul the polarisation induced in each molecule or molecular group by the external field. In the more general case when it is a question of rapidly varying fields such as those in radiation, it is necessary to consider what effect the inertia of the molecules will have on the setting up of the state of polarisation in them, and the relation between the displacement produced and the force producing it will be of a more complex type. *A priori*, we might expect that the value of ϵ will

be a function of the rate at which the field is varying and the value for steady fields may therefore be quite different from that for rapidly changing fields. This point may be illustrated in further detail by an appeal to the ideas briefly discussed above in Chapter V where the conception of dielectric polarisation was defined in terms of the electron constants of the molecules of the substance. As there explained the polarisation consists in the small relative displacements of the negative electrons in the atoms or molecules and in the statical theory it is only the quasi-elastic forces holding the electrons which are effective against the action of the applied field; but in the more general case the inertia of the electrons themselves will become effective and the relation is then necessarily of a more complex type. If we denote by m the mass of the typical electron with charge e and use all the other notation as in the previous discussion the more general equations of motion of this electron will now be of type

$$m\ddot{x} = e\,(\mathbf{E}_x + a\mathbf{P}_x) - kx,$$

and it is only in the case that $\mathbf{E}$, and therefore also $\mathbf{P}$, are independent of the time that the solution is obtained under steady conditions by the equation

$$0 = e\,(\mathbf{E}_x + a\mathbf{P}_x) - kx.$$

If we assume that the applied electric field is varying in a simple harmonic manner with a period $\frac{2\pi}{p}$, as would, for instance, be the case were it the electric part of an applied radiation field, all the functions may be taken to be dependent on the time by the imaginary exponential factor e^{ipt} and then we shall have

$$(-\,mp^2 + k)\,x = e\,(\mathbf{E}_x + a\mathbf{P}_x),$$

so that

$$x = \frac{e}{k - mp^2}(\mathbf{E}_x + a\mathbf{P}_x),$$

the displacement of each electron is therefore proportionately larger in the present case in the ratio $k : k - mp^2$. The dielectric constant is then just as before

$$\epsilon = 1 + \frac{4\pi\Sigma\,\dfrac{e^2}{k - mp^2}}{1 - \Sigma\,\dfrac{ae^2}{k - mp^2}},$$

and is therefore in general a function of p, as is in fact required by experience. If we write

$$p_0 = \sqrt{\frac{k}{m}},$$

then it is easily seen that $\frac{2\pi}{p_0}$ is the period of free vibration of the typical electron about its position of equilibrium in the molecule.

If the period of the incident light is very long compared with the different periods of free vibration of the internal electrons this formula reduces in fact to the statical form deduced above, viz.

$$\epsilon_0 = 1 + \frac{4\pi\Sigma \dfrac{e^2}{k}}{1 - \Sigma \dfrac{ae^2}{k}}.$$

536. Now the formula just determined for ϵ is precisely of the type which has been found necessary to account for all the phenomena of dispersion in solid, liquid and gaseous bodies, and although the theory cannot be said to be more than a descriptive one, it indicates the general lines along which the explanation of the phenomena associated with dispersion must be sought.

In those bodies which show an exceedingly small dispersion we might on this general theory expect some agreement in Maxwell's relation interpreted with the statical dielectric constant. The gases are the best examples of such bodies and here the agreement is remarkable; for instance in air under ordinary conditions we have

$$\sqrt{\epsilon} = 1{\cdot}000295,$$

whilst the index of refraction is 1·000294.

It has also been found that the results of experiments with long Hertzian waves can be similarly interpreted. For instance it is found that for these the index of refraction of water is 9*, whose square is exactly equal to the dielectric constant 81, statically determined.

The theory in its most general form is thus perfectly consistent with our experience, and there can therefore be no doubt whatever of the fact that light is an electromagnetic phenomenon of the specified type. A very extensive branch of physics is thus brought into line as a chapter in electric theory; its complete development would however take us beyond the scope of the present work, but reference may be made to any of the standard works on optics.

The one great advantage possessed by the electromagnetic theory of light over the older elastic solid theories is that it definitely excludes the possibility of the existence of longitudinal waves, the absence of which was rather difficult to account for as a pure matter of elastic solid theory. It also provides simple physical foundations for the necessary surface conditions governing the phenomena of reflection and refraction which will be discussed in detail at a later stage.

537. The Hertzian oscillator†. We have now seen that it is an essential consequence of Maxwell's theory that electromagnetic disturbances are pro-

* A. D. Cole, *Wied. Ann.* LVII (1896), p. 290. Cf. also Fleming, *Principles of Electric Wave Telegraphy*, p. 320.

† *Ann. d. Phys.* XXXVI. (1889), p. 1; *Electric Waves*, p. 137.

pagated as a wave motion through dielectric media. We have also seen in Chapter XI that the discharge of a condenser of capacity b through an induction a is oscillatory with a period $2\pi\sqrt{\frac{ab}{c}}$. It follows therefore that electric waves of a simple harmonic type must be passing through the dielectric medium surrounding a condenser in the act of discharging. In 1887 Hertz succeeded by a wonderful series of experiments in demonstrating the existence of these waves, and thus established the general validity of Maxwell's assumption. There were two difficulties which Hertz had to overcome in his work. The first one was to construct a suitable form of condenser to give reasonably short waves, the second was to discover a suitable means of detecting electrical and magnetic forces reversed some million times per second and only lasting for an exceedingly short time. The means by which Hertz overcame the first difficulty has been fully explained above in another connection. He used a form of condenser similar to that exhibited in the figure. A, B are two metallic plates which in Hertz's original experiments were of

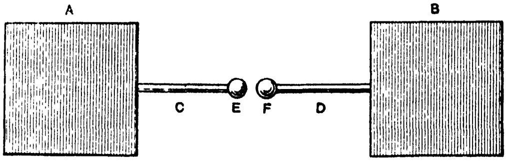


Fig. 78

zinc and about 40 cms. square: to these were soldered brass rods C, D (30 cms. long) terminating in brass balls E, F. Such a condenser has very little capacity and induction. In order to excite the waves the balls E, F are charged to very different potentials by connecting C, D to the terminals of an induction coil. When a sufficient potential is attained sparks cross the air gap between E, F which then becomes a conductor and the charges on the plates can then oscillate backwards and forwards like the charges on the coatings of a Leyden jar.

538. To obtain some idea of the radiation field in the dielectric surrounding a condenser of this type we may notice that the main and most vigorous part of the electrical motions occurs in the wires C, D and across the discharge gap EF, so that to all intents and purposes the field should be symmetrical round an axis along the wires and also about the origin mid-way between the balls E, F with the magnetic force in horizontal circles round this axis. To obtain a solution of the fundamental equations of this type we shall find it most convenient to refer the field to a system of spherical polar coordinates with the pole at the origin and axis along the axis of symmetry of the field: we shall then have

$$\mathbf{H}_r = \mathbf{H}_\theta = \mathbf{E}_\phi = 0,$$

and thus, assuming for the present that the dielectric medium surrounding the apparatus is a pure vacuum the remaining equations of Ampère give

$$\frac{1}{c}\frac{d\mathbf{E}_r}{dt} = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(r\mathbf{H}_\phi \sin\theta),$$

$$\frac{1}{c}\frac{d\mathbf{E}_\theta}{dt} = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(r\mathbf{H}_\phi \sin\theta),$$

whilst the third equation of the Faraday type becomes

$$-\frac{1}{c}\frac{d\mathbf{H}_\phi}{dt} = \frac{1}{r}\left(\frac{\partial}{\partial r}(r\mathbf{E}_\theta) - \frac{\partial \mathbf{E}_r}{\partial\theta}\right).$$

Thus if we write $\psi \equiv r\mathbf{H}_\phi \sin\theta,$

we have $\frac{1}{c}\frac{d\mathbf{E}_r}{dt} = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad \frac{1}{c}\frac{d\mathbf{E}_\theta}{dt} = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\theta},$

and on using these values in the last equation we find on putting $\mu \equiv \cos\theta$

$$\frac{1}{c^2}\frac{d^2\psi}{dt^2} = \frac{\partial^2\psi}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\psi}{\partial\mu^2}.$$

A simple solution of this equation is obtained by putting

$$\psi = R(1-\mu^2),$$

and regarding R as a function of r only: this function satisfies the equation

$$\frac{1}{c^2}\frac{d^2R}{dt^2} = \frac{d^2R}{dr^2} - \frac{2R}{r^2},$$

of which the general solution is easily verified to be

$$R = r\frac{d}{dr}\left(\frac{F_1(ct-r)}{r}\right) + r\frac{d}{dr}\left(\frac{F_2(ct+r)}{r}\right).$$

Of the two parts of this solution the first will represent an expanding wave whilst the second represents a condensing wave: it is with the first alone that we shall be concerned and we shall also find it more convenient to write

$$F(x) = \frac{\partial f(x)}{\partial x},$$

so that $$\psi = \sin^2\theta\left\{f''(ct-r) + \frac{1}{r}f'(ct-r)\right\}.$$

It is then easily verified that

$$\mathbf{E}_r = \frac{2\cos\theta}{r^2}\left\{f'(ct-r) + \frac{1}{r}f(ct-r)\right\},$$

$$\mathbf{E}_\theta = \frac{\sin\theta}{r}\left\{f''(ct-r) + \frac{1}{r}f'(ct-r) + \frac{1}{r^2}f(ct-r)\right\},$$

$$\mathbf{H}_\phi = \frac{\sin\theta}{r}\left\{f''(ct-r) + \frac{1}{r}f'(ct-r)\right\},$$

dashes being used to denote differentiation of the function f with respect to its argument.

539. Near the origin, that is near the oscillator itself, the electric field reduces to

$$\mathbf{E}_r = \frac{2f(ct)\cos\theta}{r^3}, \qquad \mathbf{E}_\theta = \frac{f(ct)\sin\theta}{r^3},$$

the other terms being very small compared with these: the magnetic field reduces to

$$\mathbf{H}_\phi = \frac{f'(ct)\sin\theta}{r^2}.$$

Here then we have the type of solution obtained. In the immediate neighbourhood of the system the electromagnetic field is identical as regards its electric part with the electrostatic field of a simple doublet of strength at any time given by $f(ct)$, while as regards its magnetic part it is identical with the field of the current produced in the changing of this doublet. It is only so far as the actual electrical motions in the vibrator described above approximate to this simple specification that the solution obtained will represent the field actually investigated by Hertz; in any case however it indicates the type of solution to be expected.

540. At a great distance from the origin it is the other terms of the solution which become appreciable and if we assume that the function f and its differential coefficients are of the ordinary type of regular function then we may put

$$\mathbf{E}_r = 0, \qquad \mathbf{E}_\theta = \frac{f''(ct-r)\sin\theta}{r},$$

whilst

$$\mathbf{H}_\phi = \frac{f''(ct-r)\sin\theta}{r},$$

and this is of course probably more representative of the conditions in the actual case than the field near the oscillator is likely to be.

541. Although it is hardly representative of the conditions realised in actual practice we may for the present assume that the dissipation of the energy in the system is so slight that the oscillations are maintained for a considerable time and as the motion is oscillatory, we may represent it in such a case by taking $\quad f(x) = A\sin px,$

so that the surrounding field is determined by the force vectors in it which at the point (r, θ, ϕ) are given by

$$\mathbf{E}_r = \frac{2A\cos\theta}{r^3}\{pr\cos p(ct-r) + \sin p(ct-r)\},$$

$$\mathbf{E}_\theta = \frac{A\sin\theta}{r^3}\{pr\cos p(ct-r) + (1-p^2r^2)\sin p(ct-r)\},$$

$$\mathbf{H}_\phi = \frac{Ap\sin\theta}{r^2}\{-pr\sin p(ct-r) + \cos p(ct-r)\},$$

all the others being zero.

Thus the intensity of the field at any point is oscillating in full accord with the vibrator with the period $\frac{2\pi}{p}$. The actual conditions are best exhibited by plotting the lines of force in it. The equations to these lines are easily obtained, being in fact in any meridian plane the lines along which ψ is constant. Several of these curves have been drawn by Hertz and are depicted below; they correspond to the instants $t = 0, \frac{\tau}{4}, \frac{\tau}{2}, \frac{3\tau}{4}$ when τ is half a complete period. From these figures we see that the lines of force running between the opposed parts of the vibrator gradually expand in all directions. This expansion goes on continuously but the ends on the vibrator itself slowly close up and finally coalesce; the line then breaks itself away and travels out into space as a closed line of force. This breaking away of the field and its travelling to a distance is the essence of the radiation emitted by a vibrator of this type.

542. The distant field in this case is specified by

$$\mathbf{E}_\theta = -\frac{p^2 A \sin\theta \sin p\,(ct - r)}{r},$$

$$\mathbf{H}_\phi = -\frac{p^2 A \sin\theta \sin p\,(ct - r)}{r},$$

so that the surfaces over which the phase of the vibration is constant are the spheres

$$r = \text{const.}$$

To use the ordinary phraseology we may call these the wave front surfaces: each of them is advancing outwards with a velocity of linear expansion equal to c.

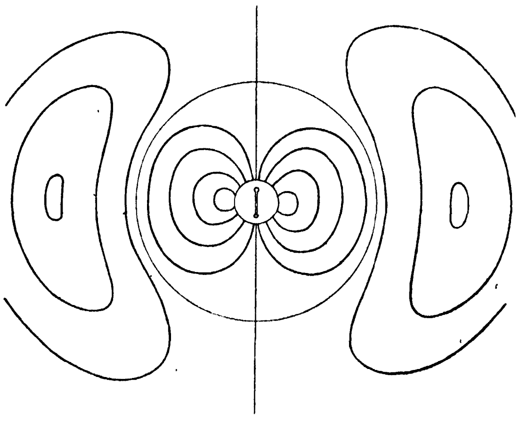


Fig. 79

We see that in the present case also the directions of both the electric and magnetic vectors at any point in the field are tangential to the wave front, and they are in addition of equal magnitude and perpendicular to one another. The electromagnetic waves emitted by a vibrator of this type are therefore transverse waves. We shall see presently that properties of the

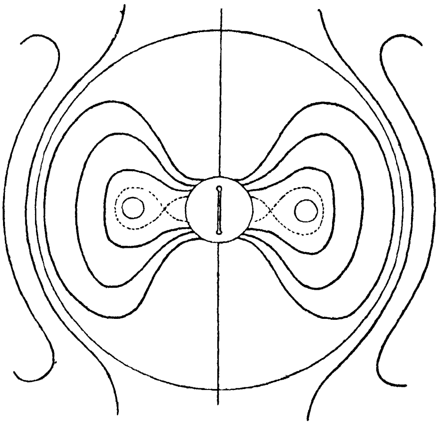


Fig. 80

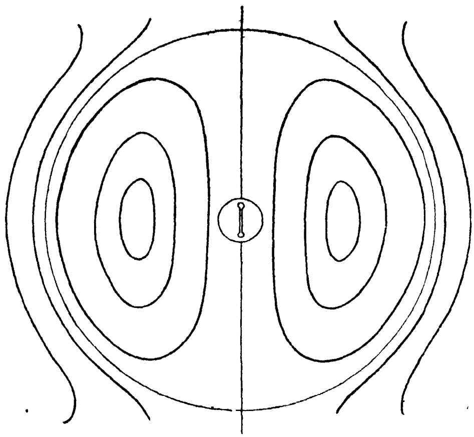


Fig. 81

waves here illustrated by a special case are characteristic of the waves produced under all circumstances and, combined with the fact that the velocity of their propagation in a vacuum is c, a velocity identical in magnitude with the velocity of light, they point to the conclusion that electromagnetic waves however they are produced, differ from light waves only in the magnitude of their wave length,

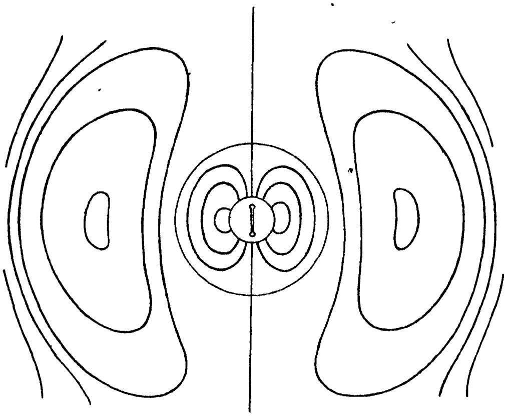


Fig. 82

543. We must now say a few words* regarding the method employed by Hertz to examine the radiation fields thus generated. The difficulty involved in the detection of these fields was at once removed by his discovery that the small rapidly alternating electric forces would produce small sparks between pieces of metal very nearly in contact of sufficient regularity to be used as an indicator of the presence of such forces and to investigate their properties. In his first experiments therefore Hertz used as a *detector* a piece of thin copper wire bent into a circle, the ends being furnished with two balls, or one ball and an adjustable screw-point. The radius of the circle for use with the vibrator described above was 35 cms. and was so chosen that the period of free electrical oscillations in it (when it is used as a condenser) might be the same as that of the vibrations in the oscillator, so as to take full advantage of the resonance effects described in the previous chapter; for this reason also it is desirable to have as small a resistance in this detector circuit as possible.

Now as to the explanation of the action of the detector. It acts in the way that all resonators do by a sort of induction process, where the on-coming wave motion continually releases the natural elastic restraint of the system and thereby induces into existence the oscillations of which it is capable. In order to start the oscillations in any electric circuit of the type under consideration in which there exists a small air gap it is first necessary to create an electric field (i.e. a state of strain) across the small air gap between its metallic ends and then to release it. The usual method of doing this is to charge the metallic ends oppositely until a discharge is effected across the gap; but if it is possible to move the necessary condition of field up through

* Cf. the article 'Electric Waves,' by J. J. Thomson, in the *Encyclopedia Britannica*, where a more detailed description of the apparatus and methods is given.

the dielectric medium the same result is produced. Thus in Hertz's experiments sparks should appear in the detector as soon as the line of centre of the balls is placed along the direction of the electric force and they will be the more vigorous the greater the intensity of the field. This was actually found to be the case and rough as these experiments necessarily were the results obtained were found to be in general agreement with the predictions of theory.

544. On some other types of electrical oscillators. Although they are of no real practical importance it is of interest theoretically to consider certain other means of generating oscillating fields of the type of those just considered; but in cases which are more susceptible of rigorous mathematical examination. Some of these cases have, it is true, been submitted to experimental examination with results in agreement, as far as it is possible to follow them, with the theoretical predictions.

If the distribution of electricity on a system in electrical equilibrium is suddenly disturbed, the electricity will redistribute itself so as to go back to the distribution it had when in electrical equilibrium. If for instance the original distribution is induced by a field which is suddenly altered or removed altogether the induced distribution will have to readjust itself, being no longer in equilibrium. This readjustment will be accomplished very rapidly owing to the free mobility of the charges, and the new distribution will soon be attained; but in it the charges will have considerable motional energy which will take them beyond the new distribution and an oscillation will result. If the damping is not too big this oscillation may continue for some time.

545. As an example of the principles here involved we may consider the oscillation of the charge induced by a uniform field of force of intensity E* on a perfectly conducting sphere (radius a) when this field is suddenly removed. The original density of the charge is

$$\sigma = \frac{3E\cos\theta}{4\pi},$$

θ being the polar angle from the radius of the sphere parallel to the lines of force of the original field. When the inducing field is removed this distribution will tend to annul itself by currents flowing over the surface of the conductor; by the symmetry of the whole affair it follows that this surface current flux will be entirely confined to the meridian planes and the external field will

* Cf. J. J. Thomson, *L. M. S. Proc.* xv. (1884), p. 197; *Recent Researches*, p. 361. Other cases have been examined by A. Lampe, *Wien. Ber.* CXII. (1903), p. 37; F. Kolaçek, *Ann. d. Phys.* LVIII. (1896), p. 271; J. J. Thomson, *Recent Researches*, p. 373 *et seq.*; J. Larmor, *Proc. L. M. S.* XXVI. (1894), p. 119; Rayleigh, *Phil. Mag.* XLIII. (1897), p. 125; R. H. Weber, *Ann. der Phys.* VIII. (1902), p. 721; Riemann-Weber, *Partielle-diff. Gleichungen, etc.* II (1901), p. 348; M. Abraham, *Ann. d. Phys.* LXVI. (1898), p. 435, *Math. Ann.* LII (1899), p. 81.

at every instant be symmetrical about the polar radius and of the type of field just investigated where the magnetic force is in circles round the radius and the electric force in the meridian planes.

We refer the whole system to a spherical polar coordinate (r, θ, ϕ) frame with the pole at the centre of the sphere and the axis along the direction of the original field and we then follow the hint suggested by the analysis of the previous paragraph and try solutions of the fundamental equations for the external (vacuum) field

$$\mathbf{E}_r = \frac{2\cos\theta}{r^2}\left(f'(ct-r) + \frac{1}{r}f(ct-r)\right),$$

$$\mathbf{E}_\theta = \frac{\sin\theta}{r}\left(f''(ct-r) + \frac{1}{r}f'(ct-r) + \frac{1}{r^2}f(ct-r)\right),$$

$$\mathbf{H}_\phi = \frac{\sin\theta}{r}\left(f''(ct-r) + \frac{1}{r}f'(ct-r)\right).$$

The corresponding vectors for the internal field are of course all zero, since the sphere is perfectly conducting ($\sigma = \infty$).

The type of solution here assumed corresponds of course to the case where the field is an expanding one and it is thus tacitly assumed that there are no reflectors anywhere to send it back again, thus producing the corresponding condensing wave.

546. The material of the sphere being perfectly conducting there will be nothing except the very slight inertia of the electrons themselves to prevent the adjustment of any specified conditions in the sphere and the inertia itself is entirely occupied in interaction with the adjusting field. This means that the distribution of charge on the surface of the sphere at any instant is precisely that which would exist on the sphere in a statical field identical with the instantaneous electric part of the varying electromagnetic field in the neighbourhood of the sphere. The electric force in the field must be therefore strictly normal to the surface of the sphere at any instant. This is the essence of the assumption of a perfect conductor.

We must therefore have

$$f''(ct-a) + \frac{1}{a}f'(ct-a) + \frac{1}{a^2}f(ct-a) = 0,$$

for all values of t. This means that the function $f(x)$ must be of the type

$$Ae^{\lambda x},$$

where $$a^2\lambda^2 + a\lambda + 1 = 0,$$

or $$\lambda a = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}.$$

We may then verify that the real part of the solution for the general field vectors corresponding to the root with $\left(+\frac{i\sqrt{3}}{2}\right)$ has

$$\mathbf{E}_r = -\frac{2A\cos\theta}{ar^2}\left\{\left(\frac{1}{2}-\frac{a}{r}\right)\cos\sqrt{3}\,\Theta + \frac{\sqrt{3}}{2}\sin\sqrt{3}\,\Theta\right\}e^{-\Theta},$$

$$\mathbf{E}_\theta = -\frac{A\sin\theta}{a^2 r}\left\{\left(\frac{1}{2}+\frac{a}{2r}-\frac{a^2}{r^2}\right)\cos\sqrt{3}\,\Theta - \frac{\sqrt{3}}{2}\left(1-\frac{a}{r}\right)\sin\sqrt{3}\,\Theta\right\}e^{-\Theta},$$

$$\mathbf{H}_\phi = -\frac{A\sin\theta}{2ar}\left\{\left(1+\frac{a}{r}\right)\cos\sqrt{3}\,\Theta - \sqrt{3}\left(1-\frac{a}{r}\right)\sin\sqrt{3}\,\Theta\right\}e^{-\Theta},$$

where we have used $$\Theta = \frac{ct-r}{2a}.$$

The external field and therefore also the conditions at the surface of the conductor are therefore simple oscillatory in type but with very large damping so that the oscillations soon die out in any case. This of course arises in a physical way from the tendency of the oscillating field to expand outwards and thus include more and more of the inertia of the surrounding aethereal field.

547. The conditions in this problem are probably more like those realised in the actual experiments of Hertz so that for a complete analysis of that problem it would be necessary to take into account the damping of the field.

This has in fact been done in a tentative manner by Pearson and Lee* who assume that the more appropriate form of vibration function defined above in a general manner is such that

$$f(x) = Ae^{-\kappa x}\sin px,$$

the constant κ, which can only be empirically determined except with such simple circumstances as those just examined, determines the rate of decay of the vibrations. The question has been experimentally examined by Bjerknes† who found that with the vibrator used by Hertz the amplitude of the vibrations fell to $\frac{1}{e}$ of the original value after a time $4t_0$, where t_0 is the period of the vibrations. The vibrations would thus become inappreciable after a few alternations.

548. On the mechanism of the establishment of radiation fields‡. We have so far confined our discussions only to the continuous propagation of electromagnetic disturbances in homogeneous radiation fields of indefinite extent. It remains therefore to examine the mode of generation of such

* *Phil. Trans.* A. CXCIII. (1900), p. 159.

† *Ann. d. Phys.* XLIV. (1891), pp. 74, 92, 513; LV. (1895), p. 121

‡ A. E. H. Love, *L. M. S. Proc.* (2) I. (1903), p. 37.

radiation fields. It is the essence of a propagation theory that the conditions of the field at any point P are affected by the conditions at any other point Q only after the time $\frac{PQ}{c}$ after the establishment of the conditions at Q. To illustrate the point in more detail let us consider the simple case of a Hertzian oscillator at the beginning of its oscillations. If we assume that the charge distribution on the oscillator before it collapses has been held there for an indefinite time previously the field surrounding the oscillator will be identical with the simple electrostatic field appropriate to the charge distribution involved. Now suppose that at the time $t = 0$ the discharge takes place and the consequent series of oscillations started. The radiation field which now begins to be generated does not however instantaneously cover the whole field because the conditions at any point in the field at a distance r from the oscillator will remain unaffected by the changes produced by the discharge for the time $\frac{r}{c}$ after the instant $t = 0$ when the discharge takes place, in other words the conditions which existed there at the time $t = 0$ remain unaltered until the disturbance in the radiation field in the surrounding aether which travels outwards in all directions with the velocity c has reached the point. This means that the new radiation field is at the instant t confined within the sphere $r = ct$ surrounding the oscillator: outside this sphere which is a wave surface for the advancing waves the old electrostatic field remains undisturbed, although of course the extent of the field covered by it is gradually diminishing.

549. The first question that naturally arises is as to the manner in which the two essentially different fields thus involved are connected across the advancing wave-front: the radiation field has resulted mainly from a collapse of the initial electrostatic field and must therefore be connected with it in some way or other; by some boundary conditions applicable at the surface of the advancing wave front. These conditions were first directly formulated by Prof. Love* by applying the fundamental equations of the theory to small circuits at the wave front.

Let us assume quite generally that the wave front in the neighbourhood of the point under investigation is practically a plane surface advancing through the field with the velocity c_1 in a direction parallel to the axis of x in a conveniently chosen coordinate system. Draw a small rectangle parallel to the plane Oxz and through which the wave front cuts (dotted in the figure): the dimensions of this rectangle which are assumed to be small compared with the radius of curvature of the wave front surface and the

* They were however previously known to Heaviside (*Electrical Papers*, II. p. 405) and Duhem (*Comptes Rendus*, t. CXXXI. (1900), p. 1171). An elegant analytical proof is given by Bateman in *Electrical and Optical Wave Motion* (C. U. Press, 1915), p. 20.

wave length of the radiation, are such that the sides parallel to Oy are of length l and those parallel to Ox of length l', which is extremely small compared with l and at the instant t is divided by the wave front into portions of lengths

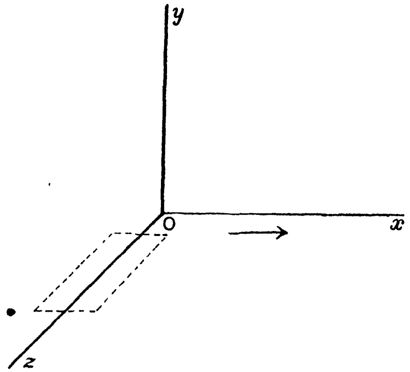


Fig. 83

ξ and $l' - \xi$. On the positive side of the wave front the field is determined by the vectors $\mathbf{E}_+$, $\mathbf{H}_+$ and on the negative side by $\mathbf{E}_-$, $\mathbf{H}_-$. Now the displacement current through the small rectangle is the rate of change of the quantity

$$\frac{l}{4\pi}\{\mathbf{E}_{y+}\,\xi + \mathbf{E}_{y-}\,(l' - \xi)\},$$

or since $\dot{\xi} = c_1$, this current is

$$\frac{l}{4\pi}\{\dot{\mathbf{E}}_{y+}\,\xi + \dot{\mathbf{E}}_{y-}\,(l' - \xi) + c_1\,(\mathbf{E}_{y+} - \mathbf{E}_{y-})\},$$

which on account of the extreme smallness of ξ and $l' - \xi$ compared with the wave length of the radiation, is practically equivalent to

$$\frac{lc_1}{4\pi}(\mathbf{E}_{y+} - \mathbf{E}_{y-}).$$

But by Ampère's relation this will be proportional to the line integral of the magnetic force round the small circuit which on account of the smallness of l' is practically equal to

$$l\,(\mathbf{H}_{z+} - \mathbf{H}_{z-}),$$

and thus
$$\frac{c}{c_1}(\mathbf{H}_{z-} - \mathbf{H}_{z+}) = \mathbf{E}_{y-} - \mathbf{E}_{y+}.$$

An application of Faraday's relation in a similar manner gives

$$\mathbf{E}_{z+} - \mathbf{E}_{z-} = -\frac{c_1}{c}(\mathbf{H}_{y+} - \mathbf{H}_{y-}).$$

If the rectangle is taken parallel to the (x, z) coordinate plane instead of the (x, y) plane as above two further conditions are obtained, viz.

$$\mathbf{H}_{y+} - \mathbf{H}_{y-} = -\frac{c_1}{c}(\mathbf{E}_{z+} - \mathbf{E}_{z-}),$$

$$\mathbf{E}_{y+} - \mathbf{E}_{y-} = \frac{c_1}{c}(\mathbf{H}_{z+} - \mathbf{H}_{z-}).$$

These four conditions break up into two pairs for they are equivalent to the condition that

$$c = c_1,$$

and also

$$(\mathbf{E}_y - \mathbf{H}_z)_+ = (\mathbf{E}_y - \mathbf{H}_z)_-,$$

$$(\mathbf{E}_z + \mathbf{H}_y)_+ = (\mathbf{E}_z + \mathbf{H}_y)_-.$$

The propagation of the wave front is thus verified to be with the velocity c and the new field inside the front is connected with the old field outside by a boundary condition which expresses that the tangential component of the vector

$$\mathbf{F} = \mathbf{E} + \frac{1}{c}[\mathbf{cH}],$$

is continuous across the surface: $\mathbf{c}$ denotes a vector defining the direction and magnitude of the velocity of transmission of the conditions in the radiation field.

The condition expressed in this form which has been deduced on the assumption of an approximately plane wave front is not necessarily restricted by this assumption and it will apply to all sufficiently extended wave front surfaces of ordinary type without discontinuity.

550. The first case examined by Love is that of the oscillations on the perfectly conducting sphere (of radius a) of the charge distribution induced by a uniform field, when that field is suddenly removed. The initial state of the aethereal field outside the sphere is that expressed by the electrostatic vector components

$$(\mathbf{E}_r, \mathbf{E}_\theta, \mathbf{E}_\phi) = \left(\frac{2E\cos\theta}{r^3}, \quad \frac{E\sin\theta}{r^3}, \quad 0\right),$$

$$\mathbf{H} = 0.$$

At the instant $t = 0$ the cause which previously maintained the field thus expressed is supposed to cease to operate. It is required to determine the subsequent state of the field to agree with this initial field at time $t = 0$ and to be such that the tangential electromotive force is continuous at the wave front which at the time t will be the sphere $r = ct + a$ and the tangential electric force at the sphere is zero. The form of solution which suggests itself is naturally of the type previously obtained in which

$$\mathbf{E}_r = \frac{2\cos\theta}{r^3}\{f(ct - r + a) + rf'(ct - r + a)\},$$

$$\mathbf{E}_\theta = \frac{\sin\theta}{r^3}\{f(ct - r + a) + rf'(ct - r + a) + r^2 f''(ct - r + a)\},$$

$$\mathbf{H}_\phi = \frac{\sin\theta}{r^3}\{rf'(ct - r + a) + r^2 f''(ct - r + a)\}.$$

This solution for the field can only apply inside the sphere $r = ct + a$: outside it the old electrostatic conditions still prevail.

At the surface of the sphere the tangential electric force is zero always so that
$$f(ct) + af'(ct) + a^2 f''(ct) = 0,$$
which provides the differential equation for the arbitrary function f: it leads to the solution

$$f(x) = Ae^{-\frac{x}{2a}}\sin\left(\frac{\sqrt{3}x}{2a} + \alpha\right),$$

and this will apply for values of $x \equiv ct - r + a$ greater than zero. The conditions at the wave front imply that

$$\mathbf{E}_{r_1} = \mathbf{E}_{r_2},$$

$$\mathbf{E}_{\theta_1} - \frac{1}{c}\mathbf{H}_{\phi_1} = \mathbf{E}_{\theta_2},$$

giving on substitution

$$f(0) = E,$$

$$f'(0) = 0,$$

and thus we must have
$$A = \frac{E}{\sin\alpha},$$

where
$$\tan\alpha = \sqrt{3}, \qquad \alpha = \frac{\pi}{3},$$

and the problem is completely determined. With this form of f it is easily verified that the expressions for the force vectors in the field can be put in the form

$$\mathbf{E}_r = \frac{4E}{\sqrt{3}}\frac{\cos\theta}{r^3}\sqrt{1 - \frac{r}{a} + \frac{r^2}{a^2}}\,e^{-\Theta}\sin(\sqrt{3}\,\Theta + \beta_r),$$

$$\mathbf{E}_\theta = \frac{2E}{\sqrt{3}}\frac{\sin\theta}{r^3}\left(1 - \frac{r}{a}\right)\sqrt{1 - \frac{r}{a} + \frac{r^2}{a^2}}\,e^{-\Theta}\sin(\sqrt{3}\,\Theta + \beta_\theta),$$

$$\mathbf{H}_\phi = -\frac{2E}{\sqrt{3}}\frac{\sin\theta}{ar^2}\sqrt{1 - \frac{r}{a} + \frac{r^2}{a^2}}\,e^{-\Theta}\sin(\sqrt{3}\,\Theta + \beta_\phi),$$

wherein
$$\Theta = \frac{ct - r + a}{2a},$$

$$\tan\beta_r = \frac{a\sqrt{3}}{a - 2r}, \qquad \tan\beta_\theta = \frac{a + r}{a - r}\sqrt{3}, \qquad \tan\beta_\phi = \frac{r\sqrt{3}}{2a - r},$$

results which differ from those given in a previous paragraph only by the phase of the motion.

It is important to notice that the field of the radiation is strongest close up near the wave front and just inside it; in fact the intensity of the field diminishes exponentially as the distance from the front is increased. In the next chapter we shall examine the energy in this field and will there show that by far the greatest proportion of the total energy radiated out is concentrated close behind the wave front.

It appears from this solution that the damped harmonic wave train can advance into a region in which the electric field is the statical one described above. It is also clear that it cannot advance into a region free from electric and magnetic forces.

551. Aided by the solution thus obtained for a simple mathematical case Prof. Love* attempted to specify an appropriate solution for the more practical case, the Hertzian oscillator, taking into account the damping which is really existent. In this case also the original field is the electrostatic field of a doublet at the origin giving a field in which

$$\mathbf{E}_r = \frac{2E\cos\theta}{r^3}, \qquad \mathbf{E}_\theta = \frac{E\sin\theta}{r^3}, \qquad \mathbf{H}_\phi = 0,$$

the radiation field which originates on the collapse of the distribution giving this statical field (presumed to take place at the time $t = 0$) will be of the usual Hertzian type in which

$$\mathbf{E}_r = \frac{2\cos\theta}{r^3}(f + rf'),$$

$$\mathbf{E}_\theta = \frac{\sin\theta}{r^3}(f + rf' + r^2f''),$$

$$\mathbf{H}_\phi = \frac{\sin\theta}{r^3}(rf' + r^2f''),$$

a specification which will at the time t hold at all points inside the wave surface which to a first approximation may be treated as the sphere $r = ct$.

We then try a solution of the appropriate type in which

$$f(x) = Ae^{-\kappa x}\sin p(x + a).$$

We have to connect this with the external statical field by the boundary conditions deduced above which in this case are

$$\mathbf{E}_{r_1} = \mathbf{E}_{r_2}, \qquad \mathbf{E}_{\theta_1} - \mathbf{H}_{\phi_1} = \mathbf{E}_{\theta_2},$$

and initially we must have $\qquad f \doteq E.$

The boundary conditions give

$$f'(0) + \frac{1}{r}f(0) = \frac{E}{r},$$

* *Proc. R. S.* LXXIV. (1904), p. 73. I am greatly indebted to Prof. Love and the Royal Society for permission to reproduce some of the diagrams illustrating this paper: they are appended to this section.

so that $$f'(0)=0, \quad f(0)=E:$$

this means that
$$E = A \sin pa, \qquad 0 = p\cos pa - \kappa \sin pa,$$

whence
$$\tan pa = \frac{p}{\kappa}.$$

Thus the arbitrary function starts in a definite initial phase which is equal to $\frac{\pi}{2}$ if there is no damping.

The field is thus in the general case defined by the vectors

$$\mathbf{E}_r = \frac{2\cos\theta}{r^3} Ae^{-\kappa(ct-r)}\left[(1-\kappa r)\sin\Theta + pr\cos\Theta\right],$$

$$\mathbf{E}_\theta = \frac{\sin\theta}{r^3} Ae^{-\kappa(ct-r)}\left[(1-\kappa r + r^2\overline{\kappa^2 - p^2})\sin\Theta + pr(1-2\kappa r)\cos\Theta\right],$$

$$\mathbf{H}_\phi = \frac{\sin\theta}{r^2} Ae^{-\kappa(ct-r)}\left[(\kappa - r\overline{\kappa^2 - p^2})\sin\Theta - p(1-2\kappa r)\cos\Theta\right],$$

where we have used $$\Theta = p(ct - r + a).$$

These formulae apply only for $r < ct$; for regions beyond the ordinary electrostatic field remains valid. The radial electric force is continuous at the front of the wave, i.e. at $r = ct$. The discontinuity of the transverse component of the electric force at the front of the wave is

$$\frac{\sin\theta}{r} \cdot A(\kappa^2 + p^2)\sin pa,$$

and this is equal as it should be to the magnetic force at the front of the wave.

552. The curves of electric force in this case are easily obtained for we know that

$$\frac{1}{c}\frac{d\mathbf{E}_r}{dt} = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad \frac{1}{c}\frac{d\mathbf{E}_\theta}{dt} = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},$$

where $$\psi = \frac{\mathbf{H}_\phi}{r\sin\theta}.$$

Thus if we write $$\psi = -\frac{1}{c}\frac{dQ}{dt},$$

then $$\mathbf{E}_r = -\frac{1}{r^2\sin\theta}\frac{\partial Q}{\partial\theta}, \qquad \mathbf{E}_\theta = \frac{1}{r\sin\theta}\frac{\partial Q}{\partial r},$$

and thus the curves of intersection of the surfaces

$$Q = \text{const.},$$

with planes through the axes of the doublet are the lines of electric force. Now it is easily verified that

$$Q = -\frac{\sin^2\theta}{r} Ae^{-\kappa(ct-r)}\left[(1-\kappa r)\sin\Theta + pr\cos\Theta\right],$$

when $ct < r$ and
$$Q = -\frac{\sin^2\theta}{r} A\sin pa,$$
when $ct > r$.

Some of these curves have been drawn in a special case by Prof. Love and are depicted below. The case taken is that for which

$$\frac{\kappa}{p} = \cdot 25 \text{ approx.},$$

and if τ is the period the curves are plotted for the times

$$t = \cdot 26\tau,\quad \cdot 385\tau,\quad \cdot 51\tau,\quad \cdot 635\tau,\quad \cdot 76\tau,\quad \cdot 885\tau,\quad 1{\cdot}01\tau, \text{ and } 1{\cdot}135\tau,$$

after the initial instant of starting. In the figures the fine continuous circle represents the wave front at the time t. The discontinuity of the field there is shown by the change of direction of the lines of force at this circle: the lines of force themselves are shown by the heavy dotted, heavy continuous, fine dotted and fine continuous lines respectively. The dotted circles that lie within the wave front are curves at which $Q = 0$ or the electric force has

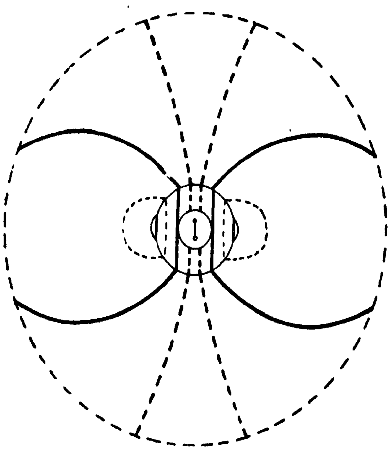


Fig. 84

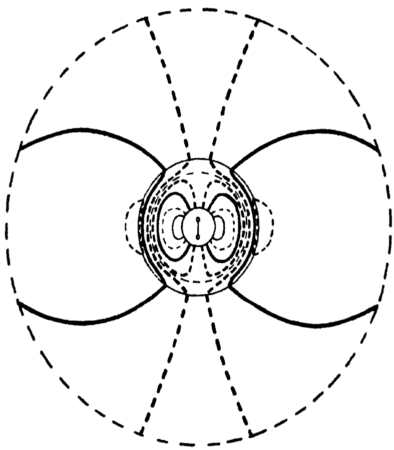


Fig. 85

no radial component. It appears that no spherical surface of the set $Q = 0$ is the front of the advancing wave train but one of these surfaces tends to coincidence with this front as the wave train advances.

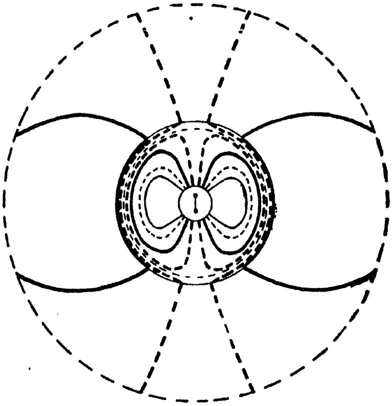


Fig. 86

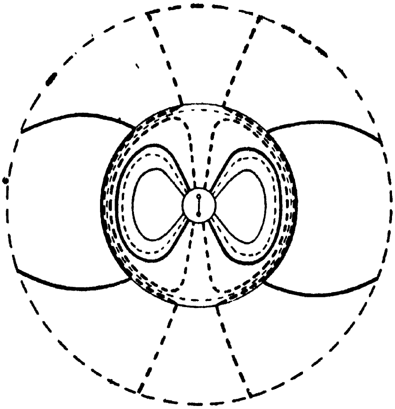


Fig. 87

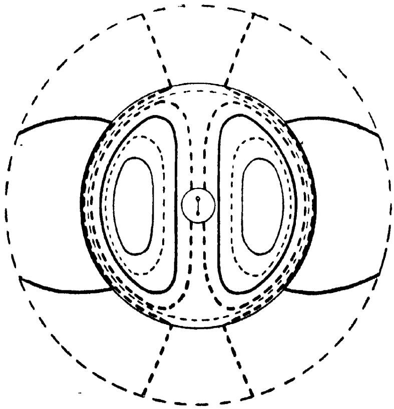


Fig. 88

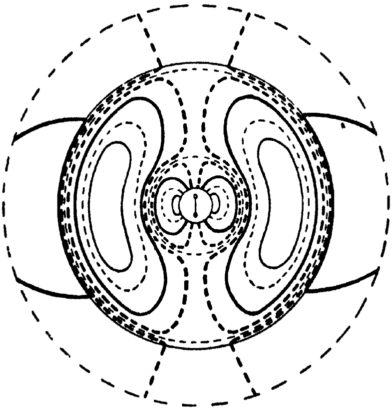
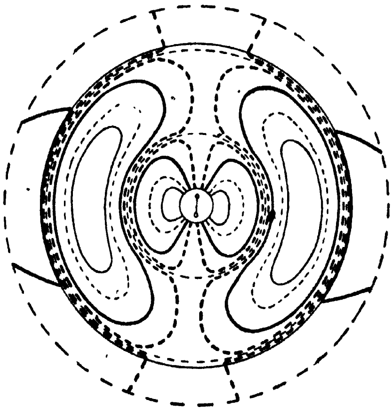
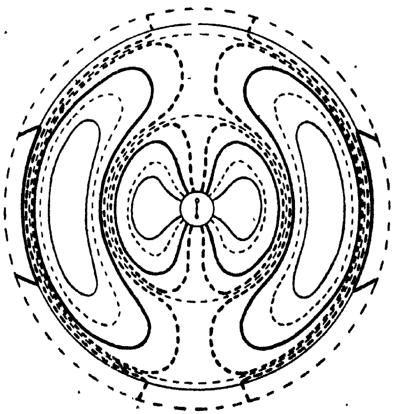


Fig. 89





553. Plane waves. We shall next turn to the consideration of certain cases of wave motion where the circumstances are much simpler than those just analysed. If the radiating system is at a very great distance from the part of the field under investigation the expanding wave front surfaces have become so large that in the part of the field where they are investigated they may be treated as practically plane surfaces. We then realise the idea of plane waves and the consequent rectilinear propagation of electromagnetic disturbances, and the analytical problems become much simplified.

Analytically the conception is obtained by a simple type of solution of the fundamental equations of propagation, which directly suggests itself. Let us choose our rectangular axes so that the z-axis is along the direction of propagation and the other two conveniently in the perpendicular plane which is parallel to the front of the wave. The general solutions for this case are then of the form

$$\left.\begin{matrix}\mathbf{E}\\\mathbf{H}\end{matrix}\right\} = Ae^{int-(a+ib)z+i\vartheta}$$

which represents plane waves of period $\frac{2\pi}{n}$ advancing in the medium with a velocity $c_1 = \frac{n}{b}$ along the positive direction of the z-axis. The solution for any vector is of course represented by the real part of the general solution thus obtained for it.

If we assume, for the isotropic medium, that both the two simple constitutive relations

$$\mathbf{C} = \sigma\mathbf{E} + \frac{\epsilon}{4\pi}\frac{d\mathbf{E}}{dt},$$

and

$$\mathbf{B} = \mathbf{H},$$

are valid even for the general case of electric waves under consideration then both vectors $\mathbf{E}$ and $\mathbf{H}$ satisfy the equation of the previous paragraph

$$\nabla^2\mathbf{E} = \frac{4\pi\sigma}{c^2}\frac{d\mathbf{E}}{dt} + \frac{\epsilon}{c^2}\frac{d^2\mathbf{E}}{dt^2},$$

and thus the above-mentioned forms for $\mathbf{E}$ and $\mathbf{H}$ are valid if

$$-\epsilon n^2 + i4\pi n\sigma = c^2(a+ib)^2,$$

or

$$-\epsilon n^2 = c^2(a^2 - b^2),$$

$$4\pi n\sigma = 2c^2ab.$$

Thus if σ is different from zero, a is so also and thus the amplitude A of each vector contains the factor e^{-az}, which means that as the waves are propagated along the positive direction of the z-axis, the amplitudes of the two vectors gradually decrease as the wave proceeds. Conductivity in the medium implies a damping of the waves, the energy being absorbed by the medium and converted into heat.

554. We can now show that these electric waves are transverse. To do this we merely prove that in the most general aeolotropic medium any vector which has the property of the ordinary stream vector of hydrodynamic theory has its direction parallel to the wave front; for if $\mathbf{V}$ is any such vector

$$\operatorname{div} \mathbf{V} = \frac{\partial \mathbf{V}_x}{\partial x} + \frac{\partial \mathbf{V}_y}{\partial y} + \frac{\partial \mathbf{V}_z}{\partial z} = 0,$$

and since all quantities in a plane wave specified as above only depend on the z-coordinate we must have

$$\frac{\partial \mathbf{V}_x}{\partial x} = \frac{\partial \mathbf{V}_y}{\partial y} = 0,$$

and therefore also
$$\frac{\partial \mathbf{V}_z}{\partial z} = 0,$$

which implies that $\mathbf{V}_z = 0$.

This is a general property of a train of plane waves whatever the constitution of the medium, whether it be crystalline or not. It means in any case that the total current and magnetic induction vectors are in the plane of the wave front. Thus in this sense all such electromagnetic waves are transverse, the fluxes associated with them being both transverse to the direction of propagation. The electric and magnetic force vectors are of course in the general case not in the wave front; this is true only when the medium is isotropic as already assumed.

555. We now assume that the magnetic and electric forces are in the wave front and thus are expressed by their components

$$(\mathbf{H}_x,\ \mathbf{H}_y,\ \mathbf{H}_z) = (A_m \cos\phi_m,\ A_m \sin\phi_m,\ 0)\, e^{int-(a+ib)z+i\theta_m},$$

$$(\mathbf{E}_x,\ \mathbf{E}_y,\ \mathbf{E}_z) = (A_e \cos\phi_e,\ A_e \sin\phi_e,\ 0)\, e^{int-(a+ib)z+i\theta_e},$$

and these must satisfy the fundamental equations of the field. Faraday's relation implies that

$$-\frac{1}{c}\frac{d}{dt}(\mathbf{H}_x,\ \mathbf{H}_y) = \frac{\partial}{\partial z}(\mathbf{E}_y,\ -\mathbf{E}_x),$$

so that
$$-in\frac{A_m}{c}\cos\phi_m = \quad A_e \sin\phi_e\,(a+ib)\,e^{i(\theta_e-\theta_m)},$$

$$-in\frac{A_m}{c}\sin\phi_m = -A_e\cos\phi_e\,(a+ib)\,e^{i(\theta_e-\theta_m)}.$$

Whence we deduce directly the two important conclusions.

(i) By division we see that

$$\tan\phi_e = -\cot\phi_m,$$

$$\phi_m = \phi_e + \frac{\pi}{2}.$$

Thus the electric and magnetic force vectors in the wave front are perpendicular to one another.

(ii) Using the fact that $\phi_m = \phi_e + \frac{\pi}{2}$ we see also that

$$-inA_m = c\,(a + ib)\,A_e\,e^{i(\theta_e - \theta_m)}.$$

Now from Ampère's circuital relation we can deduce similarly that

$$A_m c\,(a + ib)\,e^{i(\theta_m - \theta_e)} = A_e\,(\sigma + i\epsilon n),$$

so that since A_e and A_m are real we must have

$$\tan(\theta_m - \theta_e) = \frac{a}{b} = \frac{1}{2}\frac{\sigma}{n}\frac{c_1^2}{c^2},$$

where $c_1 = \frac{n}{b}$ is the velocity of propagation in the medium.

Also
$$A_m = \frac{cA_e}{n}\sqrt{a^2 + b^2}.$$

Thus there is in the general case always a phase difference $(\theta_m - \theta_e)$ between the electric and magnetic force vibrations, this phase difference vanishing only for the case when $a = 0$, i.e. $\sigma = 0$ or for a perfect non-conductor. The amplitudes of the waves are also different in the general case of absorption.

The velocity of the wave $\frac{n}{b}$ is given by

$$c_1 = \frac{c}{\sqrt{\epsilon + \frac{a^2c^2}{n^2}}},$$

which reduces in the case of non-absorbing media to

$$\frac{c}{\sqrt{\epsilon}}.$$

556. [*If we now choose the real part of the general solutions and also write

$$\Theta = nt - bz + \theta_e,$$

then we can put

$$E = A_e \cos\Theta e^{-az},$$

$$H = A_m \cos(\Theta + \theta_e - \theta_m)\,e^{-az}.$$

The electric energy per unit volume at a place is

$$W = \frac{\epsilon E^2}{8\pi} = \frac{\epsilon A_e^2}{8\pi}\cos^2\Theta e^{-2az},$$

and the mean value at the place taken over a whole oscillation is

$$\overline{W} = \frac{n}{2\pi}\int_0^{\frac{2\pi}{n}} W\,dt = \frac{\epsilon A_e^2}{16\pi}e^{-2az},$$

* The results deduced and given between brackets thus [...] depend on the forms of the energies in the field which are not properly deduced until chapter XIV.

similarly the mean kinetic or magnetic energy is

$$\bar{T} = \frac{n}{2\pi}\int_0^{\frac{2\pi}{n}} T\,dt = \int_0^{\frac{2\pi}{n}} H^2 dt = \frac{A_m{}^2}{16\pi} e^{-2az}$$

$$= \frac{A_e{}^2}{16\pi}\frac{c^2}{n^2}(a^2+b^2)\, e^{-2az}$$

$$= \frac{\epsilon A_e{}^2}{16\pi}\frac{b^2+a^2}{b^2-a^2}e^{-2az}.$$

Whence it follows that the mean magnetic energy is in general larger than the electric energy, equality occurring only in the non-conducting substances when $a = 0$. This result also exhibits clearly the way in which the energy in the wave is absorbed as it progresses, the mean total energy of the wave at any place being the sum of the electric and magnetic energies, viz.

$$\frac{\epsilon A_e{}^2}{8\pi}\frac{2b^2}{b^2-a^2}e^{-2az}.\;]$$

557. These results enable us to explain in greater detail the behaviour of metallic conductors in a radiating dielectric field discussed in the previous paragraph. They show that the propagation of the waves in the dielectric takes place without any absorption at all. As soon, however, as a disturbance reaches a conducting surface and starts off through the conducting medium the damping factor

$$e^{-az},$$

at once enters into the expression. If the conductivity is big a is large and the wave is practically damped right out before it gets far into the metal. The larger the value of a the shorter the distance the waves penetrate into the conductor, and by sufficiently large values we may neglect the penetration altogether. We can also increase a by increasing n, so that for very rapid oscillations the conducting material will always act as if it were a perfect conductor. Let us take an example to illustrate the matter further. For copper $\sigma = \frac{c^2}{1600}$ and ϵ is negligible. Now consider the incidence of waves of length 100 cms.: then

$$n = \frac{2\pi \times 3 \times 10^{10}}{10^2} = 2 \times 10^9 \text{ approx.}$$

and since ϵ is negligible

$$i4\pi n\sigma = c^2(a+ib)^2,$$

whence
$$\frac{1+i}{2}\sqrt{4\pi n\sigma} = c(a+ib),$$

or
$$2ca = 2cb = \sqrt{4\pi n\sigma},$$

$$a = \frac{1}{2}\frac{\sqrt{4\pi n\sigma}}{c} = \frac{1}{2}\sqrt{\frac{2\times 10^9 \times 4\pi}{16 \times 10^2}}$$

$$= 2 \times 10^3 \text{ approx.}$$

Thus at a depth $5 \cdot 10^{-4}$ cms. the amplitudes are reduced to $1/e$ times their initial value. The penetration is therefore very slight in a real case of this kind.

558. In many cases it is convenient to make one further simplification in the above scheme. In the analyses given above, A_e, ϕ_e and A_m, ϕ_m may vary with the time at any one place, the only conditions to be satisfied at each instant being those relations connecting them which we have obtained. We might however now imagine the medium and the original disturbance so adjusted that in the region in which we examine them the electric force vector has the same direction at every point of the field. This direction we may take as the axis of x in our coordinates; it then follows that the magnetic force vector is always parallel to the y-axis at every point of the same field. In the language of optics this means that our ray is polarised, although the plane of polarisation is undetermined by the analogy. Future evidence however points to the conclusion that the plane of polarisation is determined by the vector **H** and the wave-normal or direction of propagation. The electric force is normal to this plane.

559. On experiments with electric waves. Having definitely established in the manner briefly described above the existence of the electromagnetic waves required by Maxwell's theory Hertz proceeded at once to the further problem of proving by certain simple experiments the complete analogy which is suggested by Maxwell's analysis between these waves and waves of light. He first examined their reflexion and refraction at the interfaces between two dielectric media or between one dielectric medium and a metallic medium.

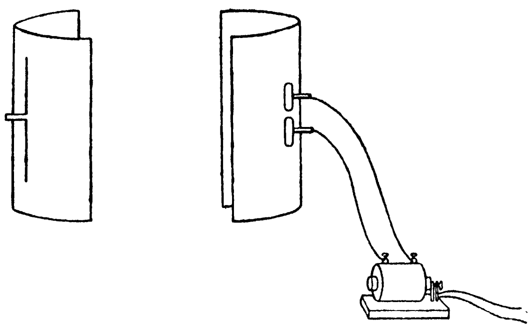


Fig. 92

For this purpose however he used another form of apparatus. The vibrator (see fig.) consisted of two equal brass cylinders 12 cm. long and 3 cm. in diameter placed with their axes coincident and in the focal line of a large zinc parabolic mirror about 2 metres high with a focal length 12·5 cm. This arrangement should on the optical analogy produce plane polarised waves

by reflection from the mirror. The detector which was placed in the focal line of an equal parabolic mirror, consisted of two lengths of wire, each having a straight piece about 50 cm. long and a curved piece about 15 cm. long bent round at right angles so as to pass through the back of the mirror. The ends which came through the mirror were connected with a spark micrometer, the sparks thus being observed from behind the mirror.

560. To show the reflexion of the waves Hertz placed the mirrors side by side, with their openings looking in the same direction and their axes converging to a point 3 metres from the wire. No sparks were observed in the detector when the vibrator was in action. When, however, a large zinc plate was placed at right angles to the line bisecting the angle between the mirrors sparks became visible, but disappeared when the metal was twisted through an angle of about 15° to either side. This experiment showed that electric waves are reflected by the metal sheet and that approximately at any rate the angle of incidence is equal to the angle of reflexion.

To show refraction Hertz used a large prism made of hard pitch with an angle of 30°. When the waves from the vibrator passed through this the sparks in the detector were not excited when the axes of the two mirrors were parallel, but appeared when the axes of the mirror containing the detector made a certain angle with the axes of the containing vibrator. When the system was adjusted for minimum deviation the sparks were more vigorous when the angle between the axes of the mirrors was 22°, corresponding on the optical analogy to an index of refraction for pitch of 1·69.

561. If a screen be made by winding wire round a large rectangular framework so that the turns of the wire are parallel to one side of the frame, and if this screen be interposed between the parabolic mirrors when placed so as to face each other there will be no sparks in the detector when the turns of the wire are parallel to the focal lines of the mirror; but if the frame is turned through a right angle so that the wires are perpendicular to the focal lines of the mirror the sparks will recommence. If the framework is substituted for the metal plate in the experiment on reflexion of electric waves, sparks will appear in the detector when the wires are parallel to the focal lines of the mirrors and will disappear when the wires are at right angles to these lines. Thus the framework reflects but does not transmit the waves when the electric force in them is parallel to the wires while it transmits but does not reflect waves in which the electric force is at right angles to the wires. The wire framework thus behaves towards the electric waves exactly as a plate of tourmaline does to waves of light.

When light polarised at right angles to the plane of incidence falls on a refracting substance at an angle $\tan^{-1}\mu$, where μ is the refractive index of the substance, all the light is reflected and none refracted. Whereas when

light is polarised in the plane of incidence, some of the light is always reflected whatever the angle of incidence. Trouton* showed that similar effects take place with electric waves. From a paraffin wall 3 feet thick, reflection always takes place when the electric force in the incident wave was at right angles to the plane of incidence, whereas at a certain angle of incidence there was no reflection when the vibrator was turned, so that the electric force was in the plane of incidence. This shows that on the electromagnetic theory of light the electric force is at right angles to the plane of polarisation.

562. The surest test of any wave theory of propagation of disturbance of any kind is obtained, if it is possible to produce some effect which depends essentially on the well-known phenomenon of the interference of two trains of the wave disturbance; it is in fact by this means that the essential wave character of the disturbance is usually exhibited. The simplest effect of this kind is the production of '*stationary waves*' by the interference of a direct and reflected train of disturbance. Hertz attempted to get stationary waves by reflection at a metallic surface. His experiments were made in a room about 15 m. long: a vibrator of the type described above was placed at one end of the room with its plates parallel to the wall and a large sheet of zinc was placed vertically against the wall at the other. The circular ring detector was held with its plane parallel to the plane of the plates of the vibrator, its centre on the line perpendicular to the zinc plate bisecting at right angles the spark gap of the vibrator. The following effects were observed when the detector was moved about. Close up to the plates there were no sparks but they began to pass feebly at a little distance from the plate and increased rapidly in brightness up to a distance of about 1·8 m. from the plate, where the maximum was attained. When the distance still further increased they diminished in brightness and vanished again at a distance of about 4 m. from the plate. When the distance was still further increased they reappeared, attained another maximum and so on. The most obvious explanation of these experiments was the one given by Hertz—that there was interference between the direct waves given out by the vibrator and those reflected from the plate, this interference giving rise, in the well-known manner, to stationary waves. The places where the electric force was a maximum were the places where the sparks were brightest and the places where the electric force was zero were the places where the sparks vanished. On this explanation the distance between two consecutive places where the sparks vanished would be half the wave length of the waves given out.

Sarasin and De la Rive† however showed that this explanation could not be correct since by using detectors of different sizes they found that the distance between consecutive places where the sparks vanished depended

* *Nature*, XXXIX. p. 391.

† *Comptes Rendus*, CXV. p. 489.

mainly upon the size of the detector, being in fact proportional to its linear dimensions (i.e. using similar shapes), and very little upon that of the vibrator. This is due to the very large damping of the oscillations in the vibrator and the very small damping of those of the detector. The rapid decay of the oscillations of the vibrator will stifle the interference between the direct and reflected wave, as the amplitude of the direct wave will, since it is emitted later, be much smaller than that of the reflected one and not able to annul its effect completely, while the well-maintained vibrations of the detector will interfere and produce the effects observed by Sarasin and De la Rive. To see this let us consider the extreme case in which the oscillations of the vibrator are absolutely dead beat. In this case an impulse, starting from the vibrator on its way to the reflector strikes against the detector and sets it in vibration; it then travels up to the plate and is reflected, the electric force in the impulse being reversed by reflection: it then again strikes against the detector which is still vibrating from the effects of the first impact; if then the phase of this vibration is such that the reflected impulse tends to produce a current round the detector in the same way as that which is circulating from the effects of the first impact, the sparks will be increased, but if the reflected impulse tends to produce a current in the opposite direction the sparks will be diminished. Since the electric force is reversed by reflection the greatest increase in the sparks will take place when the impulse finds, on its return, the detector in the opposite phase to that in which it left it; that is, if the time which has elapsed between the departure and return of the impulse is equal to an odd multiple of half the time of vibration of the detector, the distance between two spark maxima would then be $\frac{ct'}{2}$, where c is the velocity of radiation and t' the period of vibration of the detector.

Thus although these experiments bring out the essential wave characteristic of the phenomena and the possibility of producing interference of electromagnetic wave trains the circumstances in them are much more complicated than was at first supposed and the phenomenon of stationary waves is not in reality exhibited in them at all. It is possible however to produce stationary wave motion in other ways as we shall see presently.

We shall in the next chapter analyse the circumstances exhibited in these experiments in order to show that the results obtained are in accordance with theory. As the results of the experiments can, with the possible exception of the tourmaline analogy, all be explained in terms of any wave theory with the proper laws of reflection and refraction, we shall confine ourselves to the illustration of these laws by the examination of the comparatively simple circumstances connected with the reflection and refraction of plane waves at plane boundaries.

563. The mechanism of radiation. The modern theory of electrical actions ascribes all electrodynamic effects of electric currents solely to the motion of electrons: every disturbance of the aether including radiation as one type of disturbance is originated by translatory motion of electrons through the aether. Thus if we can obtain complete expressions for the aethereal disturbance initiated by and propagated from a single moving electron, we should be in a position to attempt a theory of the mechanism of radiation.

Now except at places whose distance from the nucleus of the electron is so small as to be comparable with the linear dimensions of the nucleus, it is usually sufficient to consider the electron as a point charge; and the aethereal disturbance arising from the motion of the electron is to be obtained by simple superposition of elementary disturbances arising from its transit over the successive elements of its path. Suppose then that an electron is at the point Q and after a short time δt is at Q', where $QQ' = \mathbf{v}\delta t$, $\mathbf{v}$ being its velocity; the effect of its change of position is the same as that of the creation of an electric doublet of moment $e\mathbf{v}\delta t$, with its axis along QQ'. Thus we have only to find the disturbance produced by the creation of such a doublet and then integrate the result along the path of the electron. But we have already determined above in § 537 the complete effective field for a doublet whose moment is varying in any arbitrary manner and thus the solution for the present case will be obtained by choosing the function f thus involved so that

$$\begin{aligned} f(ct) &= 0 && \text{when } t < t_0, \\ f(ct) &= e\int_0^{\delta t_0} \mathbf{v}dt && t_0 < t < t_0 + \delta t_0, \\ f(ct) &= 0 && t_0 + \delta t_0 < t. \end{aligned}$$

564. Before however making this transformation we may notice that the field specified in the previous paragraph can be regarded as defined in the ordinary way by a scalar static potential

$$\phi = \frac{\cos\theta}{r}\left\{f'(ct-r) + \frac{f(ct-r)}{r}\right\},$$

together with the kinetic vector potential whose components are

$$\mathbf{A}_r = \frac{\cos\theta}{r} f'(ct-r),$$

$$\mathbf{A}_\theta = \frac{\sin\theta}{r} f'(ct-r).$$

These expressions are simplified by reference to a general rectangular coordinate system when they may be written in the form

$$\phi = -\frac{\partial}{\partial s}\left\{\frac{\mathbf{f}(ct-r)}{r}\right\},$$

$$\mathbf{A} = \frac{\mathbf{f}'(ct-r)}{r},$$

$\mathbf{f}$ denoting in the complete vector sense the moment of the doublet; the magnitude is of course f. The differential $\frac{\partial}{\partial s}$ denotes the gradient of the function in the direction of the axis of the doublet.

The conditions at any point in the general field at the time t thus depend essentially on the conditions of the vibrator at the time $t' = t - \frac{r}{c}$ previously; this is the *effective time* for the relative position of vibrator and field-point. This result is of course the essence of a propagation theory.

A slight change in the charge distribution on the vibrator, specified by the function f, during the small interval of time δt results in a spherical shell of disturbance travelling out with the velocity c of radiation so that at any time t after the instant of its generation it lies between the concentric spheres $r = ct$ and $r = c(t + \delta t)$. Moreover the total field integrated across the small thickness of the shell is, except as regards the static electric part which is not really propagated, independent of the thickness of the shell which of course depends on the time taken to establish it.

565. Returning now to our moving charge we see at once that the slight displacement of it during the small time δt_0 results in a shell of a spherical disturbance which travels out with the velocity of light from the centre at which it was generated and in which the field is precisely of the type just investigated, in which however the function f is as specified above; this field is thus determined by

$$\phi = \frac{\partial}{\partial s}\left(\frac{e}{r}\right)\delta s,$$

$$\mathbf{A} = \frac{e\delta\mathbf{v}}{cr},$$

where however it must be remembered that as regards the field at a distance r from the electron at time t_0 these values are effective only at the instant $t = t_0 + \frac{r}{c}$, or in other words the effective field at the time t arises from the motion and position of the electron at the previous time $t_0 = t - \frac{r}{c}$, r being the radial distance from the field point to the effective position of the electron at this previous instant. We denote this by enclosing the quantities concerned by square brackets so that we may write

$$\phi = \left[\frac{\partial}{\partial s}\left(\frac{e}{r}\right)\delta s\right] = \left[\delta\,\frac{e}{r}\right],$$

$$\mathbf{A} = \left[\frac{e\delta\mathbf{v}}{cr}\right].$$

It follows therefore by integration along the whole of the path from the last effective position backwards to infinity that the two potentials

$$\phi = \frac{e}{[r]},$$

$$\mathbf{A} = \left[\frac{e\mathbf{v}}{cr}\right],$$

define completely the field of the single moving charge.

566. These are the so-called *retarded potentials* of electron theory. It is however usual to express them to the next order of approximation. The above values are calculated on the assumption that the velocity of the electron is very small compared with the velocity of light, so that the change of position of the electron during the effectively small time δt is negligible. If we proceed to the next order of approximation and take into account the motion of the electron during this interval of time δt we shall see that the shell of disturbance sent out by the electron during it will not be uniformly thick but will in reality lie between two eccentric spheres with their centres in the initial and final effective positions of the electron, i.e. at a distance $[\mathbf{v}\delta t_0]$ apart. The thickness of the shell along any radius r will now be approximately $[(c - \mathbf{v}_r)\,\delta t]$, $\mathbf{v}_r$ being the component of the electronic velocity along that radius. But it is clear that the aggregate of the field sent out in any direction in the shell will be independent of the thickness and therefore of whether it is on the whole of uniform thickness or not, so that the field will be intrinsically stronger in the thinner parts of the shell and correspondingly weaker in the thicker parts: this means that the field vectors in the part of the shell along the radius r will be increased from the values in the uniform shell in the ratio

$$\frac{e}{c - [\mathbf{v}_r]} = \frac{1}{1 - \frac{[\mathbf{v}_r]}{c}}.$$

It follows therefore that to the second order of approximation the field is given by*

$$\phi = \frac{e}{\left[r\left(1 - \frac{\mathbf{v}_r}{c}\right)\right]},$$

$$\mathbf{A} = \frac{e[\mathbf{v}]}{c\left[r\left(1 - \frac{\mathbf{v}_r}{c}\right)\right]},$$

and these are the values usually quoted.

* A. Liénard, *L'eclairage électrique*; E. Wiechert, *Arch. néerlandaises* (2) v. (1900), p. 54; K. Schwarzschild, *Gött. Nachr.* (1903). Cf. also Schott, *Electromagnetic Radiaticn*, p. 22; Bateman, *Electrical and Optical Wave Motion*, Ch. VIII; De la Rive, *Archives de Genève* (1907), p. 433.

567. The mode of deduction of the last expressions for the retarded potentials is perhaps not so rigorous as is desirable with such fundamental formulae. They can however easily be deduced in a more rigorous manner from a slight modification of certain general formulae already obtained. In Chapter IX we have obtained general expressions for the scalar and vector retarded potentials in any electrodynamic field and using the expressions there obtained we see that if the total charge distribution in any field is of density ρ and if the current flux is due to the motion with velocity $\mathbf{v}$ of this charge then the scalar and vector potentials of the field may be written in the form

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int e^{i\mu\left(t-\frac{r}{c}-\tau\right)}\frac{dQ\,d\tau\,d\mu}{r}$$

$$\mathbf{A} = \frac{1}{2\pi c}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int e^{i\mu\left(t-\frac{r}{c}-\tau\right)}\frac{\mathbf{v}\,dQ\,d\tau\,d\mu}{r}\ .$$

where $dQ = \rho dv$ is the charge element in the small volume dv at the time τ. When the charge is of amount e and concentrated in a small volume element round the point $(x_e y_e z_e)$ these expressions reduce to

$$\phi = \frac{e}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\mu\left(t-\frac{r}{c}-\tau\right)}\frac{d\tau\,d\mu}{r}$$

$$\mathbf{A} = \frac{e}{2\pi c}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\mu\left(t-\frac{r}{e}-\tau\right)}\frac{\mathbf{v}\,d\tau\,d\mu}{r}\ .$$

Now change in these integrals the variable τ to τ' where

$$\tau' = \tau + \frac{r}{c}$$

so that
$$d\tau' = d\tau\left(1 + \frac{1}{c}\frac{dr}{d\tau}\right) = d\tau\left(1 - \frac{\mathbf{v}_r}{c}\right),$$

then we get, for instance,

$$\phi = \frac{e}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\mu(t-\tau')}\frac{d\tau'\,d\mu}{\left[r\left(1-\frac{\mathbf{v}_r}{c}\right)\right]_{t=\tau'-\frac{r}{c}}}$$

since the limits for τ' are the same as those for τ when $(\mathbf{v}) < c$. But the double integral in this last case is a proper Fourier integral whose value is

$$\frac{2\pi}{\left[r\left(1-\frac{\mathbf{v}_r}{c}\right)\right]_{\tau'=t}}$$

so that
$$\phi = \frac{e}{\left[r\left(1-\frac{\mathbf{v}_r}{c}\right)\right]}.$$

Similarly we find that

$$\mathbf{A} = \frac{e\,[\mathbf{v}]}{\left[cr\left(1 - \frac{\mathbf{v}_r}{c}\right)\right]}$$

where in the last two expressions square brackets serve to indicate the values of the functions affected at the time $\left(t - \frac{r}{c}\right)$. These are the same expressions as found above.

568. The whole circumstances in the field of the moving charge can now be readily deduced and the expressions for the electric and magnetic force vectors at any point obtained by simple differentiation of these potentials. It is however easier to deduce them from the general results obtained in Chapter IX on the introduction of the retarded potentials. In fact if we again put in these expresssions $\mathbf{C}_1 = \rho\mathbf{v}$ and carry out the volume integration and then transform the integrals as we have just done for the potentials we find that

$$\mathbf{E} = \frac{e\mathbf{r}_1}{\left[r^2\left(1 - \frac{\mathbf{v}_r}{c}\right)\right]} + \frac{d}{dt}\left[\frac{e\left(\mathbf{r}_1 - \frac{\mathbf{v}}{c}\right)}{cr\left(1 - \frac{\mathbf{v}_r}{c}\right)}\right]$$

whilst

$$\mathbf{H} = \left[\frac{e\,[\mathbf{v}\mathbf{r}_1]}{cr^2\left(1 - \frac{\mathbf{v}_r}{c}\right)}\right] + \frac{d}{dt}\left[\frac{e\,[\mathbf{v}\mathbf{r}_1]}{c^2 r\left(1 - \frac{\mathbf{v}_r}{c}\right)}\right]$$

wherein $\mathbf{r}_1$ denotes the unit vector in the direction of the radius r.

We must now bear in mind that the quantities inside the large square brackets are functions of $\tau = t - \frac{r}{c}$ as well as (x, y, z) explicitly; but

$$\frac{d\tau}{dt} = \frac{1}{\left[1 - \frac{\mathbf{v}_r}{c}\right]}$$

and hence

$$\frac{1}{c}\frac{d}{dt}\left[r\left(1 - \frac{\mathbf{v}_r}{c}\right)\right] = 1 - \left[\frac{(\dot{\mathbf{v}}\mathbf{r}) + c^2 - \mathbf{v}^2}{c^2\left(1 - \frac{\mathbf{v}_r}{c}\right)}\right]$$

and

$$\frac{1}{c}\frac{d\mathbf{r}_1}{dt} = \left[\frac{\frac{\mathbf{r}_1\mathbf{v}_r}{c} - \frac{\mathbf{v}}{c}}{r\left(1 - \frac{\mathbf{v}_r}{c}\right)}\right].$$

Hence finally we have

$$\mathbf{E} = \frac{e}{c^2}\left[-\frac{\dot{\mathbf{v}}}{r}\left(\frac{d\tau}{dt}\right)^2 + \frac{\left(\mathbf{r}_1 - \frac{\mathbf{v}}{c}\right)(c^2 - \mathbf{v}^2 + (\dot{\mathbf{v}}\mathbf{r}))}{r^2}\left(\frac{d\tau}{dt}\right)^3\right]$$

$$\mathbf{H} = \frac{e}{c^2}\left[\frac{[\dot{\mathbf{v}}\mathbf{r}_1]}{r}\left(\frac{d\tau}{dt}\right)^2 + \frac{[\mathbf{v}\mathbf{r}_1]\,(c^2 - \mathbf{v}^2 + (\dot{\mathbf{v}}\mathbf{r}))}{cr^2}\left(\frac{d\tau}{dt}\right)^3\right].$$

569. We first notice that $$\mathbf{H} = [\mathbf{r}_1\mathbf{E}]$$
so that the magnetic force is everywhere perpendicular to the electric force and to the radius from the field point to the effective position of the charge element at the instant. On the other hand

$$(\mathbf{r}_1\mathbf{E}) = \left[\frac{e\,(c^2 - \mathbf{v}^2)}{c^2 r^2 \left(1 - \frac{\mathbf{v}_r}{c}\right)}\right]$$

so that the electric force is not transverse to the radius vector unless $c = |\mathbf{v}|$; but the deviation from perpendicularity becomes smaller and smaller as the distance from the charge increases.

Again since $$1 - \frac{\mathbf{v}_r}{c} = 1 - \frac{(\mathbf{v}\mathbf{r}_1)}{c} = \left(\mathbf{r}_1, \mathbf{r}_1 - \frac{\mathbf{v}}{c}\right)$$

it is easy to verify that*

$$\left(\mathbf{r}_1 - \frac{\mathbf{v}}{c}\right)(\dot{\mathbf{v}}\mathbf{r}_1) - \dot{\mathbf{v}}\left(1 - \frac{\mathbf{v}_r}{c}\right) = \left[\mathbf{r}_1, \left[\mathbf{r}_1 - \frac{\mathbf{v}}{c}, \dot{\mathbf{v}}\right]\right]$$

so that $$\mathbf{E} = \frac{e}{rc^2}\left(\frac{d\tau}{dt}\right)^3 \left[\frac{1}{r}\left(\mathbf{r}_1 - \frac{\mathbf{v}}{c}\right)(c^2 - \mathbf{v}^2) + \left[\mathbf{r}_1\left[\mathbf{r}_1 - \frac{\mathbf{v}}{c}, \dot{\mathbf{v}}\right]\right]\right].$$

Also since $$\mathbf{H} = [\mathbf{r}_1\mathbf{E}]$$

we have $$[\mathbf{H}\mathbf{r}_1] = -[\mathbf{r}_1[\mathbf{r}_1\mathbf{E}]] = \mathbf{E} - \mathbf{r}_1(\mathbf{r}_1\mathbf{E})$$

and thus also $$\mathbf{E} = [\mathbf{H}\mathbf{r}_1] + \frac{e\mathbf{r}_1}{c^2r^2}(c^2 - \mathbf{v})\left(\frac{d\tau}{dt}\right)^2.$$

The part of the electric force not depending on the acceleration and the predominant part in the field near the electron is

$$\mathbf{E}_s = \frac{e}{c^2}\left[\left(\mathbf{r}_1 - \frac{\mathbf{v}}{c}\right)(c^2 - \mathbf{v}^2)\left(\frac{d\tau}{dt}\right)^3\right]$$

whilst the corresponding part of the magnetic force is

$$\mathbf{H}_s = \frac{1}{c}[\mathbf{v}\mathbf{E}_s].$$

Now the vector $\left[\mathbf{r}_1 - \frac{\mathbf{v}}{c}\right]$ is parallel to the direction of the radius from the field point to what would be the instantaneous position of the moving charge (as distinct from its effective position) if it be assumed that it has moved from its effective position with the constant velocity $[\mathbf{v}]$ that it then had.

If the motion of the particle is with constant velocity in a straight line this is the whole field and as such will be more fully reviewed in the sequel. We are at present more fully concerned with the part of the field depending on the acceleration, which is far more dispersed than the present part which is concentrated close up round the electron.

* Cf. Introduction, § 6.

570. If the motion of the particle is accelerated the part of the field depending on the accleration which predominates at large distances from the particle and is insignificant in the rest of the field is determined by

$$\mathbf{E} = \frac{e}{rc^2}\left(\frac{d\tau}{dt}\right)^3 \left[\mathbf{r}_1, \left[\mathbf{r}_1 - \frac{\mathbf{v}}{c}, \dot{\mathbf{v}}\right]\right]$$

and $$\mathbf{E} = [\mathbf{H}\mathbf{r}_1].$$

Thus in this part of the field the electric and magnetic forces are both perpendicular to the radius vector from the effective position of the particle to the field point under review; they are also perpendicular to one another and are therefore of equal magnitude.

At large distances from the particle therefore the field, which is completely specified by these components alone, is like a proper field in radiation; the wave front surfaces are eccentric spheres each having its centre at the position of the particle when it was generated and the two force vectors lie in these surfaces.

If the velocity of the motion is small then we have practically

$$\mathbf{E} = \frac{e\dot{\mathbf{v}}}{rc^2},$$

and $$\mathbf{E} = [\mathbf{H}\mathbf{r}_1]$$

so that the electric force is tangential to the meridian circles and the magnetic force to the parallels of latitude, the polar axis being parallel to the effective acceleration.

We shall see presently that this means that there is a flux of energy normal to the wave front and away from the electron : this flux amounts to

$$\frac{e^2}{4\pi^2 c^3 r^2}\sin^2\theta\left[\frac{d\mathbf{v}}{dt}\right]^2 *$$

per unit area per unit time so that on the whole there is the amount

$$\frac{2}{3}\frac{e^2}{c^3}\left[\frac{d\mathbf{v}}{dt}\right]^2$$

per unit time that travels away and is lost to the electron.

When an electron is put into motion it sends out a stream of radiation which lasts as long as its velocity is being accelerated : in the process of setting up the velocity $\mathbf{v}$ from rest, there is a total loss of energy by radiation equal to

$$\frac{2}{3}\frac{e^2}{c^3}\int\left[\frac{d\mathbf{v}}{dt}\right]^2 dt.$$

When the velocity has become constant there is no more radiant energy sent out from it, though the previous sheets of radiation will continue to travel on into the more distant stagnant aether, leaving behind them the

* θ is the latitude of the place.

steady magnetic field of the uniformly moving electron: but that field, which thus becomes established as a trail or residue of the shell of radiation arising from the original initiation of the motion of the electron, does not itself involve any sensible amount of energy except in the immediate neighbourhood of the electron.

571. This analysis has an important application to the explanation of the radiation from an incandescent body. We have seen generally how long Hertzian waves can be produced by a process which consists essentially in the production of a rapid oscillatory motion of electrical charges. Now we have already been led to the conclusion that each element or molecule of a material body contains as an essential part in its constitution a number of electrons and positive charges which under ordinary conditions take up a sort of equilibrium configuration inside it: if we can produce a disturbance in this steady configuration the individual electrons will emit radiation of a type depending on their motion. Thus if we knew the motions of the electrons we could specify completely the type of radiation emitted by the body; but this is just what we do not know. We are still unable to specify completely the type of mechanism governing the electronic motions inside an atom, and we can therefore only offer tentative suggestions such as that given above on page 475. We can however infer from an examination of the radiation itself certain details concerning its mode of generation and it is on this evidence that the suggestive mechanisms are being constructed. It is for example found that the radiation from a gas whose density is not too big consists mainly of a limited number of distinctly separated harmonic constituents with periods and intensities characteristic of the substance of which the gas is composed. This suggests that the vibrations of the electrons giving rise to the radiation must be very nearly simple harmonic; and this suggests again that the electrons are vibrating in the molecule about certain definite positions of equilibrium to which they are bound by certain quasi-elastic forces proportional to their displacement from the position. This is of course the simplest possible idea and has already been introduced on a previous occasion, but further evidence seems to indicate the impossibility of its validity: it would for instance seem to imply that one electron cannot be responsible for more than three of the harmonic constituents of the radiation, and an almost incredibly large number of electrons would thus have to be assumed to exist in the molecules of certain substances. It appears however that no completely satisfactory explanation of these difficulties has yet been offered and we shall therefore content ourselves with this simple explanation.

In striking contrast with the radiation from a gas, the radiation from an incandescent solid or liquid presents as a general rule nothing of a periodic character, for it arises from the independent and irregular disturbances of countless molecules and electrons: it thus has the appearance of a formless

mass of radiant disturbance advancing with the velocity of light: it is possible however even in these cases to have certain more or less predominant constituents of a definite period, but at high temperatures these are completely covered by the irregular radiation which is of a purely thermal character. It is a problem in these theories to determine how, if the thermal radiation of a substance is resolved by a prism into its harmonic spectrum, the energy of the total radiation is distributed among the harmonic constituents thus separated out: some aspects of this problem will be discussed at the end of chapter XIV.

572. Before concluding this paragraph reference must be made again to the so-called Roentgen or X-rays. These rays were first observed by Roentgen in the neighbourhood of a discharge tube when the vivid green phosphorescence is exhibited on the walls of the tube, and they have been found also as an important constituent (the γ-rays) of the radiation emitted by radio-active substances. These rays exhibit a remarkable resemblance to light. Their rectilinear propagation, as evidenced by the sharp shadow thrown by bodies which intercept them, their power of affecting a photographic plate and their power of passing through solid bodies are obvious examples of this resemblance. But there are equally striking differences between Roentgen rays and rays of light. They are not refracted in their passage from one medium to another: they show some sort of reflexion, but the laws governing it are totally different from those of the corresponding phenomenon in light.

The generally accepted view of this radiation as originated from the discharge tube is that first proposed by Stokes: it is composed of thin spherical sheets of disturbance sent out into the aether by the sudden impacts of the rapidly moving electrons of the kathode stream against the walls of the tube: it may also in part be due to the shocks imparted to the molecules forming the walls of the tube. The similar radiation from radio-active substances would then have its origin in part in the sudden generation of the rapid motion in the electrons thrown off from those substances and again in part in the readjustment of the remaining molecule to its new conditions after the electron has left.

In so far as these sheets of radiation are due to sudden but transient disturbance of the electrons in the molecules themselves, the magnetic force belonging to them alternates in direction in crossing each thin shell of pulse so that the average value taken across it is null. In so far as they are due to the sudden start or stoppage of the kathode particles or electrons, each of which is a single moving electron, this balanced alternation of magnetic force across the thickness of the sheet does not hold; the force may be in the same direction all the way across. As during the progress of the emission or impact the accelerations of the kathode particles arrested or emitted and of the disturbed electrons of the molecules will be presumably of the same order of

magnitude, we would naturally conclude from the formula expressing the radiation in terms of the acceleration of the electron that these are both concerned in the emission of radiant energy, and the fact that the radiation is found to contain certain constituents characteristic of the substances on which the particles collide or from which they are thrown off supports this view.

In addition to the thin pulse arising from the sudden shock imparted to the molecules of the substance stopping or starting the electron, we would expect to find also more continuous radiation due to their state of vibration which would ensue: this would be represented by the phosphorescent light which accompanies the phenomenon.

573. As regards the X-radiation the present explanation has received remarkable confirmation in the last few years in a wonderful series of experiments originated by Laue and extensively developed by many workers, among the most prominent of whom are Prof. Bragg and his son*. In these experiments beams of carefully sifted homogeneous X-rays are reflected or refracted by crystalline media, and it is found that the regular arrangement of the molecules of such substances makes them behave towards the radiation more or less like a three-dimensional optical grating. The radiation passing across each molecule sets the electrons in that molecule in rapid vibration and these in their turn emit the secondary radiation which is observed as the transmitted or reflected beam, the regular arrangement of the molecules or vibrating centres giving rise to a measurable regularity in the radiation. It has thus been found possible not only to discover the arrangement of molecules in the crystals but also to establish definitely the periodic character of the X-radiation, even to the extent of obtaining an accurate estimate of its frequency. The wave length of the radiation is characteristic of the exciting substance but is much shorter than the radiation in the visible spectrum.

574. The previous discussion suggests that we have to deal in actual practice not with the single electrons but with whole groups of them more or less tightly bound to the elements of the ponderable matter or moving about freely in the interstices between these elements: and since the formulation of this more general problem brings out further aspects of the radiation from incandescent bodies, it seems desirable to give at least its barest outlines†. We first suppose that the motion of the electron under consideration is confined to a certain very small region v, one point of which is chosen for origin of coordinates. Referred to the axes of coordinates thus chosen let $\mathbf{r}_e$ be the

* Cf. W. L. Bragg, *Proc. Camb. Phil. Soc.* XVII. (1913), p. 43; *Proc. R. S.* A. LXXXVIII. (1913), p. 428; M. Laue, *München. Ber.* (1912), p. 363; *Ann. d. Phys.* XLI. (1913), p. 989; XLII. (1913), p. 397; P. P. Ewald, *Phys. Zeitschr.* (1913), p. 465; L. S. Ornstein, *Amsterdam. Proc.* (1913); M. Born u. T. von Karman, *Phys. Zeitschr.* (1912), p. 297.

† Cf. Lorentz, *The Theory of Electrons*, p. 55.

position vector of the electron so that its velocity is $\dot{\mathbf{r}}_e$ and its acceleration $\ddot{\mathbf{r}}_e$. We shall regard all these quantities as so small that we may neglect any terms involving their squares and products. Next let $\mathbf{r}$ denote the coordinate vector of a point P at some distance from the origin of coordinates, outside the small surface considered, for which we want to determine the field. Now if Q is the effective position of the electron as regards the field at the point P at time t, the distance PQ will differ from r only by terms of the first order, and the effective time t_0 will differ from the time $t - \frac{r}{c}$ in the same way. The changes of position and motion of the electron in a very small time being infinitely small of the second order we may regard Q as the effective position at the instant $t - \frac{r}{c}$ and the velocity there as the velocity at this time. Moreover

$$\frac{1}{PQ} = \frac{1}{r} - ([\mathbf{r}_e]\,\nabla)\,\frac{1}{r},$$

because the difference between the distances PQ and PO is equal to the difference between the vectors $\mathbf{r}$ and $\mathbf{r}_e$ taken at P or Q. The square brackets of course serve to indicate the values of the quantities affected at the instant $t - \frac{r}{c}$. Thus if we use also

$$\frac{1}{1 - \frac{[\mathbf{v}_r]}{c}} = 1 + \frac{[\mathbf{v}_r]}{c},$$

we have
$$\phi = \frac{e}{4\pi}\left\{\frac{1}{r} - ([\mathbf{r}_e]\,\nabla)\,\frac{1}{r} + \frac{[\mathbf{v}_r]}{cr}\right\}.$$

Now as regards the last term in the expression we may write

$$\frac{[\mathbf{v}_r]}{c} = \frac{1}{cr}[(\mathbf{r}\,.\,\mathbf{v})]$$
$$= \frac{[(\mathbf{r}\,.\,\dot{\mathbf{r}}_e)]}{cr}$$

and this is
$$= - \operatorname{div}\,[\mathbf{r}_e],$$

since for example
$$\frac{\partial\,[x_e]}{\partial x} = \frac{\partial\,[x_e]}{\partial t_0}\cdot\frac{\partial t_0}{\partial r}\cdot\frac{\partial r}{\partial x}$$
$$= [\dot{x}_e]\left(-\frac{1}{c}\cdot\frac{x}{r}\right).$$

Thus we have finally for the scalar potential at the external point of the field

$$\phi = e\left\{\frac{1}{r} - \operatorname{div}\left(\frac{[\mathbf{r}_e]}{r}\right)\right\}.$$

The expression for the vector potential is similarly deduced and is

$$\mathbf{A} = \frac{e}{c}\left\{(1 - (\nabla\mathbf{r}_e)\right\}\frac{[\mathbf{v}]^*}{r}.$$

* In this formula the operator ∇ is presumed to affect all quantities following it.

The radiation field which predominates at large distances and in which we find the flow of energy of which we have already spoken, is determined by the second term in ϕ and by the vector potential. At smaller distances it is superposed on the field represented by the first term of ϕ, which is the static potential of the electron at rest.

575. Now suppose that there are a number of electrons and elements of charge inside the small volume v under consideration. The field of each of these charge elements will then be exactly of the type thus specified and the total field of them all together will be simply obtained by superposition of their separate fields. Thus if Σ is used to denote a sum over all the elements of charge, we have in this total field

$$\phi = \frac{(\Sigma e)}{r} - \operatorname{div}\left(\frac{\Sigma\,[e\mathbf{r}_1]}{r}\right)$$

and to the first order the vector potential is given by

$$\mathbf{A} = \frac{[\Sigma e\mathbf{v}]}{r} = \frac{[\Sigma e\dot{\mathbf{r}}_e]}{r}.$$

But if the volume element under consideration is very small and contains only the constitutional electrons and charges of the molecules of matter inside, so that the total charge is zero $\Sigma e = 0$, and then

$$\Sigma e\mathbf{r}_e$$

is the vector quantity which we have previously recognised as the polarisation of the volume element: we may denote this by

$$\mathbf{P}\,dv$$

and then we see that
$$\phi = -\operatorname{div}\frac{[\mathbf{P}]}{r}\,dv,$$

$$\mathbf{A} = \frac{[\dot{\mathbf{P}}]}{cr}\,dv.$$

These relations would also hold in the case of a single uncharged molecule if the appropriate value of the vector $\mathbf{P}$ is implied. They show that the single molecule or small volume element of a material body will be a centre of radiation whenever the polarisation $\mathbf{P}$ is changing: the distant field will again be determined by

$$\mathbf{E}_\theta = \mathbf{H}_\phi = \frac{\ddot{\mathbf{P}}\sin\theta}{cr},$$

with the usual spherical polar frame of reference with the pole at the centre of the element and axis parallel to the direction of polarisation. Thus however rapidly the individual electrons may be rushing about inside the molecule or element of volume there will be no radiation if the acceleration of the polarisation or the vectorial sum $(\Sigma e\dot{\mathbf{v}})$ taken over them all is constantly zero*

* Cf. Larmor, *Aether and Matter*, p. 228.

The importance of this result is that it shows that certain groups of rapidly moving electrons may, in spite of their rapid individual motion, yet be very permanent as a self-contained dynamical system. The condition for permanency is that there should be no dissipation of the energy of the motions, and if the individual motions are subject to the above restriction this condition is certainly satisfied. For example, the orbital motion of two electrons of equal inertia and opposite charge, round each other, the accelerations reinforce each other instead of cancelling, so that this simple type is not a permanent molecular conformation, though it is easy to construct other types that would be possible.

576. Although it has little or no connection with the present aspect of the subject it is of fundamental importance in the general theory to consider one other special case of the analytical results of the present section. Suppose that the charges in the element of volume under consideration are moving so that the polarisation of the element is continually zero. The vector potential of the field of the element is then determined solely by the term

$$-\frac{1}{r}\Sigma e\,(\nabla\mathbf{r}_e)\,\mathbf{v} = -\frac{1}{r}\Sigma e\,(\nabla\mathbf{r}_e)\,\dot{\mathbf{r}}_e$$

the time being however $\left(t-\frac{r}{c}\right)$ instead of t. Again we have

$$\Sigma e\,(\nabla\mathbf{r}_e)\,\dot{\mathbf{r}}_e = \operatorname{curl}\frac{1}{2}\Sigma e\,[\mathbf{r}_e\dot{\mathbf{r}}_e] - \frac{1}{2}\frac{d}{dt}\Sigma q\,\{(\nabla\mathbf{r}_e)\,\mathbf{r}_e\}$$

so that it is the electrons moving in permanent orbits that are mainly effective. Moreover for these the second sum on the right is negligibly small, so that if we use

$$\mathbf{I}dv = \frac{1}{2c}\Sigma e\,[\mathbf{r}_e\dot{\mathbf{r}}_e]$$

the vector potential of the field of the element is practically

$$\mathbf{A} = [\operatorname{curl}\mathbf{I}]\frac{dv}{r}$$

which is just the same as if the element were magnetically polarised to intensity **I**.

This result suggests an assumption, which we have already hinted, that all magnetism is the result of small whirling motions of the electrons contained in the ultimate molecules or molecular groups of matter, and it has in fact been interpreted in this way by Larmor and Lorentz. The connection of the idea with Weber's theory of magnetism is obvious.

Although this view of the matter is now generally accepted as providing a sufficient account of the general phenomena of magnetism, it cannot yet be regarded as definitely established in fact. We shall however generally assume that it is a sufficient theoretical basis for our future discussions.

CHAPTER XIII

THE REFLEXION, REFRACTION AND CONDUCTION OF ELECTRIC WAVES

577. On the reflexion and refraction of electric waves at an interface between two different dielectric media. We have so far merely treated analytically of the propagation of electromagnetic waves in homogeneous bodies. But what happens when such a wave reaches the boundary between two different media? The interface between the two media then becomes the seat of fresh sources of disturbance of a fictitious nature, of course, as they do not bring in any new energy. Thence arises the new effect, viz. the reflexion: not the whole but only a part of the wave disturbance enters the new medium, the rest is thrown back and forms the reflected beam. The disturbance outside the second medium is thus due to the superposition of the primary and secondary waves and if there be just the one interface, this state of things continues. This secondary wave may itself be very complex because the disturbance going into the second medium may reach the interface again at other parts and there suffer reflexion and transmission anew. It is however sufficient to treat the simpler case, the conditions for which are secured by making the second medium extend indefinitely beyond an infinite plane face which it has in common with the first medium.

The general equations of § 520 combined with the conditions which must hold between the fields on either side of the boundary at the surface will enable us to determine the problem in this case. For the present we shall however slightly modify the notation of the previous chapter and put the z-axis along the normal to the boundary surface between the two media, which we shall assume to be a surface of indefinite extent lying in the (x, y) plane of coordinates. The plane of incidence, i.e. the plane containing the normal to the surface and the direction of propagation of the plane wave falling on the surface, is taken as the (y, z) plane. The direction of propagation makes an angle α with the normal; this is the *angle of incidence*, to adopt an optical term. From the part of this surface where this wave-train falls there arise two other plane wave trains, one of which goes on into the second medium in a direction making an angle β with the normal; the other is reflected back on the other side of the normal at an angle α'.

578. We shall confine ourselves to the case when the two media are absolutely non-conducting and non-magnetic. We first treat the case when the magnetic oscillations are perpendicular to the plane of incidence. The

electric force is then in this plane, so that $\mathbf{E}_y$, $\mathbf{E}_z$, $\mathbf{H}_x$ are the only components of the two vectors differing from zero; the dynamical equations are thus

$$\frac{\epsilon}{c}\frac{d\mathbf{E}_y}{dt} = \frac{d\mathbf{H}_x}{dz}, \qquad \frac{\epsilon}{c}\frac{d\mathbf{E}_z}{dt} = -\frac{d\mathbf{H}_x}{dy},$$

$$\frac{1}{c}\frac{d\mathbf{H}_x}{dt} = \frac{\partial \mathbf{E}_z}{\partial y} - \frac{\partial \mathbf{E}_y}{\partial z}.$$

We now assume that the incident wave is a simple periodic train specified by

$$(\mathbf{E}_{i_y}, \mathbf{E}_{i_z}, \mathbf{H}_{i_x}) = (A_i \cos \alpha, \ -A_i \sin \alpha, \ -A_i \sqrt{\epsilon})\, e^{i(nt - b(y \sin \alpha + z \cos \alpha))},$$

which must satisfy the equation

$$\nabla^2 \mathbf{E} = \frac{\epsilon}{c^2}\frac{d^2\mathbf{E}}{dt^2},$$

so that $$b^2 = \frac{n^2\epsilon}{c^2}, \qquad \frac{n}{b} = \frac{c}{\sqrt{\epsilon}}.$$

For the reflected wave we must have

$$(\mathbf{E}_{r_y}, \mathbf{E}_{r_z}, \mathbf{H}_{r_x}) = (A_r \cos \alpha', \ -A_r \sin \alpha', \ -A_r \sqrt{\epsilon})\, e^{i(nt - b'(y \sin \alpha' - z \cos \alpha'))},$$

which also satisfies the above equation so that

$$b'^2 = \frac{n^2\epsilon}{c^2} = b^2.$$

Moreover as the two waves (incident and reflected) must agree on the boundary as regards variation along it we must have $\alpha = \alpha'$.

For the refracted wave we have

$$(E_{s_y}, E_{s_z}, H_{s_x}) = (A_s \cos \beta, \ -A_s \sin \beta, \ -H_s \sqrt{\epsilon})\, e^{i(nt - b_1(y \sin \beta + z \cos \beta))}.$$

Certain assumptions are of course already involved in the forms here adopted. The functions chosen all satisfy the fundamental electromagnetic equations between themselves. Moreover we have assumed the frequency of the vibration for each wave to be the same; but this must be so because we are dealing with forced vibrations, and so the frequency n remains constant under all conditions. We have taken the polarisation in each beam to be similar, this is of course necessitated by the general theory, particularly as regards the boundary conditions which compares corresponding components of the fields on either side of the surface.

579. The boundary conditions state that the tangential components of the electric and magnetic force vectors in the total field must be continuous as we go through the surface; but in order that these conditions may be satisfied, the exponential function in each beam must be the same for $z = 0$. This requires

(i) as above $\alpha = \alpha'$; the angles of incidence and reflexion are equal and

(ii) also $$b_1 \sin \beta = b \sin \alpha,$$

or $$\frac{b_1}{b}=\frac{\sin\alpha}{\sin\beta}=\sqrt{\frac{\epsilon_1}{\epsilon}}=\frac{c_1}{c},$$

c and c_1 being the velocities of propagation in the two media. This is the exact analogue of Snell's law in optics: the relation

$$\frac{c_1}{c}=\sqrt{\frac{\epsilon_1}{\epsilon}},$$

is Maxwell's relation between the dielectric constants of the two media and the index of refraction for light between them (c_1/c).

With these the boundary conditions imply that

$$(A_i-A_r)\cos\alpha=A_s\cos\beta,$$

$$A_i\sqrt{\epsilon}+A_r\sqrt{\epsilon}=A_s\sqrt{\epsilon_1},$$

whence $$A_r\left(\frac{\cos\alpha}{\cos\beta}+\frac{\sin\beta}{\sin\alpha}\right)=A_i\left(\frac{\cos\alpha}{\cos\beta}-\frac{\sin\beta}{\sin\alpha}\right)$$

or $$A_r=\frac{\tan(\alpha-\beta)}{\tan(\alpha+\beta)}A_i,$$

which is another formula of theoretical optics; it is Fresnel's formula* for the reflected amplitude.

580. If $$\frac{\cos\alpha}{\cos\beta}=\frac{\sin\beta}{\sin\alpha}\quad\text{or}\quad\sin 2\alpha=\sin 2\beta,$$

then $A_r=0$ and for this particular direction there is no reflected light; it is when

$$2\alpha=\pi-2\beta\quad\text{or}\quad\alpha+\beta=\frac{\pi}{2},$$

i.e. when the refracted ray is perpendicular to the reflected wave direction. The angle α which satisfies this condition is called the *polarisation angle*; it is such that

$$\sqrt{\frac{\epsilon_1}{\epsilon}}=\frac{\sin\alpha}{\sin\beta}=\tan\alpha.$$

When the angle of incidence is $\tan^{-1}\sqrt{\frac{\epsilon_1}{\epsilon}}$ and the magnetic force is perpendicular to the plane of incidence all the light passes through into the second medium, none whatever being reflected.

581. If the magnetic oscillation were in the plane of incidence then we should use

$$(\mathbf{E}_{i_x},\ \mathbf{H}_{i_y},\ \mathbf{H}_{i_z})=(A_i,\quad A_i\sqrt{\epsilon}\cos\alpha,\ -A_i\sqrt{\epsilon}\sin\alpha)\,e^{int-ib\,(y\sin\alpha+z\cos\alpha)},$$

$$(\mathbf{E}_{r_x},\ \mathbf{H}_{r_y},\ \mathbf{H}_{r_z})=(A_r,\ -A_r\sqrt{\epsilon}\cos\alpha,\ -A_r\sqrt{\epsilon}\sin\alpha)\,e^{int-ib\,(y\sin\alpha-z\cos\alpha)},$$

and

$$(\mathbf{E}_{s_x},\ \mathbf{H}_{s_y},\ \mathbf{H}_{s_z})=(A_s,\quad A_s\sqrt{\epsilon_1}\cos\beta,\ -A_s\sqrt{\epsilon_1}\sin\beta)\,e^{int-ib_1(y\sin\beta-z\cos\beta)}.$$

* *Mém. de l'Acad.* XI. (1832), p. 393; *Ann. chim.* XLVI. 1843. Cf. *Œuvres*, I. p. 767

The boundary conditions again give

$$b \sin \alpha = b_1 \sin \beta,$$

so that Snell's law is still true. Moreover

$$A_i + A_r = A_s,$$

and
$$\sqrt{\epsilon}\,(A_i - A_r) \cos \alpha = A_s \sqrt{\epsilon_1} \cos \beta,$$

whence again we deduce Fresnel's second formula for the reflected amplitude

$$A_r = -A_i \frac{\sin \alpha - \beta}{\sin \alpha + \beta}.$$

Thus in the reflected wave the electric force at the surface is reversed, the magnetic one however retains its direction.

These formulae of Fresnel have been shown to be in good agreement with experiment if in the first case the plane of polarisation is perpendicular and in the second parallel to the plane of incidence. The magnetic force must therefore lie in the plane of polarisation and the electric force perpendicular to it. Although this theory thus determines this question of the plane of polarisation quite definitely it still leaves open the question as to which is the actual working force in the propagation of the wave. According to the modern view of these things the electric force appears as the most probable direct working agent in the electromagnetic propagation.

If in this second case $\alpha = 0$ then $\beta = 0$ also and we get then, neglecting the sign

$$A_i \frac{\alpha - \beta}{\alpha + \beta} = A_r,$$

and since then
$$\frac{c_1}{c} = \frac{\alpha}{\beta},$$

this gives
$$A_r = A_i \frac{c - c}{c_1 + c},$$

a formula which is also true under the same circumstances in the first case treated.

In this case it is impossible to make the reflected amplitude zero because

$$\frac{\sin(\alpha - \beta)}{\sin(\alpha + \beta)}$$

can never be zero.

582. If $\epsilon_1 < \epsilon$ then

$$\frac{\sin \alpha}{\sin \beta} < 1,$$

so that when α gets beyond a certain limit the equation

$$\sin \alpha = \sqrt{\frac{\epsilon_1}{\epsilon}} - \beta$$

can no longer be satisfied by any real value of β, since for such

$$\sin \beta \leq 1.$$

If however we put

$$\sin\beta = p, \qquad \cos\beta = \pm\sqrt{1-p^2},$$

then for such cases as mentioned

$$\cos\beta = \pm\, i\sqrt{p^2-1}, \qquad p > 1.$$

The exponential factor which occurs in both cases in the refracted wave is then of the form

$$e^{int - ib_1(py \pm iz\sqrt{p^2-1})},$$

and we must take the lower sign because the quantities would otherwise increase with increasing z. The oscillations in the second medium thus decrease rapidly on account of the factor

$$e^{-z\sqrt{p^2-1}}.$$

583. If the magnetic oscillations are parallel to the plane of incidence then the boundary conditions give in this case

$$(A_i - A_r)\cos\alpha = -A_s i\sqrt{p^2-1},$$

$$(A_i + A_r)\sqrt{\epsilon} = A_s\sqrt{\epsilon_1},$$

or $$A_i\left(\cos\alpha + i\sqrt{\frac{\epsilon}{\epsilon_1}}\sqrt{p^2-1}\right) = A_r\left(\cos\alpha - i\sqrt{\frac{\epsilon}{\epsilon_1}}\sqrt{p^2-1}\right).$$

If now we put $$\frac{\cos\alpha}{\sqrt{\epsilon}} = \rho\cos\theta, \qquad \frac{\sqrt{p^2-1}}{\sqrt{\epsilon_1}} = \rho\sin\theta,$$

then $$\tan\theta = \sqrt{\frac{\epsilon}{\epsilon_1}}\cdot\frac{\sqrt{p^2-1}}{\cos\alpha}, \qquad \rho = \sqrt{\frac{\cos^2\alpha}{\sqrt{\epsilon}} + \frac{p^2-1}{\sqrt{\epsilon_1}}},$$

and then $$A_i e^{i\theta} = A_r e^{-i\theta},$$

or $$A_r = A_i e^{2i\theta}.$$

Thus the amplitude of the reflected light is equal to that of the incident light, although the one is out of phase with the other by 2θ where

$$\theta = \tan^{-1}\sqrt{\frac{\epsilon}{\epsilon_1}}\sqrt{\frac{p^2-1}{\cos^2\alpha}}.$$

584. If on the other hand the magnetic oscillation is in the plane of incidence

$$A_i + A_r = A_s,$$

$$(\sqrt{\epsilon}\,A_i - \sqrt{\epsilon_1}\,A_r)\cos\alpha = -A_s i\sqrt{p^2-1}\sqrt{\epsilon_1}$$

$$= -(A_i + A_r)\,i\sqrt{p^2-1}\sqrt{\epsilon_1},$$

here we put

$$\sqrt{\epsilon}\cos\alpha = \rho\cos\theta_1\sqrt{\epsilon_1}, \quad \sqrt{p^2-1} = \rho\sin\theta_1,$$

and have then again $$A_r = A_i e^{2i\theta_1},$$

where $$\theta_1 = \tan^{-1}\frac{\sqrt{p^2-1}\sqrt{\epsilon_1}}{\sqrt{\epsilon}\cos\alpha}.$$

The difference of phase in the two cases is

$$2(\theta - \theta_1),$$

but the combination of oscillations which are perpendicular to one another with a definite difference of phase results in an elliptically polarised beam. Thus in the case of total reflexion of a general plane polarised train of waves the reflected wave is elliptically polarised.

These are the principal results for the passage of a beam across the interface between two dielectric media: they are in all cases identical with the well-known results of physical optics and in this connection have been fully verified by experiment; in the more difficult circumstances with the much longer electric waves the tests as mentioned above are entirely favourable in a general way to the theory, although the same amount of accuracy is not attainable.

585. On the reflexion of electric waves at a conducting surface. In the previous paragraph we have treated of the reflexion and refraction of electric or light waves at the plane surface separating two different dielectric media: the case when one or both media have conducting properties can be similarly discussed and leads to similar conclusions, the only essential difference being due to the damping action now introduced. We shall therefore content ourselves by giving the analysis for the simplest case of this kind, which is the one from which Hertz tried to get standing waves. We consider the incidence normally of a plain wave travelling in a dielectric medium on to an infinite block of metal with a plane face.

We may take the incident wave to be specified by

$$(\mathbf{E}_{i_x}, \mathbf{H}_{i_y}) = (\mathbf{A}_i, -\mathbf{A}_i\sqrt{\epsilon})\, e^{i(nt-bz)},$$

in which

$$b^2 = \frac{n^2\epsilon^2}{c^2},$$

and then the reflected and refracted beams are determined by

$$(\mathbf{E}_{r_x}, \mathbf{H}_{r_y}) = (A_r, A_r\sqrt{\epsilon})\, e^{i(nt+bz)}$$

and

$$(\mathbf{E}_{s_x}, \mathbf{H}_{s_y}) = \left(A_s, -A_s\frac{b_1}{n}\right) e^{i(nt+b_1z)},$$

wherein

$$4\pi\frac{in\sigma}{c^2} = -b_1^2.$$

provided of course the displacement current can be neglected,

$$b_1^2 = -i4\pi\frac{n\sigma}{c^2}, \qquad b_1 = -\frac{1-i}{2}\sqrt{4\pi\frac{n\sigma}{c^2}}.$$

The wave in the metal has therefore the exponential factor

$$e^{-z\sqrt{\pi\frac{n\sigma}{c^2}}}$$

as a damping factor to its amplitude. The amplitude therefore diminishes rapidly, as previously explained, as we go down in the metal.

586. If we write
$$a = \sqrt{\pi \frac{n\sigma}{c^2}},$$
then $b_1 = -a(1-i)$ and the exponential factor for the wave in the metal is
$$e^{-az}\, e^{i(nt-az)}.$$
The boundary conditions give besides that
$$A_i + A_r = A_s,$$
$$A_i - A_r = \frac{b_1}{b} A_s,$$
so that
$$A_s = \frac{2b}{b+b_1} A_i,$$
$$A_r = \frac{b-b_1}{b+b_1} A_i.$$
Thus the amplitudes of the various beams are in the ratios determined by
$$\frac{A_s^2}{4b^2} = \frac{A_r^2}{(b-a)^2+a^2} = \frac{A_i^2}{(b+a)^2+a^2},$$
but they are different in relative phase determined by the imaginary parts of the amplitudes as defined. The total electric force intensity in the field just outside the metal is
$$(A_r + A_i)\, e^{i(nt)},$$
or
$$\frac{2b}{b+b_1} A_i e^{i(nt)},$$
and in the general case of short waves is very small because b/b_1 is in general small if the conductivity is big. We may even approximate and put this equal to
$$\frac{2b}{b_1} A_i e^{i(nt+bz)}.$$
The same form holds for the electric force in the internal field with the proper exponent
$$\begin{aligned}
\mathbf{E}_{s_x} &= \frac{2b}{b_1} A_i e^{-az}\, e^{i(nt-az)} \\
&= -\frac{2b}{a(1-i)} A_i e^{-az}\, e^{i(nt-az)} \\
&= -\sqrt{2}\,\frac{b}{a} A_i e^{-az}\, e^{i\left(nt-az+\frac{\pi}{4}\right)} \\
&= -2\sqrt{\frac{2n}{\sigma}}\,\epsilon A_i e^{-az}\, e^{i\left(nt-az+\frac{\pi}{4}\right)}.
\end{aligned}$$

[The heat developed in the metal in such cases reckoned per unit area is
$$H = \sigma \int_0^{+\infty} E_{s_x}{}^2 dx = 8n\epsilon A_i{}^2 \int_0^{+\infty} e^{2az} \cos^2\left(nt + az + \frac{\pi}{4}\right) dz,$$

where we have chosen the real part of the expression for E_s as representing the field. On the average this reduces to

$$H = \frac{2A_i^2 n\epsilon}{a}.$$

The energy degraded into heat thus forms a negligible fraction of that incident on the conductor. Thus in the general case the waves are turned back without sensible loss by degradation, and for an ideal good conductor the surface layer is at a node of the electric force. There is a superficial current in the conductor but there is no sensible electric resistance, the small electric force near the node establishing the necessary current without production of heat.]

587. It can also be seen that it is impossible to reduce the amplitude of the reflected beam to zero: in other words it is impossible to have a perfect absorber or absolutely black body bounded by an abrupt interface; but that does not preclude the possibility of a molecular constitution of the interface, of a loose and gradual kind such as may exist in lamp-black for example, which might absorb light as soft porous bodies absorb sound*.

We may notice also that the total field in the dielectric is completely specified as regards its electric part by the force component

$$A_r e^{i(nt+bz)} + A_i e^{i(nt-bz)},$$

which to the first order in $\frac{b}{b_1}$ is the same as

$$- A_i e^{int} (e^{ibz} - e^{-ibz})$$
$$= - A_i e^{int} \sin bz,$$

so that in this case the characteristic stationary wave condition is produced; but of course we have assumed that there is no damping so that the amplitude A_i of the incident wave remains constant in time. If this were not the case the solution would not be of the simple form here specified.

588. The experimental investigation of the optical properties of metals has led to results of a far reaching importance in the general theory of metallic conduction. The theory of the measurements usually made is rather too complicated for us to enter into in this work, but a few of the results may suffice to explain the points which they bring out. The indices of refraction of metals have been determined by Kundt by using very thin metallic prisms of small angle, and although the method is not capable of any great exactness his results agree well with those deduced by more elaborate arrangements: the values of the following three metals are quoted:

Silver	·18,
Gold	·37,
Steel	2·41.

* Rayleigh, *Theory of Sound*, Ed. 2, §§ 350–1.

The reflecting powers of metals, i.e. the ratio of the reflected to incident energy for normal incidence has been determined by Hagen and Rubens* : they find for

Silver	...	...	95·3 %,
Gold	...	...	85·1 %,
Steel	...	...	58·5 %,

and from these the coefficient of absorption is determined from the formula

$$R = \frac{n^2 + \kappa^2 + 1 - 2n}{n^2 + \kappa^2 + 1 + 2n},$$

so that for the above metals κ is given by the values

Silver ...	...	...	3·67,
Gold ...	...	...	2·86,
Steel ...	...	...	3·4.

But the dielectric constant of the metal on the ordinary definition is given by

$$\epsilon = n^2 - \kappa^2,$$

so that in each of the above cases it will be negative; but a negative dielectric constant has no sense. The clue to the difficulty is provided in a further result obtained by Hagen and Rubens† : the absorption coefficient of the metal is determined by the formula

$$\kappa = \sqrt{\frac{\pi n \sigma}{c^2}},$$

and a knowledge of κ would enable us to determine σ: in this way it was found that for long heat waves the absorption is completely determined by the ordinary ohmic resistance but that for shorter waves the value of σ decreased as a function of the wave length; this points to the conclusion that the constitutional relation of the theory, viz. the relation between the electric force and electric conduction current, has to be modified for application to rapidly alternating fields.

589. To find what is to replace this relation we must refer back to our previous general discussion of the mechanism of metallic conduction. It was there found possible to interpret the phenomena of conduction in metals in terms of the ordinary conceptions of the electron theory and a formula for the conductivity of the metal was obtained by statistical considerations involving the ordinary electron constants of the metal. In this deduction we assumed however that the applied electric field driving the current is stationary. If the field is varying the deduction must be slightly modified but the general principles of the analysis remain: we shall therefore content ourselves by

* *Ann. d. Phys.* I. (1900), p. 352; VIII. (1902), p. 1.

† *Ann. d. Phys.* XI. (1903), p. 873.

just indicating the essential steps of the analysis which differ from those adopted above in the case where the field is varying harmonically with period $\frac{2\pi}{p}$*.

The function which determines the distribution of motions among the electrons satisfies quite generally the differential equation

$$\mathbf{F}_x \frac{\partial f}{\partial \xi} + \mathbf{F}_y \frac{\partial f}{\partial \eta} + \mathbf{F}_z \frac{\partial f}{\partial \zeta} + \xi \frac{\partial f}{\partial x} + \dots + \frac{\partial f}{\partial t} + \frac{f}{\tau_m} = \frac{f_0}{\tau_m},$$

where the notation is exactly as previously and

$$\tau_m = \frac{l_m}{u}.$$

With the approximation adopted in the previous case and on the assumption that the physical conditions of the metal are uniform throughout its volume we have

$$f = f_0 \left[1 + 2q \int_0^\infty e^{-\frac{\tau}{\tau_m}} \frac{d\tau}{\tau_m} \int_{t_1=t-\tau}^{t_1=t} (\mathbf{F}_x\xi + \mathbf{F}_y\eta + \mathbf{F}_z\zeta)_1 dt_1\right],$$

the suffix 1 serving to indicate the values of the functions at the time t_1. For the present purposes it will be sufficient to use

$$\mathbf{F}_x = \frac{eE_0}{m} e^{ipt}, \qquad \mathbf{F}_y = \mathbf{F}_z = 0,$$

so that

$$f = f_0 \left[1 + \frac{2qeE}{m} \frac{e^{ipt}\tau_m \xi}{1 - ip\tau_m}\right].$$

590. It follows then by exactly the same argument that the electric current density is given by

$$C = 2\sqrt{\frac{2}{3\pi}} \frac{Ne^2 l_m}{mu_m} E_0 \int e^{ipt} \left[\int_0^\infty \frac{2^q u e^{-qu^2} du}{1 - i\frac{pl_m}{u}}\right]$$

$$= 2\sqrt{\frac{2}{3\pi}} \frac{Ne^2 l_m}{mu_m} E_0 e^{ipt} \int_0^\infty \frac{\left(1 + i\frac{pl_m q^{\frac{1}{2}}}{z^{\frac{1}{2}}}\right)}{1 + \frac{p^2 l_m{}^2 q}{z}} e^{-z} dz$$

$$= \sigma_1 E_0 e^{ipt} + ip\sigma_2 E_0 e^{ipt},$$

where

$$\sigma_1 = \sigma_0 \int_0^\infty \frac{ze^{-z} dz}{1 + \frac{p^2 l_m{}^2 q}{z}},$$

$$\sigma_2 = \sigma_0 l_m q^{\frac{1}{2}} \int_0^\infty \frac{\sqrt{z}e^{-z} dz}{1 + \frac{p^2 l_m{}^2 q}{z}}.$$

* Cf. Livens, *Phil. Mag.* xxx. (1915), p. 112.

Thus the relation between the current density and electric force is

$$C = \sigma_1 E + \sigma_2 \frac{dE}{dt}.$$

And this is the more general relation of which we have spoken.

If we use this relation instead of the simpler one given above we shall find in exactly the same way that

$$\epsilon = n^2 - \kappa^2 = -\sigma_2,$$

$$\kappa = \frac{\sigma_1}{p},$$

and now we have a completely effective account of the facts observed by Kundt and Hagen and Rubens. The inertia of the free electrons provides a contribution to the dielectric constant of a negative amount and which may therefore in a real case be the predominating part. Moreover the more correct formula for the conductivity is one that decreases with the wave length from the ordinary Ohmic form to which it reduces in the case of $p = 0$.

591. The diffraction of electric waves. We have so far treated of the reflexion and refraction of electric waves only at the indefinitely extended plane interface between two different media. The cases where the bounding surface is neither plane nor of indefinite extent are of equal importance both from the theoretical and from the practical point of view and, in spite of the fact that the analysis involved in it is much more than ordinarily complex it seems desirable to indicate the details of one or two problems. The only tractable cases are of course those in which the bounding interface between the different media assumes some simple geometrical shape: we shall here deal only with those cases in which it is either in the shape of a circular cylinder of indefinite extent in both directions in its axis or of a sphere. In each case the medium outside the enclosed surface will be assumed to be free aether and the medium inside a simple homogeneous isotropic dielectric with constant ϵ.

The first case to be examined is that in which a simple harmonic plane polarised train of waves in the aether is incident normally on a circular cylinder composed of the simple dielectric medium*. Problems of this type naturally divide themselves into two distinct classes, according as the direction of the electric force vector in the incident radiation is parallel or perpendicular to the axis of the cylinder; any other case can be made up of a combination of two such cases. The analysis of the two types of problem is similar and we examine the first one in detail.

* Rayleigh, *Phil. Mag.* XII. (1881), p. 81; Seitz, *Ann. d. Phys.* XVI. (1905), p. 746; Ignatowsky, *Ann. d. Phys.* XVIII. (1906), p. 495. Debye, *Phys. Zeitschr.* (1908), p. 775; Schaefer, *Ann. de Phys.* XXXI. (1910), p. 462; Nicholson, *Proc. L. M. S.* (2), XI. p. 104.

592. The field is referred to a cylindrical polar coordinate frame (r, θ, z) with the z-axis in the direction of the cylinder axis and the polar plane $(\theta = 0)$ in the direction of the propagation of the incident waves. The general equations of the field are then of the type

$$\frac{4\pi r\mathbf{C}_r}{c} = \frac{\partial \mathbf{H}_z}{\partial \theta} - \frac{\partial}{\partial z}(r\mathbf{H}_\theta), \qquad -\frac{r}{c}\frac{d\mathbf{H}_r}{dt} = \frac{\partial \mathbf{E}_z}{\partial \theta} - \frac{\partial}{\partial z}(r\mathbf{E}_\theta),$$

$$\frac{4\pi\mathbf{C}_\theta}{c} = \frac{\partial \mathbf{H}_r}{\partial z} - \frac{\partial \mathbf{H}_z}{\partial r}, \qquad -\frac{1}{c}\frac{d\mathbf{H}_\theta}{dt} = \frac{\partial \mathbf{E}_r}{\partial z} - \frac{\partial \mathbf{E}_z}{\partial r},$$

$$\frac{4\pi r\mathbf{C}_z}{c} = \frac{\partial}{\partial r}(r\mathbf{H}_\theta) - \frac{\partial \mathbf{H}_r}{\partial \theta}, \qquad -\frac{r}{c}\frac{d\mathbf{H}_z}{dt} = \frac{\partial}{\partial r}(r\mathbf{E}_\theta) - \frac{\partial \mathbf{E}_r}{\partial \theta},$$

where, inside the cylinder, $$4\pi\mathbf{C} = \epsilon\frac{d\mathbf{E}}{dt},$$

and outside $$4\pi\mathbf{C} = \frac{d\mathbf{E}}{dt}.$$

If the electric force in the incident wave is parallel to the axis of the cylinder it is natural to try a general solution of the field equations which satisfies the same condition; we put

$$\mathbf{E}_r = \mathbf{E}_\theta = 0$$

and then it immediately follows from the third of the second set of equations that

$$\mathbf{H}_z = 0$$

in addition. The general equations thus reduce to

$$-\frac{r}{c}\frac{d\mathbf{H}_r}{dt} = \frac{\partial \mathbf{E}_z}{\partial \theta}, \qquad -\frac{1}{c}\frac{d\mathbf{H}_\theta}{dt} = -\frac{\partial \mathbf{E}_z}{\partial r}$$

with $$\frac{4\pi r}{c}\mathbf{C}_z = \frac{\partial}{\partial r}(r\mathbf{H}_\theta) - \frac{\partial \mathbf{H}_r}{\partial \theta}.$$

It follows that $$\frac{4\pi}{c^2}\frac{d\mathbf{C}_z}{dt} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathbf{E}_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \mathbf{E}_z}{\partial \theta^2}$$

where, inside the cylinder $$4\pi\frac{d\mathbf{C}_z}{dt} = \epsilon\frac{d^2\mathbf{E}_z}{dt^2},$$

whilst outside the same relation holds with $\epsilon = 1$.

593. The main problem is now, just as always before, to solve the characteristic equations for the interior and exterior fields and to fit the solutions up at the boundary interface and at a large distance away. The necessity for agreement at the interface suggests that the type of functions are the same in the two fields, while the types themselves are determined by the fact that the external field must agree at very distant points with the undisturbed field.

The undisturbed field is that of a simple harmonic wave train in which the vibrations are plane polarised in the principal axial plane of reference. Thus in this field we may put

$$\mathbf{E}_z = Ee^{ip(ct - r\cos\theta)}$$

and the total external field must agree with this value at a large distance from the cylinder. The form of this function is not however suitable to the problem in hand and consequently it must be expanded in the more appropriate equivalent form*

$$\mathbf{E}_z = Ee^{ipct}\left[J_0(pr) + 2i^n \sum_1^\infty J_n(pr)\cos n\theta\right]$$

wherein each term is known to be a suitable normal solution of the fundamental characteristic equation. J_n is Bessel's function of the first kind and nth order.

594. The method is now obvious. The general type solution of the characteristic equation of the external field, viz.,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathbf{E}_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \mathbf{E}_z}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_z}{\partial t^2}$$

is $$e^{ipct}[A_n J_n(pr)\cos n\theta + B_n K_n(pr)\cos n\theta]$$

where K_n is Bessel's function of the second kind which is regular at infinity.

The same form of solution applies to the internal field but with $p_1 = \sqrt{\epsilon}p$ in place of p.

Now the total conditions in the field must be regular except for the imposed conditions of the incident radiation field; it follows that the internal field can only involve Bessel functions of the first kind, while the additional part of the external field can only involve those of the second kind. We may therefore generally assume for the external field the form

$$\mathbf{E}_z = Ee^{ipct}\left[J_0(pr) + A_0 K_0(pr) + 2i^n \sum_{n=1}^\infty \{J_n(pr) + A_n K_n(pr)\}\cos n\theta\right]$$

whilst for the internal field we assume that

$$\mathbf{E}_z = Ee^{ipct}\left[B_0 J_0(p_1 r) + 2i^n \sum_{n=1}^\infty B_n J_n(p_1 r)\cos n\theta\right];$$

and these two fields must now be fitted up at the surface of the cylinder whose equation may be taken to be

$$r = a.$$

The surface conditions are that the tangential electric and magnetic forces are continuous across the boundary. This means that $\mathbf{E}_z$ and $\frac{\partial \mathbf{E}_z}{\partial r}$ are continuous at $r = a$ or that

$$J_n(pa) + A_n K_n(pa) = B_n J_n(p_1 a)$$
$$J_n'(pa) + A_n K_n'(pa) = B_n J_n'(p_1 a)$$

* Heine, *Kugelfunktionen*, t. I. p. 82; Whittaker, *Modern Analysis* (2nd Ed. 1915), p. 351.

for all values of n. These equations determine all the unknown coefficients of the field functions so that the whole circumstances in the fields are at least theoretically determinate. In practice however the coefficients thus deduced are much too complex to be of any real service so that resort has to be had to approximate values, depending on the relative magnitudes of the wave-length of the incident radiation and of the radius of the cylinder. We consider here only the case when the radius of the cylinder is so small compared with the wave length that all terms above $(ap)^2$ can be neglected.

595. To the second order of approximation when (pa) is small

$$J_0 = 1 - \frac{p^2a^2}{2}, \qquad K_0 = \log \frac{2}{\gamma pa}.$$

The equations to determine A_0 and B_0 are then

$$1 + A_0 \log \frac{2}{\gamma pa} = B_0$$

$$\frac{pa}{2} - A_0 \cdot \frac{1}{pa} = \epsilon B_0 \frac{pa}{2}.$$

Thus

$$A_0 = \frac{\frac{\epsilon - 1}{2} p^2a^2}{1 - \epsilon \frac{p^2a^2}{2} \log \frac{2}{\gamma pa}}$$

$$= \frac{\epsilon - 1}{2} p^2a^2$$

approximately, whilst

$$B_0 = 1 - \frac{1 - \epsilon}{2} p^2a \; \log \frac{2}{\gamma pa}.$$

Thus to this order of approximation the internal field is exactly identical, except as regards velocity of propagation, with the original field reduced in the ratio

$$1 : 1 - (1 - \epsilon) \frac{p^2a^2}{2} \log \frac{2}{\gamma pa}.$$

The external field consists mainly of the original field, on which however is superposed the field of the scattered radiation in which

$$\mathbf{E}_z = (\epsilon - 1) \frac{p^2a^2}{2} EK_0 (pr)\, e^{ipct}.$$

The field of this scattered radiation is, to the present order of approximation, symmetrical round the axes of the cylinder.

596. The case in which the electric force in the initial radiation field is perpendicular to the axis of the cylinder is similarly treated; but in approximating to the values of the coefficients it is now necessary to proceed to the next higher power of the product (ap) and also to include the next

term in the expansion of the field functions in their normal form. In this case there is only one component of the magnetic force, viz. that parallel to the axes of the cylinder and it is most convenient now to regard this component as the fundamental vector of the theory. We now find just as above that the external field is determined by the original undisturbed field on which is superposed the field of the scattered radiation in which the magnetic force is

$$\mathbf{H}_z = a^2 p^2 \frac{\epsilon - 1}{\epsilon + 1} e^{ipct} HK_1(pr) \cos\theta$$

and the electric force analogously.

In this case the distribution of the scattered radiation is not symmetrical round the cylinder. The intensity in it is greatest in the positive and negative directions of propagation and it vanishes entirely in the perpendicular directions. The scattered radiation thus appears to be partially polarised.

597. The case* in which the radiation is incident on a sphere is quite similarly treated but the analysis is necessarily more complicated even than that of the cylindrical cases. We shall give the main outlines for the case when the sphere is composed of uniform dielectric material with constant ϵ.

In this case† spherical polar coordinates are used, with the polar axis in the direction in which the radiation is incident on the sphere, and the origin at the centre of the sphere. The general field equations are then of the type

$$\frac{4\pi r^2 \sin\theta}{c} \mathbf{C}_r = \frac{\partial}{\partial\theta}(r \sin\theta \mathbf{H}_\phi) - \frac{\partial}{\partial\phi}(r\mathbf{H}_\theta),$$

$$\frac{4\pi r \sin\theta}{c} \mathbf{C}_\theta = \frac{\partial \mathbf{H}_r}{d\phi} - \frac{\partial}{\partial r}(r \sin\theta \mathbf{H}_\phi),$$

$$\frac{4\pi r}{c} \mathbf{C}_\phi = \frac{\partial}{\partial r}(r\mathbf{H}_\theta) - \frac{\partial \mathbf{H}_r}{\partial\theta},$$

and

$$-\frac{r^2 \sin\theta}{c} \frac{d\mathbf{H}_r}{dt} = \frac{\partial}{\partial\theta}(r \sin\theta \mathbf{E}_\phi) - \frac{\partial}{\partial\phi}(r\mathbf{E}_\theta),$$

$$-\frac{r \sin\theta}{c} \frac{d\mathbf{H}_\theta}{dt} = \frac{\partial \mathbf{E}_r}{\partial\phi} - \frac{\partial}{\partial r}(r \sin\theta \mathbf{E}_\phi),$$

$$-\frac{r}{c} \frac{d\mathbf{H}_\phi}{dt} = \frac{\partial}{\partial r}(r\mathbf{E}_\theta) - \frac{\partial \mathbf{E}_r}{\partial\theta},$$

wherein $$\mathbf{C} = \frac{\epsilon}{4\pi} \frac{d\mathbf{E}}{dt}$$

inside the sphere and $$\mathbf{C} = \frac{1}{4\pi} \frac{d\mathbf{E}}{dt}$$

outside.

* Rayleigh, *Phil. Mag.* XLI. (1871), pp. 107, 274, 447; XII (1881), p. 81; *Collected Papers*, I. pp. 87, 104, 518.

† Rayleigh, *Phil. Mag.* XLIV. (1897), pp. 28–52; *Collected Papers*, IV. p. 321; *Proc. Roy. Soc.* LXXXIV. (1910), p. 25; XC. (1914), p. 219. Cf. also Love, *Proc. L. M. S.* XXX. (1899), p. 308.

598. Again the solutions are divided into two classes: in the first the radial component of the electric force vanishes and in the second the radial component of the magnetic force vanishes. If these two cases are respectively denoted by 1 and 2 then it is easy to verify by examining each case separately and superposing the results that the general solution of the equations for internal space may be written in the form

$$\mathbf{E}_r = \frac{\partial^2\Pi_2}{\partial r^2} - \frac{\epsilon}{c^2}\frac{\partial^2\Pi_2}{\partial t^2}, \qquad \mathbf{H}_r = \frac{\partial^2\Pi_1}{\partial r^2} - \frac{\epsilon}{c^2}\frac{\partial^2\Pi_1}{\partial t^2},$$

$$\mathbf{E}_\theta = \frac{1}{r}\frac{\partial^2\Pi_2}{\partial r\partial\theta} - \frac{1}{cr\sin\theta}\frac{\partial^2\Pi_1}{\partial\phi\,\partial t}, \qquad \mathbf{H}_\theta = \frac{1}{r}\frac{\partial^2\Pi_1}{\partial r\,.\,\partial\theta} + \frac{\epsilon}{cr\sin\theta}\frac{\partial^2\Pi_2}{\partial\phi\partial t},$$

$$\mathbf{E}_\phi = \frac{1}{r\sin\theta}\frac{\partial^2\Pi_2}{\partial\phi\partial r} + \frac{1}{cr}\frac{\partial^2\Pi_1}{\partial\theta\partial t}, \qquad \mathbf{H}_\phi = \frac{1}{r\sin\theta}\frac{\partial^2\Pi_1}{\partial r\partial\phi} - \frac{\epsilon}{rc}\frac{\partial^2\Pi_2}{\partial\theta\partial t},$$

where Π_1 and Π_2 are both solutions of the characteristic equation

$$\frac{\partial^2\Pi}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Pi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Pi}{\partial\phi^2} = \frac{\epsilon}{c^2}\frac{\partial^2\Pi}{\partial t^2}.$$

Outside the sphere the same type of solution holds but now $\epsilon = 1$.

The main problem is now to obtain the appropriate regular solutions of this equation in the two parts of the field and then to make them agree with each other at the boundary of the sphere, and with the applied field at a large distance away, where the disturbance created by the sphere is necessarily inappreciable.

The typical normal solution* of the fundamental equation is of the type

$$\Pi = R_n S_n e^{ipct}$$

where S_n is a surface harmonic of order n and R_n is a function of r which satisfies the equation

$$\frac{d^2R_n}{dr^2} + \left\{p_1{}^2 - \frac{n(n+1)}{r^2}\right\} R_n = 0$$

wherein $p_1{}^2 = \epsilon p^2$. The general solution of this equation is known to be of the type

$$r^{n+1}\left(\frac{1}{r}\frac{d}{dr}\right)^n\left(\frac{R_0}{r}\right)$$

where
$$\frac{d^2R_0}{dr^2} + p_1{}^2R_0 = 0$$

so that
$$R_0 = A_0\sin(p_1r) + B_0e^{-ip_1r}.$$

The type solution of the general equation is therefore

$$R_n = A_nR_{n_1} + B_nR_{n_0}$$
$$= A_nr^{n+1}\left(\frac{1}{r}\frac{d}{dr}\right)^n\left\{\frac{\sin p_1r}{r}\right\} + B_nr^{n+1}\left(\frac{1}{r}\frac{d}{dr}\right)^n\left\{\frac{e^{-ip_1r}}{r}\right\}$$

the former part being regular at the origin and the latter part at infinity.

* Whittaker, *Modern Analysis* (2nd Ed.), Ch. XVIII; Bateman, *Electrical and Optical Wave Motion*, Ch. III.

599. We consider the case of the incidence of a simple harmonic wave train which is polarised so that the electric force is in the plane $\phi = 0$: the functions defining it will be assumed to depend on the time and position by the exponential factor

$$e^{ip(ct+r\cos\theta)}$$

or what is the same thing*

$$e^{ipct}\left[\sum_{n=0}^{\infty}(2n+1)i^{3n}\frac{R_n P_{n_i}(\cos\theta)}{r}\right]$$

where P_n is Legendre's function of order n, and

$$R_{n_i}(z) = z^{n+1}\left(\frac{1}{z}\frac{d}{dz}\right)^n\left(\frac{\sin z}{z}\right).$$

The radial component of the electric force is then of the form

$$\begin{aligned}\mathbf{E}_r &= E\sin\theta\cos\phi\, e^{ipct}\, e^{ipr\cos\theta}\\ &= -\frac{E\cos\phi}{ipr}e^{ipct}\frac{\partial}{\partial\theta}e^{ipr\cos\theta}\\ &= -\frac{Ee^{ipct}\cos\phi}{ipr}\left[\sum_1^{\infty}(2n+1)\frac{i^{3n}R_{n_i}}{r}\frac{\partial P_n}{\partial\theta}\right]\end{aligned}$$

and the other components are analogous. Thus the corresponding Π functions for the initial field are

$$\Pi_1 = \frac{Ee^{ipct}\sin\phi}{p^2}\sum_1^{\infty}\left[\frac{2n+1}{n(n+1)}i^{3n+1}R_{n_i}(pr)\frac{\partial P_n}{\partial\theta}\right]$$

$$\Pi_2 = \frac{Ee^{ipct}\cos\phi}{p^2}\sum_1^{\infty}\left[\frac{2n+1}{n(n+1)}i^{n3+1}R_{n_i}(pr)\frac{\partial P_n}{\partial\theta}\right].$$

600. These functions suggest the typical forms to be assumed in the more general circumstances. In fact, following the usual method, we can now try a general solution for the external field determined by functions Π_1 and Π_2 which differ from those just given by having

$$R_{n_i}(pr) + A_n' R_{n_0}$$

and

$$R_{n_i}(pr) + A_n'' R_{n_0}$$

in place of R_{n_i} in the two functions respectively. For the internal field on the other hand we replace $R_{n_i}(pr)$ by $R_{n_i}(p_1 r)$ in both functions, but with additional constant multipliers B_n', B_n'' to distinguish them in the two cases.

The internal and external fields thus specified have now to agree at the boundary of the sphere ($r = a$). The conditions there are that the tangential electric and magnetic forces are continuous, and they are easily seen

* Rayleigh, *Theory of Sound*, II. p. 272; Heine, *Kugelfunktionen* (1878), I. p. 82.

to be satisfied if $\epsilon\Pi_1$, Π_2, $\frac{\partial \Pi_1}{\partial r}$ and $\frac{\partial \Pi_2}{\partial r}$ are continuous at $r = a$. From the continuity of Π_1 we have that

$$R_{n_i}(pa) + A_n' R_{n_0}(pa) = \epsilon B_n' R_{n_i}(p_1 a)$$

and
$$\frac{\partial}{\partial a}\{R_{n_i}(pa)\} + A_n' \frac{\partial}{\partial a}\{R_{n_0}(pa)\} = B_n' \frac{\partial}{\partial a}\{R_{n_i}(p_1 a)\}$$

so that A_n', B_n' are completely determinate. The equations derived from the continuity of Π_2 determine in a similar manner the constants A_n'', B_n''.

It is easy to prove that the new series obtained to represent the internal and additional part of the external field are themselves convergent so that a completely effective representation of these fields is obtained. Again however the solutions, although theoretically complete, are too complex to convey any practical meaning and we have to resort to approximations. One particular aspect of these approximations will alone be considered.

601. If the radius of the sphere is small compared with the length of the wave then the argument (pa) of all the functions is small: thus we can use the approximations

$$R_{n_i}(x) = \frac{x^{n+1}}{1.3\ldots(2n+1)}, \qquad R_{n_0}(x) = \frac{1.3\ldots(2n-1)}{x^n} e^{-ix}$$

which are true for small values of x. To the first order in the product (pa) the scattered wave is represented therefore by the first term in each of the series with the unknown constants determined as

$$B_1' = \frac{\epsilon - 1}{\epsilon + 2} \cdot \frac{2a^3}{3}, \qquad B_1'' = 0.$$

Thus in this case the scattered wave is specified as regards its electrical constituents by the components

$$\mathbf{E}_r = 0$$

$$\mathbf{E}_\theta = \frac{\epsilon - 1}{\epsilon + 2} \frac{a^2 p^2 E e^{ip(ct-r)}}{r} \sin\phi \cos\theta$$

$$\mathbf{E}_\phi = \frac{\epsilon - 1}{\epsilon + 2} \frac{a^2 p^2 E e^{ip(ct-r)}}{r} \cos\phi.$$

The disturbance in it is zero in the direction

$$\phi = \frac{\pi}{2}, \qquad \theta = \frac{\pi}{2},$$

which is parallel to the direction of the electric force in the undisturbed incident radiation. Thus when a plane train of non-polarised waves falls upon a small dielectric sphere, the light scattered in any direction perpendicular to the direction of incidence will be completely polarised.

It is also important to notice that the intensity of the scattered radiation depends on the wave length of the light $(2\pi/p)$ and under similar circumstances the radiation of short wave length is much more intensely scattered than that of long wave length.

602. An interesting application of these results has been made by Rayleigh* to explain the blue colour of the sky. The light coming from the sun arrives at our atmosphere as a parallel beam of radiation: in traversing the atmosphere it is scattered by reflexion at the numerous small particles which are present there. The particles may be the molecules of the atmospheric gases themselves or also small globules of water or other vapour; in any case the majority of them are probably of sufficiently small dimensions to justify an application of the above results. We should thus expect on theoretical grounds that the blue constituents of the white light (the short wave lengths) would be more strongly scattered than any other. The light from the sky, which is none other than the light scattered from the small particles of matter in the atmosphere, should therefore be blue. The theoretical polarisation phenomena have also been very satisfactorily verified by the facts, although here it is necessary to go to the next order of approximation in the theory to obtain complete agreement.

The more general theory of the scattering of incident radiation by a spherical obstacle with arbitrary optical properties which can be developed on the above lines also admits of some very interesting applications in the study of the colours exhibited by metal glasses, metallic films and colloidal solutions or suspensions of metals†. The problem is also of interest in its dynamical aspects in connection with the theory of comets' tails.

603. It may be remarked in conclusion that the general methods of analysis used in this section are not necessary if only the approximate results finally obtained are required. In fact the form obtained for the approximate solutions shows that they could have been written down at sight. Consider the case of the sphere just examined. The wave length of the radiation is very large compared with the radius of the sphere, so that at any instant the sphere may be considered as existing in a uniform field; the statical effect of such a field is to induce polarisation in the spherical dielectric which is equivalent, as regards outside points, to a doublet of strength

$$\frac{\epsilon - 1}{\epsilon + 2} E a^3 e^{ipct}$$

* *l. c.* p. 529.

† Maxwell Garnett, *Phil. Trans.* A. CCIII. (1904), p. 385; CCV. (1905), p. 237; G. Mie, *Ann. d. Phys.* XXV. (1908), p. 377; R. Gans, *ibid.*; XXIX. (1909), p. 280; XXXVII. (1912), p. 881. Other problems of a similar type have been worked out by J. J. Thomson, *Recent Researches*, p. 437; J. W. Nicholson, *Proc. L. M. S.* (2), IX. (1910), p. 67; XI. (1912), p. 277; Debye, *Ann. d. Phys.* XXX. (1909), p. 57. Full references are given by Bateman, *Electrical and Optical Wave Motion.*

at the centre at time t. The radiation field due to this varying doublet, which, as such, is equivalent to a simple Hertzian oscillator, is identical with the field determined above as associated with the radiation scattered from the sphere.

604. On the propagation of electric waves along conductors: Rayleigh's method. We have so far discussed the radiation in fields of unlimited extent, so that any local electromagnetic disturbance could be transmitted throughout the whole of the field subject merely to the conditions of reflexion and refraction at interfaces of discontinuity in the transmitting medium. Unfortunately however this unlimited nature of the field necessarily implies cubical expansion of the volume covered by the radiation field and the consequent very large diminution of the intensity of the field in any part of the field under observation, a process which is admirably illustrated by the example discussed in § 544. It thus becomes necessary

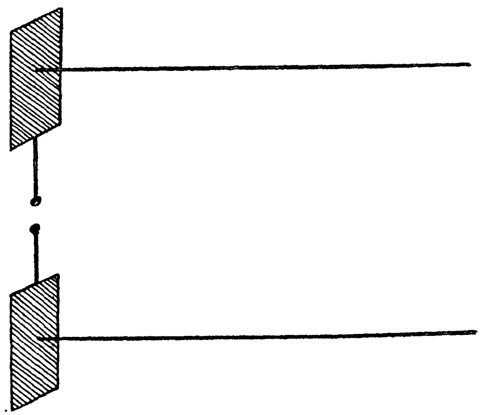


Fig. 93

both from a practical and theoretical point of view to introduce some means of preventing this indefinite spreading of the field. This is easily accomplished in many ways. Hertz discovered that by placing a long straight conducting wire with one end near the radiator, he was able as it were to induce the field of radiation to flow out in the direction of the wire: the wire acts as a sort of guide for the radiation which then travels out along it and does not spread out sideways. In his successful experiments he therefore conducted the waves from the vibrator described above by a long straight wire soldered perpendicularly on to one of the condenser plates in it; the other plate and the end of this wire were then earthed. The oscillating current in the conducting circuit thus flowed through the condenser out along the wire and back through the earthed connections, which in the

actual case were the walls of the room in which the experiments were performed: owing however to the fact that the complementary or return circuit was provided in the walls, the results were still somewhat irregular. A still better plan was adopted by Lecher* by discharging the second plate not directly to earth but first through a second long wire leading out from it close to and parallel to the wire from the other plate. By this means the field in the air is still more localised, the lines of electric force going from one conductor to the other, instead of a long way off to the walls of the room. The results of the experiments with this apparatus were consequently much more regular and precise.

605. The aether is a thing that can vibrate somewhat like a jelly and the electromagnetic condition is propagated through it in waves. The function of the conductors is to annul the elasticity so that in the above form of apparatus the vibrations are like those in a jelly filling the whole of space with two straight cylindrical holes in it (the conductors). The two holes have a complementary function, the strain caused by the one being released on the other, instead of having to extend to infinity before being relieved. Thus the waves which would otherwise spread out in all directions are guided by the wires so that the activity is confined to the medium immediately surrounding them.

There are of course other equally obvious and some still more effective ways of restricting the radiation fields; they may for example be enclosed by conducting media, which if the conductivity is good are practically impermeable. This method is of practical importance in connection with the propagation of electromagnetic wave signals along cables or in telephone wires. A cable for our purposes consists essentially of a cylindrical metallic conductor surrounded at a small distance by a coaxial cylindrical metallic sheath (in submarine cables this conducting sheath is the water itself), the space between being occupied by a uniform dielectric substance. According to our theory any disturbance created in the dielectric medium between the metallic surfaces will run along in that medium without however spreading very much because it cannot penetrate far through the metallic boundaries. The field and propagation are therefore confined to the dielectric shell.

606. Let us now examine some of these cases in greater detail. We shall commence by an examination of the general circumstances of the propagation of electromagnetic disturbances of the type just discussed, the propagation taking place only in the direction of the conductors which are all parallel and cylindrical. We shall simplify the problem as much as possible by neglecting the complicated state of affairs in the field up near the vibrator.

* *Ann. d. Phys.* XLI. (1890), p. 850. Cf. also D. Mazzotto, *Il Nuovo Cimento* (4). VI. (1897), p. 172.

We shall in fact treat all the conductors as very long and consider only the circumstances in them at some distance from either end. The vibratory discharge of condenser at one end of the conductors will then make oscillatory currents run along the conductors, but of such amounts that the total current crossing any surface perpendicular to the direction of propagation is zero. If this latter condition were not fulfilled some other conductor outside the system (the return circuit) would have to carry the complementary current and the irregularity observed in Hertz's experiments would present itself. The wave length of the disturbance is presumed to be large compared with the dimensions of the cross section of the conductors and also with their distance apart.

It then follows by symmetry that the conduction current is directed along the length of the conductors, there being no cross flow in a steady state; and if the disturbance is periodic, but not too fast, this conduction current is by far the more important part of the total current flow, and compared with it the others may certainly be neglected. Of course in the surrounding dielectric field there is no conduction current and the displacement current is all there is, but this is excessively small compared with the conduction current in the metallic parts. If therefore we interpret everything in terms of the vector potential in the field, there will only be one appreciable component of that quantity, viz. that parallel to the axis which is equal at any point to

$$\mathbf{A} = \frac{1}{c}\int \frac{C_1 dv}{r},$$

C_1 denoting the volume density of the electric flux in the wires. From this the electric force is determined by

$$\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{A}}{dt} - \operatorname{grad}\psi,$$

and the magnetic force by

$$\mathbf{H} = \operatorname{curl}\mathbf{A}.$$

It follows that the magnetic lines of force of the field round the conductors are round about in planes perpendicular to the direction of the wires. The electric force has its main electrokinetic part along the direction of the wires; but its static part is practically in the perpendicular planes, because it is contributed in the ordinary way by the charges accumulated; and we have assumed that the wave length of the disturbance is long compared with the distance apart and dimensions of the wires so that the distribution is uniform along at any place; or at least the gradient of ψ along the direction of propagation is small compared with its value perpendicular.

607. We have next to investigate the field in the different regions. In the interior of each medium the total current of Maxwell's theory, comprised of

the conduction and displacement currents, is connected with the magnetic field in that medium by the universally valid circuital reaction of Faraday,

$$\frac{4\pi}{c}\mathbf{C} = \text{curl}\,\mathbf{H},$$

and in the general case $\mathbf{C}$ consists of two parts (i) the conduction current which is proportional to the electromotive force

$$\mathbf{C}_1 = \sigma\mathbf{E},$$

the constant σ being the specific conductivity of the medium, (ii) the displacement current which is

$$\mathbf{C}_2 = \frac{\epsilon\dot{\mathbf{E}}}{4\pi}.$$

Thus
$$\text{curl}\,\mathbf{H} = 4\pi\frac{\sigma}{c}\mathbf{E} + \frac{\epsilon}{c}\dot{\mathbf{E}},$$

or interpreting it in terms of $\mathbf{A}$,

$$\text{curl curl}\,\mathbf{A} = -4\pi\frac{\sigma}{c^2}\frac{d\mathbf{A}}{dt} - \frac{\epsilon}{c^2}\frac{d^2\mathbf{A}}{dt^2} - \left(4\pi\frac{\sigma}{c^2}\frac{d}{dt} + \frac{\epsilon}{c^2}\frac{d^2}{dt^2}\right)\text{grad}\,\psi.$$

Since now there is only one appreciable component of the vector potential, viz. that along the direction of propagation, and since moreover the gradient of ψ in this direction is negligible this leads to an equation for the one component of the form

$$\nabla^2\mathbf{A} - \frac{\partial^2\mathbf{A}}{\partial z^2} = \frac{1}{c^2}\left(4\pi\sigma\frac{d}{dt} + \epsilon\frac{d^2}{dt^2}\right)\mathbf{A},$$

z denoting the coordinate along the direction of propagation. This equation has of course different forms in the different regions.

In the conductors the displacement current is negligible and the equation is of the form

$$\nabla^2\mathbf{A} - \frac{\partial^2\mathbf{A}}{\partial z^2} = 4\pi\frac{\sigma}{c^2}\frac{d\mathbf{A}}{dt}.$$

In the dielectric region between the conductors on the other hand there is no conduction current ($\sigma = 0$) and the equation reduces to

$$\nabla^2\mathbf{A} - \frac{\partial^2\mathbf{A}}{\partial z^2} = \frac{\epsilon}{c^2}\frac{d^2\mathbf{A}}{dt^2}.$$

Of course in the general case the total current is

$$\sigma\mathbf{E} + \frac{\epsilon\dot{\mathbf{E}}}{4\pi},$$

so that in these problems the utmost generality is obtained by the substitution of complex coefficients for real ones.

We conclude that in the conductors the quality expressed by the quantity $\mathbf{A}$ is propagated, like heat conduction, by diffusion, not by waves; instead

of a wave pulse going through it merely soaks in. In the dielectric regions on the other hand this quality is propagated in simple undamped wave forms with a velocity $c/\sqrt{\epsilon}$, ϵ being the dielectric constant of the medium.

608. If the metal were a perfect conductor σ would be infinite, but if σ is large $d\mathbf{A}/dt$ must be very small and actually zero if the current is alternating. This means that $\mathbf{A}$ is constant at all points in the conductors; but in this case the current density is zero at all internal points so that we arrive at the conclusion that in a perfect conductor the current is confined to the surface of the metal, it does not soak in at all. In this case and when the alternations are not too rapid the equation satisfied by $\mathbf{A}$ in the dielectric reduces to

$$\frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} = 0,$$

the axes being chosen so that the z-axis is along the direction of the field. This indicates a very simple state of affairs. $\mathbf{A}$ is constant in all conductors and satisfies the equation for the ordinary electrostatic potential in the dielectric medium between: the determination of $\mathbf{A}$ is thus reduced to an electrostatic problem.

Moreover the current density on any sheet multiplied by c/ϵ corresponds to the charge density of the electrostatic problem: this follows at once from the form for $\mathbf{A}$ which expresses it as the Newtonian potential of the instantaneous current distribution. The distribution of current would thus be analogous to the distribution of charge in the electrostatic problem: the total current flowing over the cross section of any conductor in the plane under consideration would for example be

$$J_r = -\frac{c}{4\pi}\int_r \frac{\partial \mathbf{A}}{\partial n}\, ds,$$

the integral being taken round the boundary of the section.

609. The energies also correspond, in fact, if ϕ is the electrostatic potential of the charged conductors, the static potential energy per unit length of conductors is

$$\tfrac{1}{2}\Sigma Q_r \phi_r,$$

where $$Q_r = -\frac{\epsilon}{4\pi}\int_r \frac{\partial \phi}{\partial n}\, ds,$$

whilst the corresponding energy in the electromagnetic case, which is got from the general formula

$$T = \frac{1}{2c}\int (\mathbf{AC})\, dv,$$

is practically $$\frac{1}{2c}\Sigma J_r \mathbf{A}_r,$$

where J_r is given as above: this can be written in the form

$$\frac{1}{2\epsilon}\Sigma\left(\frac{\epsilon J_r}{c}\cdot\mathbf{A}_r\right),$$

so that the energies are in the ratio $\epsilon : 1$. In the case of a single conductor whose capacity per unit length is b and induction a we have that

$$J_r = \frac{\mathbf{A}_r}{ca},$$

whilst $$Q_r = b\phi_r.$$

The electromagnetic and electrostatic energies for corresponding cases are thus respectively

$$\frac{1}{2}\frac{\mathbf{A}_r^2}{ac^2}, \qquad \frac{b}{2}\phi_r^2,$$

whence we see that $$\frac{c^2}{\epsilon} = \frac{1}{ab},$$

or $$\frac{1}{\sqrt{ab}} = c_1 = \frac{c}{\sqrt{\epsilon}},$$

which verifies that the velocity of propagation is the velocity of light: this is the result obtained by Kirchhoff's method on a former occasion*.

610. The special condition of perfect conductivity might at first sight appear to destroy the essential wave propagation characteristic of the phenomena since it leads to a characteristic equation for **A** of the form

$$\frac{\partial^2\mathbf{A}}{\partial x^2} + \frac{\partial^2\mathbf{A}}{\partial y^2} = 0.$$

It is however very easy to see that even in this special case the field is still in essence propagated. We have in fact under the assumptions mentioned

$$\mathbf{E}_x = -\frac{\partial\psi}{\partial x}, \qquad \mathbf{E}_y = -\frac{\partial\psi}{\partial y},$$

and $$\mathbf{H}_x = \frac{\partial\mathbf{A}}{\partial y}, \qquad \mathbf{H}_y = -\frac{\partial\mathbf{A}}{\partial x}.$$

Thus from the first of Ampère's circuital equations we get

$$-\frac{\epsilon}{c}\frac{d\psi}{dt} = -\frac{\partial A}{dz},$$

and from the second of Faraday's equations we get similarly

$$-\frac{1}{c}\frac{dA}{dt} = \frac{\partial\psi}{\partial z},$$

whence it immediately follows that

$$\frac{\epsilon}{c^2}\frac{d^2\psi}{dt^2} = \frac{d^2\psi}{dz^2},$$

and $$\frac{\epsilon}{c^2}\frac{d^2A}{dt^2} = \frac{d^2A}{dz^2}.$$

* Cf p. 456.

Confining ourselves to the case of progressive waves these equations are solved by

$$\psi = \psi_0 f(c_1 t - z),$$
$$A = A_0 f(c_1 t - z),$$

where $c_1 = \dfrac{c}{\sqrt{\epsilon}}$ and ψ_0 and A_0 are appropriate functions of x and y satisfying the equations

$$\frac{\partial^2 \psi_0}{\partial x^2} + \frac{\partial^2 \psi_0}{\partial z^2} = 0,$$

and

$$\frac{\partial^2 A_0}{\partial x^2} + \frac{\partial^2 A_0}{\partial y^2} = 0.$$

This gives the essence of the affair; the conditions in any plane perpendicular to the direction of propagation are similar to those which would hold in a statical theory: the lines of electric force running out normally from the conductors beginning on positive charges and ending on negative charges, just as in the electrostatic problem; the direction of the gradient of the magnetic vector potential at each point of the field is identical with that of the line of force there, so that the lines of magnetic force are the orthogonal trajectories of the lines of electric force. This state of affairs then travels through the dielectric medium with the velocity c_1. The field of radiation is therefore of the typical transverse type with the electric and magnetic forces of equal intensity and perpendicular to one another in the wave front.

611. As an example we may consider the very simple case of two concentric cylinders (radii a and b) with the dielectric field between them. The total currents along the conductors are equal and opposite, the one being the exact complement of the others just as the charges are in the statical case. The vector potential in the field is then

$$A = \left(2 \log \frac{b}{r}\right) f(c_1 t - z),$$

and the magnetic force is in concentric circles round the axis of the cylinders and the electric force is radial.

As a second example we may quote the results for Lecher's* modification of Hertz's original apparatus for the production of electric waves. Let us assume that the two wires are circular. A cross section of the field is represented in the figure and we know that if we put in the limiting points of the coaxal system of circles determined by the two circular sections and denote by r_1 and r_2 the distances, in the plane of the paper from these points the electrostatic potential of these conductors uniformly charged oppositely is $c \log \dfrac{r_1}{r_2}$: thus the vector potential in the present radiation field has the one component

$$\mathbf{A} = \left(\log \frac{r_1}{r_2}\right) f(c_1 t - r).$$

* *Ann. d. Phys.* XLI. (1890), p. 850.

In this case the lines of electric force are the arcs of circles through the limiting points and the lines of magnetic force are the orthogonal circles.

This simple method of deducing results for radiation problems was suggested by Rayleigh* but of course it only applies for the case of perfect

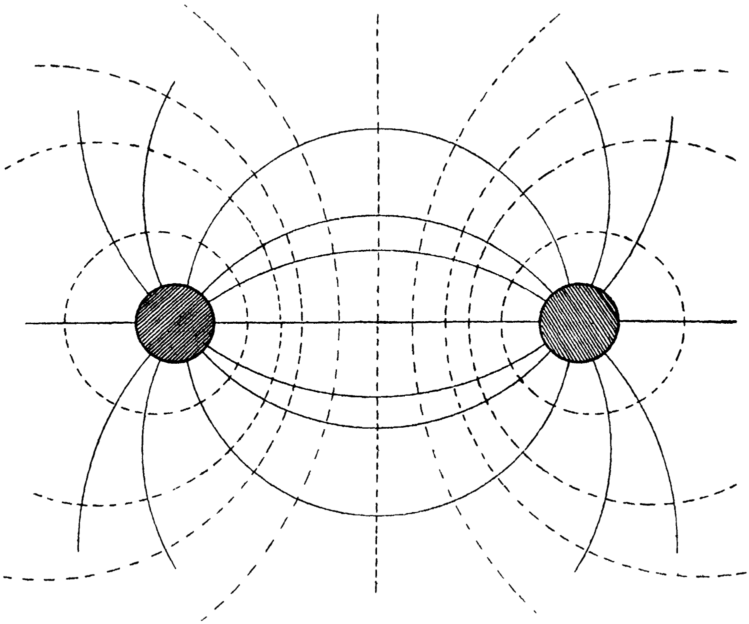


Fig. 94

conductors. The results will be true as approximations when the conduction is not perfect, provided that the oscillation is quick enough to ensure that the current is confined to a thin layer near the surface of the conductor. The more general problems are soluble in certain cases but involve rather complicated analysis.

612. The propagation of waves along cables: general theory. In the previous paragraph we have considered the propagation of waves along wires and cylindrical conductors under the assumption of perfect conductivity in the metal: the cases where this condition is not fulfilled are much more difficult to analyse but it is essential that we should have some idea of the circumstances in such a case in order that we may form an estimate of the degree of approximation of the results obtained above. We shall now consider the general problem in the one particular case which is of real practical importance, viz. the propagation of waves along a circular cylindrical cable consisting of

* *Phil. Mag.* LIV. (1897), p. 199.

an inner and outer conductor with a shell of dielectric between them. The analysis for this case is rather complicated and we shall find it more convenient to approach the problem tentatively by first examining a very simple problem which brings out most of the important additional points of the theory and is easily analysed.

We first consider therefore the propagation of electric waves in a dielectric slab between two parallel conducting slabs extending on both sides

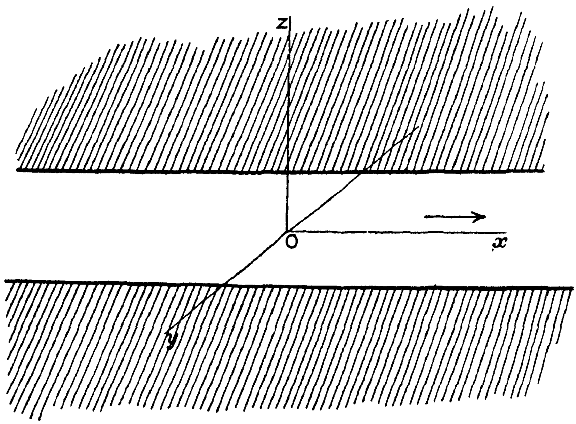


Fig. 95

to infinity. This would represent fairly well the circumstances of the propagation in the dielectric shell of the cable as above provided the thickness of the shell at any place is not too large compared with the diameter of a cross section.

613. We choose axes as in the figure. The z-axis is normal to the plane surfaces and the x-axis in the medial plane of the slab in the direction of propagation of the waves. The magnetic force is thus parallel to Oy. The general equations of the theory thus reduce to the form

$$\frac{1}{c}\left(4\pi\sigma\mathbf{E}_x + \epsilon\frac{d\mathbf{E}_x}{dt}\right) = -\frac{d\mathbf{H}_y}{dz},$$

$$\frac{1}{c}\left(4\pi\sigma\mathbf{E}_z + \epsilon\frac{d\mathbf{E}_z}{dt}\right) = \frac{d\mathbf{H}_y}{dx},$$

and

$$-\frac{1}{c}\frac{d\mathbf{H}_y}{dt} = \frac{\partial\mathbf{E}_z}{\partial x} - \frac{\partial\mathbf{E}_x}{\partial z},$$

whence $\mathbf{H}_y$ satisfies the usual fundamental equation

$$\frac{1}{c^2}\left(4\pi\sigma\frac{d\mathbf{H}_y}{dt} + \epsilon\frac{d^2\mathbf{H}_y}{dt^2}\right) = \frac{\partial^2\mathbf{H}_y}{\partial x^2} + \frac{\partial^2\mathbf{H}_y}{\partial z^2},$$

which assumes different forms in the various regions. In the conducting regions the displacement current is negligible and $\mathbf{H}_y$ satisfies the equation

$$\frac{4\pi\sigma}{c^2}\frac{d\mathbf{H}_y}{dt} = \frac{\partial^2 \mathbf{H}_y}{\partial x^2} + \frac{\partial^2 \mathbf{H}_y}{\partial z^2},$$

and in the dielectric medium $\sigma = 0$ so that $\mathbf{H}_y$ satisfies the equation

$$\frac{\epsilon}{c^2}\frac{d^2\mathbf{H}_y}{dt^2} = \frac{\partial^2 \mathbf{H}_y}{\partial x^2} + \frac{\partial^2 \mathbf{H}_y}{\partial z^2}.$$

Following the usual plan an appropriate solution of these equations can be specified by

(i) in the dielectrics

$$\mathbf{H}_{y_1} = A_1 e^{i(nt+ax)} \cosh bz,$$

where

$$-\frac{n^2\epsilon}{c^2} = -a^2 + b^2,$$

(ii) in the conductor on the positive side

$$\mathbf{H}_{y_2} = A_2 e^{i(nt+ax) - b_1(z-d)},$$

where $2d$ is the thickness of the slab and

$$\frac{4\pi i n\sigma}{c^2} = -a^2 + b_1^2,$$

(iii) in the conductor on the other side a similar symmetrical form holds, it need not further be specified.

The surface conditions of continuity of the tangential electric and magnetic forces show that

$$A_2 = A_1 \cosh bd,$$

and

$$\frac{b_1 A_2}{4\pi\sigma} = \frac{ib}{n\epsilon} A_1 \sinh bd,$$

so that

$$\frac{b_1}{4\pi\sigma} = \frac{ib}{n\epsilon} \tanh bd.$$

The usual assumptions made above that the length of the wave is long compared with the thickness of the slab enable us to approximate to the value of tanh bd and we can write

$$\frac{b_1}{4\pi\sigma} = \frac{ib^2 d}{n\epsilon},$$

or

$$b^2 = -\frac{ib_1 n\epsilon}{4\pi\sigma d}.$$

614. Let us examine these results in a particular case. In copper $\sigma = \dfrac{c^2}{1600}$ and if we take a wave length $\lambda = 100$ cms. then

$$n = 3 \,.\, 10^8.$$

and thus since a turns out to be a fraction of the order 10^{-2} we must have

$$b_1^2 = \frac{i4\pi n\sigma}{c^2},$$

this part of b_1^2 being by far the largest. Thus

$$b_1 = \frac{1+i}{2}\sqrt{\frac{4\pi n\sigma}{c^2}},$$

which gives precisely the same damping factor as previously discussed on more than one occasion. The propagation of the field directly into the conductors is therefore damped off in precisely the same rapid manner as previously exhibited.

In this case

$$\begin{aligned} b^2 &= -\frac{ib_1 n\epsilon}{\sigma d} \\ &= \frac{1-i}{2}\sqrt{\frac{n^2\epsilon^2}{4^2\pi^2\sigma^2 d^2}\cdot\frac{n4\pi\sigma}{c^2}} \\ &= \frac{1-i}{2}\cdot\frac{n\epsilon}{4\pi\sigma c d}\sqrt{4\pi n\sigma}, \end{aligned}$$

and is thus of the order 10^{-10}, and is extremely small. Again

$$a^2 = \frac{n^2\epsilon}{c^2} + b^2,$$

and since b^2 is small compared with $\frac{n^2\epsilon}{c^2}$ we can write

$$\begin{aligned} a &= \frac{n\sqrt{\epsilon}}{c}\left(1 + \frac{1}{2}\frac{c^2 b^2}{n^2\epsilon}\right) \\ &= \frac{n\sqrt{\epsilon}}{c} + a_1(1-i), \end{aligned}$$

where $$a_1 = \frac{1-i}{4d}\sqrt{\frac{n}{\sigma}},$$

and is of the order $2 \cdot 10^{-6}$.

615. The wave motion in the dielectric is thus propagated in the specified direction with a velocity

$$\frac{n}{\frac{n\sqrt{\epsilon}}{c} + a_1} = \frac{c}{\sqrt{\epsilon}}\left(1 - \frac{ca_1}{n\sqrt{\epsilon}}\right),$$

which differs from $\frac{c}{\sqrt{\epsilon}}$ but very slightly. The velocity of propagation in the medium is hardly affected by the presence of the conductors. The chief influence of these is however in their damping effect which is no longer negligible. There is an exponential factor in the wave form in the dielectric

$$e^{-a_1 x},$$

which defines the mode of decay of the disturbance as it is propagated. The energy is gradually used up against friction in the conductors, even the supply located in the dielectrics being called upon. In the particular problem quoted the damping comes out to be such that in a distance of the order about 10^4 metres the amplitude is reduced to $\frac{1}{e}$ of its initial value.

In the general case however we see that with given materials the shorter rapid oscillations are the least damped : very rapid ones are not appreciably damped at all. The reason for this is obvious for in such cases the current is confined to a very thin layer at the surface of the conductors. There is no current in the body of the conductor, simply charge oscillation at its surface. We do not of course mean that there is no resistance at all to a surface current of this nature; it is merely the excessive rapidity of the oscillations that makes the resistance unimportant.

If we consider the real part of the above solutions only it is easily verified that the energy dissipated in the conductors at any place (per unit breadth along the y-axis) is equal to

$$\frac{cA_1^2}{2\sqrt{n\sigma}}e^{-2a_1x},$$

and this again illustrates the above remarks very vividly.

616. An approximate solution for the actual circumstances in the cable could now be obtained by wrapping this solution round a cylindrical shell as already explained, the results here obtained applying per unit length round the cable thus formed; owing however to the intrinsic importance of the problem it seems advantageous at least to indicate the various steps in the analysis*. For this purpose we first examine the case where there is no outer conductor. In this case and in fact in any case involving coaxal circular cylinders it is most convenient to refer the field to cylindrical polar coordinates (r, θ, z) with the axes along the axes of the cylinders.

Under the usual circumstances of propagation along the cylindrical conductor the field is symmetrical round the axis and in the case of the type above examined where the magnetic force is in circles round the axis the general field equations assume the form

$$\frac{\epsilon}{c}\frac{d\mathbf{E}_z}{dt} + \frac{4\pi\sigma}{c}\mathbf{E}_z = \frac{1}{r}\frac{d}{dr}(r\mathbf{H}_\theta)$$

$$\frac{\epsilon}{c}\frac{d\mathbf{E}_r}{dt} + \frac{4\pi\sigma}{c}\mathbf{E}_r = -\frac{d\mathbf{H}_\theta}{dz}$$

$$-\frac{1}{c}\frac{d\mathbf{H}_\theta}{dt} = \frac{\partial\mathbf{E}_r}{\partial z} - \frac{\mathbf{E}\partial_z}{\partial r}.$$

* Cf. Rayleigh, *Phil. Mag.* (5), XXI. (1886), p. 381; Heaviside, *Elect. Papers*, II. pp. 39 and 168; J. Stefan, *Wien. Ber.* XCV. (1887), p. 917; *Ann. d. Phys.* XLI. (1890), p. 400.

These equations are satisfied by the forms

$$\mathbf{E}_z = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Pi}{\partial r}\right), \qquad \mathbf{E}_r = \frac{\partial^2\Pi}{\partial r\partial z},$$

$$\mathbf{H}_\theta = -\frac{\epsilon}{c}\frac{\partial^2\Pi}{\partial r\partial t} - \frac{4\pi\sigma}{c}\frac{\partial\Pi}{\partial r},$$

when Π satisfies the equation

$$\frac{\epsilon}{c^2}\frac{d^2\Pi}{dt^2} + \frac{4\pi\sigma}{c^2}\frac{d\Pi}{dt} = \frac{\partial^2\Pi}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Pi}{\partial r}\right).$$

The equations in the interior of the wire are obtained as usual by neglecting the displacement current (i.e. by putting $\epsilon = 0$) whereas in the field outside the conductor there is no conduction current (or $\sigma = 0$). We shall also assume for simplicity that $\epsilon = 1$ in the external medium.

At the surface of the conductor the boundary conditions imply the continuity of the tangential components of the electric and magnetic forces.

617. To represent the propagation of simple periodic waves along the cylinder we try the solution

$$\Pi = e^{ip(ct-lz)}\chi(r),$$

wherein l is a constant to be determined and which determines the velocity and damping of the wave propagation. The equation for χ is the simple Bessel equation

$$\frac{d^2\chi}{dx^2} + \frac{1}{x}\frac{d\chi}{dx} + \chi = 0,$$

where the independent variable x has the respective forms

$$x = pr\sqrt{k^2 - l^2}, \qquad k^2 = -\frac{i4\pi\sigma}{pc},$$
$$x = pr\sqrt{1 - l^2},$$

in the solutions corresponding to the conductor and the dielectric medium. We shall write

$$\nu_0 = p\sqrt{k^2 - l^2}, \qquad \nu_1 = p\sqrt{1 - l^2}.$$

In the interior of the conductor we must choose the particular Bessel function which is finite on the axis, that is we must take

$$\chi = CJ_0(\nu_0 r),$$

where $$J_0(x) = \frac{1}{\pi}\int_0^\pi e^{ix\cos a}\,da.$$

Outside in the dielectric we must use the second type of function and take

$$\chi = DK_0(\nu_1 r),$$

where $$K_0(\nu_1 r) = \frac{1}{i}\int_0^{i\infty} e^{ix\cos a}\,da = \int_1^\infty \frac{e^{ix\xi}\,d\xi}{\sqrt{\xi^2 - 1}}.$$

618. Thus inside the conductor the field is completely determined by the vectors

$$\mathbf{E}_z = \nu_0^2 C J_0(\nu_0 r)\, e^{ip(ct-lz)},$$
$$\mathbf{E}_r = -\, il\nu_0 C J_0'(\nu_0 r)\, e^{ip(ct-lz)},$$
$$\mathbf{H}_\theta = -\,\frac{4\pi\nu_1\sigma}{c} C J_0'(\nu_0 r)\, e^{ip(ct-lz)},$$

whilst outside in the dielectric field they are

$$\mathbf{E}_z = \nu_1^2 D K_0(\nu_1 r)\, e^{ip(ct-lz)},$$
$$\mathbf{E}_r = -\, il\nu_1 D K_0'(\nu_1 r)\, e^{ip(ct-lz)},$$
$$\mathbf{H}_\theta = -\, ip\nu_1 D K_0'(\nu_1 r)\, e^{ip(ct-lz)}.$$

The boundary conditions that the force components $\mathbf{E}_z$ and $\mathbf{H}_\theta$ are continuous across the surface of the cylinder give

$$\nu_0^2 C J_0(\nu_0 a) = \nu_1^2 D K_0(\nu_1 a)$$

and
$$\frac{4\pi\nu_1\sigma}{c} C J_0'(\nu_0 a) = -\, ip\nu_1 D K_0'(\nu_1 a),$$

whence on elimination of C and D

$$\frac{c\nu_0}{4\pi\sigma}\frac{J_0(\nu_0 a)}{J_0'(\nu_0 a)} = \frac{\nu_1}{ip}\frac{K_0(\nu_1 a)}{K_0'(\nu_1 a)},$$

which is the transcendental equation for the constant l which determines the complete circumstances of the propagation. This equation cannot, of course, be generally solved, but the approximate solution can be obtained in most cases of real practical importance. The algebraical processes involved are however rather complicated although they are quite straightforward, and since the results obtained are precisely of the character of those obtained above in the simple problem it is perhaps not necessary to give them out in full here. They are very fully discussed in an elaborate paper on this subject by Sommerfeld*. The propagation takes place mainly in the dielectric with a velocity nearly equal to that of radiation, and there is a slight penetration into the conductor and consequent dissipation of energy, the main effect of which is to introduce a slight damping effect in the wave propagation.

The complete problem † of the cable where the cylindrical conductor above discussed is enclosed by a second coaxial one can now be directly solved.

* *Ann. d. Phys.* LXVII. (1899), p. 233. Cf. also J. J. Thomson, *Proc. L. M. S.* XVII. (1886), p. 310; *Recent Researches*, § 259; Abraham, *Encyklop. d. math. Wiss.* Bd. v. 2 (1910), p. 526; Larmor, *Proc. 5th Int. Congress of Math.* I. (1912), p. 206.

† Cf. J. J. Thomson, *Proc. R. S.* XLVI. (1889), p. 1; *Recent Researches*, p. 262; F. Harms, *Ann. d. Phys.* XXIII. (1907), p. 44. Other cases have been examined by Mie, *Ann. d. Phys.* II. (1900), p. 201; W. B. Morton, *Phil. Mag.* I. (1901), p. 563; J. W. Nicholson, *Phil. Mag.* (1909) and *Phil. Mag.* XIX. (1910), p. 77; H. C. Pocklington, *Proc. Camb. Phil. Soc.* IX. (1897), p. 324; D Hondros, *Ann. d. Phys.* XXX. (1909), p. 905; *Dissertation*, Munich (1909).

The field in the interior of the core will involve a Bessel function of the first kind and that in the metallic sheath, which is presumed to extend to infinity all round, will depend on a function of the second kind. The field in the dielectric shell between will involve both kinds of functions. The two boundary conditions at each surface will determine the four constants thus involved. The details of the method are now obvious and they need not further detain us.

CHAPTER XIV

GENERAL ELECTRODYNAMIC THEORY

619. The energy in the electromagnetic field. Since we became convinced of the impossibility of perpetual motion there has always been connected with our conceptions of natural phenomena the idea of that something which we call energy. The kinetic and potential energies of matter were the kinds of energy first recognised and for this reason it is customary to try to associate any new form of energy which turns up with something which in its properties is akin to matter. Thus arose for example the idea of the material aether which formed the basis of the older wave theory of light and which ascribed to this aether elastic and inertia properties and then regarded the energy of the light waves as composed of the kinetic and potential energy of the medium. The main object of Maxwell's electric theory is to adopt the idea of an elastic aether to explain the electrodynamical actions of electromagnetic systems, although care has been taken not to attribute to this aether any nature analogous to that of ordinary matter and the theory is in fact quite independent of any definite assumptions or hypotheses we may make as to its constitution. Thus far therefore we merely use the word aether as a convenient means of describing those properties of space which are concerned in electromagnetic phenomena, those properties being mathematically expressed by the general equations of Maxwell's theory, which contain in themselves not only the totality of the older laws of electrostatic and electrodynamic phenomena, but also the laws for the propagation of light and electric waves in space.

In reality the universe consists of matter and the electromagnetic field in the aether. If the electromagnetic field were not present our eyes would not indicate to us the presence of the matter. In fact natural phenomena in general appear to us as a result of the interaction between matter and the aether. A simple materialistic conception of things regards the interaction of the material constituents as the essence of the affair and regards the electromagnetic theory merely as an auxiliary means of formulating the laws of these interactions; matter is the only actuality. We may however with equal justification regard the matter from the other point of view. We can regard the electromagnetic aether as the only actuality and matter as a special manifestation of a 'condition' in this aether. Such a view is of course one-sided but it is helpful in preventing us from going to the other extreme. There is in fact one point in its favour; our knowledge of electromagnetic phenomena is much more precise and extensive than our knowledge of matter.

For the present however we shall not definitely take up either view, although we shall as before often fall back on particular analogies which point to certain definite conclusions as to the nature of this aether. In any case we have as before a definite something to which to attach the energy which is associated with any electromagnetic field, and our present object is to discuss the distribution and variation of this energy location. The method to be pursued involves a simple application of the energy principle with as few additional hypotheses as possible.

620. This energy principle seems to suggest that energy is to be considered as a definite substantial something. In any self-contained electromagnetic or mechanical system the total quantity of energy is always the same, just like the quantity of matter in a self-contained material system. About the matter we know moreover that it can only move continuously in space, a jump from one place to another without crossing the intervening space being excluded. A sudden discontinuous translation of energy from one point of space to another is however *à priori* not impossible. A distance action theory of gravity would, for example, conceive it as possible that one body can accelerate another, i.e. give it energy, without reference to the intermediate space. Maxwell's theory on the other hand does not admit of any discontinuous energy translations of this nature, its underlying idea being in fact that the energy in the electromagnetic field is transferred continuously from point to point even with a finite velocity.

The general theory of the energy streaming in the electromagnetic field was first developed by Poynting on the basis of Maxwell's theory and we shall now go through the important points of the subject although on rather different lines from those adopted by Poynting*. We shall proceed tentatively and attempt to get as much information from our few hypotheses as possible.

621. The energy of the electromagnetic field must be continuously distributed throughout that field in the aether; the only way of getting away from this is by assuming action at a distance and this practically asserts that we cannot trace the energy. Now for all dynamical purposes it is necessary to know not only the total amount of the energy in the field but also how it is distributed in the field. For example in a dynamical theory of an elastic solid it is always necessary to know the potential energy w per unit volume at any point (a quadratic function of the strains in the element dv at the place) as well as the total potential energy

$$W = \int w\,dv,$$

before using the results in any mechanical discussion. The important point

* *Phil. Trans.* A, CLXXV. (1884). The present treatment follows the lines sketched by Larmor in *Phil. Trans.* A, CXC. (1897), p. 285, and developed in further detail in his lectures.

is that we must be sure that this represents the actual distribution and not only the total amount. We might, for example, in the process of obtaining W, have integrated by parts and so got

$$W = \int_f w_n' \, df + \int w' \, dv,$$

where the surface integral is taken over a surface f bounding the system. If this surface is indefinitely extended we can neglect this part and thus

$$W = \int w' \, dv,$$

and $w' \neq w$. But by integrating by parts we always mix up the energy from different parts of the system to get that at the typical volume element; the energy in any element would then depend on all the distant elements and we should not then have a proper local distribution. In the case of media like the aether this is one of the complexities to be met with and we have to find out as best we can the proper distribution of energy; but in any case we are never absolutely certain that our simply obtained energy distributions have not after all been obtained by some such process as integration by parts and do not therefore represent the true distribution required.

622. With these preliminary remarks let us now consider the conservation of the total energy in any electrodynamic field. In this case we interpret the general principle in the form that the diminution of energy inside any closed surface in the field is equal to the flow of energy outwards across the surface. In other words if E is the total energy inside, $-dE/dt$ is equal to the flux of energy outwards over the surface; and we ought to be able to express this flux as a surface integral in the form

$$\int_f \mathbf{S}_n \, df,$$

$\mathbf{S}_n$ denoting the outward normal component of the vector determining the energy flux, which will of course have magnitude and direction like the flux of anything else.

The total energy E of course represents the available electrodynamic energy in the field. If W represents the potential energy and T the kinetic energy then

$$\frac{dE}{dt} = \frac{dW}{dt} + \frac{dT}{dt} + F = -\int_f \mathbf{S}_n \, df,$$

wherein F represents the electromagnetic energy dissipated in the space considered per unit time, either directly into heat or in the performance of mechanical work on the masses with free charges or polarisations.

623. In this relation we are fairly sure of the form of two of the quantities involved. The potential energy is very generally expressed by

$$W = \int_v dv \int_0^P (\mathbf{E} \,.\, \delta\mathbf{P}) + \frac{1}{8\pi}\int_v \mathbf{E}^2 dv,$$

where $\mathbf{P}$ is the polarisation intensity of the molecules of the medium produced by the electric force $\mathbf{E}$. The first term represents the work done against the material reactions to the setting up of the polarisation and is stored up as internal potential energy in the polarisation of the medium being simply the organised part of the energy of elastic stress. The second part represents the purely electrical part of the potential energy associated with the polarisations in the molecules representing energy of strain in the aethereal medium due to the presence and configuration of the electrical charges. We have of course excluded the existence of hysteretic effects in the polarisation of the medium so that it is generally reversible. The polarisation is presumed to be an elastic affair involving no dissipation.

Again we know that the energy dissipated per second is

$$\int (\mathbf{E}\mathbf{C}_1)\, dv,$$

$\mathbf{C}_1$ representing the part of the total current depending on the motion of the free electrons. This current consists of two essentially distinct parts concerned respectively with the true conduction electrons and with those giving rise to the convection currents. In both cases the electric force acts on the free electrons and increases their velocities, but in the former case this increase is dissipated by collision into irregular heat motion whereas in the latter it is converted into mechanical energy of effectively non-electric nature. In the most general case the convection current will arise in the motion of charged bodies and polarised media.

The expression for the kinetic energy is not considered as so certain as the above two expressions which appear to be very generally valid. We can however leave it over for the present as the theory determines possible forms for it.

624. We now have

$$\frac{dW}{dt} = \frac{1}{4\pi}\int_v (\mathbf{E} \,.\, \dot{\mathbf{E}})\, dv + \int_v (\dot{\mathbf{P}} \,.\, \mathbf{E})\, dv$$

or

$$= \int_v (\mathbf{E} \,.\, \dot{\mathbf{D}})\, dv,$$

and

$$F = \int (\mathbf{E} \,.\, \mathbf{C}_1)\, dv$$

$$= \int \left(\mathbf{E} \,.\, \mathbf{C} - \dot{\mathbf{D}}\right) dv$$

$$= \int (\mathbf{E} \,.\, \mathbf{C})\, dv - \int (\mathbf{E}\dot{\mathbf{D}})\, dv.$$

Therefore we have $$\frac{dW}{dt} + F = \int (\mathbf{E} \cdot \mathbf{C})\, dv,$$

and thus we must have

$$\frac{dT}{dt} = \frac{dE}{dt} - \int (\mathbf{E} \cdot \mathbf{C})\, dv,$$

and then if Maxwell's ideas on the nature of the electromagnetic actions are tenable we must be able to express $\frac{dE}{dt}$ as a surface integral as above, so that

$$\frac{dT}{dt} = -\int_f \mathbf{S}_n df - \int (\mathbf{E} \cdot \mathbf{C})\, dv.$$

Various possibilities are now open to us. This is all we can learn about the distribution of the kinetic energy from the energy principle alone. We can however make further simple hypotheses and thereby gain additional insight into the matter.

625. (i) The most natural hypothesis is obtained by transforming the last integral by the substitution

$$\frac{4\pi}{c}\mathbf{C} = \operatorname{curl} \mathbf{H}.$$

We get then

$$\int (\mathbf{E} \cdot \operatorname{curl} \mathbf{H})\, dv = -\int [\mathbf{EH}]_n df + \int (\mathbf{H} \cdot \operatorname{curl} \mathbf{E})\, dv$$

$$= -\int [\mathbf{EH}]_n df - \frac{1}{c}\int \left(\mathbf{H} \cdot \frac{d\mathbf{B}}{dt}\right) dv,$$

so that $$\frac{dT}{dt} = -\int_f \mathbf{S}_n df + \frac{c}{4\pi}\int [\mathbf{EH}]_n df + \frac{1}{4\pi}\int \left(\mathbf{H} \cdot \frac{d\mathbf{B}}{dt}\right) dv.$$

And now we might take

$$\mathbf{S} = \frac{c}{4\pi}[\mathbf{EH}],$$

and then we should have

$$\frac{dT}{dt} = \frac{1}{4\pi}\int_v \left(\mathbf{H} \cdot \frac{d\mathbf{B}}{dt}\right) dv.$$

or $$T = \frac{1}{4\pi}\int dv \int_0^B (\mathbf{H} d\mathbf{B}),$$

provided of course that the integrand is a perfect differential, otherwise we could attach no meaning to it. This means that this is a possible form for T if the induced magnetism is reversible; if the magnetisation were reversible and $(\mathbf{H} d\mathbf{B})$ were not a complete differential then we could have perpetual motion.

We might thus take the kinetic energy as distributed throughout the space with a volume density

$$\frac{1}{4\pi}\int^{B}\mathbf{H}d\mathbf{B} = \frac{1}{8\pi}(\mathbf{B}^2 - 16\pi^2\mathbf{I}^2) - \int_0^H(\mathbf{I}d\mathbf{H})$$

which in empty space gives a density

$$\frac{1}{8\pi}\mathbf{B}^2.$$

This is Maxwell's form of the kinetic energy in the electromagnetic field; it thus regards all currents and magnetism as having associated with them kinetic energy. This suggests that the magnetic induction is a type of velocity in the aether and that $\frac{1}{8\pi}$ is a coefficient of inertia. This would be a natural assumption to make and brings the theory into line with our ordinary notions of elastic bodies.

This form for T has the one great advantage of expressing the energy at a place directly in terms of the field vectors specifying the condition of the aether at that place; it is therefore the natural specification for T.

626. On this hypothesis we have

$$-\frac{dE}{dt} = \frac{c}{4\pi}\int_f [\mathbf{EH}]_n\, df,$$

for this is what remains when we identify $\frac{dT}{dt}$ with the other part of the complete expression. This means that the vector

$$\mathbf{S} = \frac{c}{4\pi}[\mathbf{E}\,.\,\mathbf{H}]$$

represents the flux of energy; the integral of the flux of this vector across any surface represents the rate of change of energy inside the surface. The resultant flux of energy at any point is

$$\frac{c}{4\pi}(\mathbf{H}\,.\,\mathbf{E}\sin\widehat{\mathbf{HE}}),$$

and is directed perpendicular to both $\mathbf{H}$ and $\mathbf{E}$; the energy flows perpendicular to both forces in the field.

This is Poynting's result and this vector $\mathbf{S}$ is usually called after him. It is however only right on the hypothesis that the kinetic energy is distributed in the medium with a density $\frac{1}{4\pi}\int_0^B(\mathbf{H}d\mathbf{B})$ per unit volume; but even then we are not absolutely sure that it is right because we might have added to it some other vector quantity which would however integrate out when taken all over the surface f. However, following a usual practice in physics,

it is best to adhere to the simplest hypothesis. This idea of Poynting's is the simplest certainly, and the addition of anything else is merely a gratuitous complication which is not, after all, necessary. The actual phenomena strongly suggest that the flux of energy is correctly represented by this vector.

627. (ii) There is however no definite and precise reason why we should take the matter this way; we might have adopted some other scheme. The only other one of any importance is obtained by performing the first integration by parts in some other way. We found that

$$\frac{dT}{dt} = \frac{dE}{dt} - \int (\mathbf{E}\,.\,\mathbf{C})\,dv,$$

and we integrated by the substitution $\frac{4\pi}{c}\mathbf{C} = \operatorname{curl}\mathbf{H}$. We might however follow another course and introduce the vector potential $\mathbf{A}$ by the substitution

$$\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{A}}{dt} - \operatorname{grad}\psi,$$

and then we have

$$\begin{aligned}\int (\mathbf{E}\,.\,\mathbf{C})\,dv &= -\frac{1}{c}\int\left(\frac{d\mathbf{A}}{dt}\,.\,\mathbf{C}\right)dv - \int(\mathbf{C}\,.\operatorname{grad}\psi)\,dv \\ &= -\frac{1}{c}\int\left(\frac{d\mathbf{A}}{dt}\,.\,\mathbf{C}\right)dv + \int\psi\,.\operatorname{div}\mathbf{C}\,dv - \int_f \psi\mathbf{C}_n\,df,\end{aligned}$$

and since $$\operatorname{div}\mathbf{C} = 0,$$
we see that

$$\int (\mathbf{E}\,.\,\mathbf{C})\,dv = -\frac{1}{c}\int\left(\mathbf{C}\frac{d\mathbf{A}}{dt}\right)dv - \int_f \psi\mathbf{C}_n\,df,$$

so that now we have

$$\frac{dT}{dt} = \frac{dE}{dt} + \int_f \psi\mathbf{C}_n\,df + \frac{1}{c}\int\left(\mathbf{C}\frac{d\mathbf{A}}{dt}\right)dv.$$

We might now take

$$-\frac{dE}{dt} = \int_f \mathbf{S}_n\,df = \int_f \psi\mathbf{C}_n\,df,$$

and then we should have

$$T = \frac{1}{c}\int dv\int_0^A (\mathbf{C}d\mathbf{A}).$$

628. This is the general form of a result which has received very influential support from some quarters* and there is a good deal to be said for it. Consider the case of a conduction current flowing in complete circuits. If we consider that what goes on is the electric force pulling the electrons then the work done per unit time is precisely this result This is to a certain

* Cf. Macdonald, *Electric Waves*, Chs. IV, V, VIII.

extent a reasonable hypothesis; but it entirely neglects the aether; it says that the electrons are the things to which the energy is attached and the work is that done by the electric force pulling the electrons about. On the aether theory however the energy of an electric particle is really distributed in the field all round it. We thus want to distribute the energy in the aether and take the electrons merely as the key, the singular point which binds or locks up a portion of the energy; on this idea the electron appears merely as a nucleus of strain which locks up a portion of the energy; the actual nature of the process or mechanical action by which this is accomplished is at present unknown; but we do not need any more precise knowledge of it.

There is another disadvantage to this scheme; there is no obvious physical explanation of the functions involved in it. In fact the mathematical definition of the functions involved is not complete, so that the functions themselves cannot, without further arbitrary restrictions, represent definite physical entities. Even if we did choose, say, the instantaneous potentials of Maxwell's theory then their definition in the integral form shows that the value of each of them at any point depends on all the other elements of the field, so that if we take a distribution of the kinetic energy like this it is practically importing action at a distance into the theory. In a theory of action in and through a medium any quality at one place in the medium depends only on the conditions at the place and not on the other places remote from it, the quality being propagated from place to place with a finite velocity. In such a theory therefore this form for T is an unlikely one.

However we cannot say that either form is wrong; it all depends on the point of view adopted. The chief point to be noticed is that we get different distributions of magnetic energy according to the assumptions made. The particular value adopted is after all merely a matter of preference, not proof. We propose however to adopt in our future investigations the form first given as this is in agreement with the work of most authors.

629. It is important to notice that in the above argument very little has been assumed respecting the constitutive properties of the material media distributed throughout the field. We presume merely the non-existence of hysteretic effects in the electric and magnetic phenomena. The results admit therefore of very general application. Of course the form

$$T = \frac{1}{8\pi}\int (\mathbf{H}\,.\,\mathbf{B})\,dv,$$

assumes a linear relation between $\mathbf{B}$ and $\mathbf{H}$.

In the case when there are no magnetic substances about the kinetic energy is distributed throughout the field with a density

$$\frac{1}{8\pi}\mathbf{B}^2,$$

and in this case it belongs entirely to the aether, because it is independent of the nature of the substances present. The inertia of the electromagnetic field thus appears to be in the aether.

If there are magnetic substances about and we are dealing with slow electric changes so that the magnetism can follow the field the total available magnetic energy is

$$\frac{1}{4\pi}\int dv \int_0^B (\mathbf{H}d\mathbf{B}) = \frac{1}{8\pi}\int (\mathbf{B}^2 - 16\pi^2\mathbf{I}^2)\, dv - \int dv \int_0^H (\mathbf{I}d\mathbf{H}).$$

It thus consists of the total magnetic energy of the electromagnetic field less that part of it which is concerned with the bodily forces on the magnetic media and which is mechanically available only in so far as the presence of these media increases the available energy associated with the currents and charges of the system.

Even when magnetic substances are present they have generally nothing whatever to do with radiation phenomena. The magnetisation of the medium being an affair of the molecules takes a comparatively considerable time to establish, and the alternations in radiation are far too quick for it to follow*. This is the reason why even iron may be treated as non-magnetic in the electromagnetic theory of light.

It used to be argued against this theory, that according to it, the reflexion of light from the surface of an iron magnetic cannot be altered by altering the magnetisation in any way, a fact which is in direct contradiction to experiment. However the magnetism in this case does come in, but in another way, as a second order effect, and the rotation of the plane of polarisation on reflexion from a magnetised mirror is a second order phenomena involving the magnetisation of the metal.

The potential energy of course always depends to a certain extent on the matter present in the field, the energy of polarisation of the medium always forming an essential part in its complete specification.

630. The physical significance of the energy flux or Poynting vector may be illustrated by a simple application of practical importance. A long straight cylindrical conductor carries a current J. The conductor is surrounded by an electromagnetic field and we require the nature of the flow of energy in this field.

The magnetic lines of force in the surrounding field are in circles round the wire. As regards the electric force we know that just inside the conductor it is directed along the axis and is just sufficient to drive the current, this being secured by a slight initial accumulation of charge on the conductor. But this force cannot change in going across the surface of the conductor

* This applies only to the appreciable cases of ferromagnetism; the ordinary intramolecular phenomena are far too small.

so that just outside it is also tangential and equal to the internal value. Thus at a place just outside the conductor the flux of energy is into the conductor normally and is equal per unit area to

$$cH \,.\, E.$$

Thus the total energy flowing into unit length of the conductor from the external field is the integral

$$c\int_s H \,.\, E ds,$$

taken round a ring on the conductor. By symmetry this is equal to

$$cE\int H ds,$$

and
$$\int H ds = \frac{J}{c},$$

so that the energy flowing in per unit length is

$$EJ,$$

which is the energy dissipated into heat in the unit length of the conductor, by Joule's law. We thus see that when a current is flowing along a wire energy flows in sideways from the external field in just sufficient quantities to account for the energy converted into heat. From this point of view the energy appearing in the form of heat is supplied from the aether.

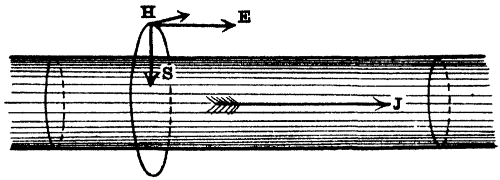


Fig. 96

631. Let us now assume that the wire is a long straight cylindrical one with a circular cross section of radius a. Then if the current flow is uniform all along the wire the magnetic force will be in circles round its axis and the electric force directed along the axis. To obtain a closer insight into the field thus specified we shall find it convenient to refer the field to a system of cylindrical polar coordinates (r, θ, z) with the axis along the axis of the cylinder. The only components of the field vectors at any point which are not zero are $\mathbf{E}_z$ and $\mathbf{H}_\theta$ and these are symmetrical round the axes of the field. In the conductor these two components are connected by the relations

$$\frac{4\pi\sigma}{c}\mathbf{E}_z = \frac{1}{r}\frac{d}{dr}(r\mathbf{H}_\theta)$$

$$-\frac{1}{c}\frac{d\mathbf{H}_\theta}{dt} = \frac{\partial \mathbf{E}_z}{\partial r},$$

to which the fundamental field equations of Ampère and Faraday reduce under the special circumstances of the present problem. It is of course assumed that it is possible to neglect the displacement current in comparison with conduction current; this is justified in most cases of any real importance.

It follows then that

$$\frac{4\pi\sigma}{c^2}\frac{d\mathbf{E}_z}{dt}=\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathbf{E}_z}{dr}\right).$$

A particular solution of this partial differential equation is*

$$\mathbf{E}_z = Ce^{ipt}J_0(x),$$

where we have written

$$x=\nu r, \qquad \nu^2=-\frac{4\pi i\nu\sigma}{c^2},$$

and where $J_0(x)$ satisfies Bessel's equation

$$J_0''(x)+\frac{1}{x}J_0'(x)+J_0=0.$$

Since the field cannot become infinite inside the cylinder J_0 must be taken as the Bessel function of the first kind, viz.

$$J_0(x)=\frac{1}{\pi}\int_0^\pi e^{ix\cos a}\,da.$$

The density of the current flux at any point in the interior of the cylinder is given by

$$\mathbf{C}_z=\sigma\mathbf{E}_z=\sigma Ce^{ipt}J_0(x),$$

and its distribution over the cross section is thus determined. Remembering the particular approximate forms of the function J_0 when its argument is small and large we notice that for very slowly alternating currents the distribution is practically uniform across the section but for very rapid oscillations it is confined to a very thin layer at the surface.

632. To excite this field in the interior of the conductor we must apply along its outer surface a field of strength

$$\mathbf{E}_{0_z}=Ce^{ipt}J_0(\nu'a),$$

a being the radius.

The total current flowing through a section of the cylinder is

$$J=2\pi\int_0^a\sigma\mathbf{E}_z r\,dr$$

$$=\frac{c^2}{2ip}\left\{r\frac{\partial\mathbf{E}_z}{\partial r}\right\}_{r=a},$$

or

$$J=\frac{c^2Ce^{ipt}}{2ip}\{xJ_0'(x)\}_{x=\nu a}.$$

* Cf. Rayleigh, *Phil. Mag.* (5), XXI. (1886), p. 381; O. Heaviside, *Electrical Papers*, II. pp. 39, 168; J. Stefan, *Wien. Ber.* XCV. (1887), p. 917; *Ann. d. Phys.* XLI. (1890), p. 400; Abraham, *Encykl. d. Math. Wissensch.* V. 18, p. 514.

We may thus write the relation between the complex expressions for J and $\mathbf{E}_{0_z}$ in the form

$$\mathbf{E}_{0_z} = RJ + \frac{1}{c^2} L \frac{dJ}{dt}$$

$$= J\left(R + \frac{ip}{c^2} L\right),$$

whence it follows that

$$R + \frac{ip}{c^2} L = \frac{2ip}{c^2} \left\{\frac{J_0(x)}{xJ_0'(x)}\right\}_{x=\nu a}.$$

From this relation we can get the two real quantities R and L.

The physical meaning of these coefficients R and L can now be directly deduced by an application of Poynting's theorem as above explained. If we now denote by E_{0_z}, H_θ, J the real parts of their respective complex representations as above, we shall find for the energy which enters per unit time and length into the conductor from the external field is

$$-2\pi\,(r\mathbf{S}_r)_{r=a} = \frac{c}{2}\,(\mathbf{E}_z\,.\,r\mathbf{H}_\theta)_{r=a}$$

$$= \mathbf{E}_z J.$$

But from the above this is

$$\mathbf{E}_z J = RJ^2 + \frac{d}{dt}\left(\frac{1}{2c^2} LJ^2\right).$$

Suppose now we integrate this equation over a complete oscillation; the energy entering the conductor must then just be equal to the heat developed in the circuit as Joule's heat: this is given by the first term on the right of the above equation. The second term on the right of this equation which disappears on integration over the whole oscillation must then give the increase of the magnetic energy of the field of the current inside the wire. According to this explanation it is usual to call R the *effective resistance* and L the *effective self-inductance* (per unit length) of the conductor for the particular period of the oscillating current.

633. If the period of oscillation is long or the radius of the wire small νa will be small and we may use the approximate form for J_0 which expresses it as an ascending power series; it is then found that

$$R = R_0\left(1 + \frac{(\theta a^2)^2}{12} - \frac{(\theta a^2)^4}{180} + \dots\right),$$

$$L = \frac{1}{8\pi}\left(1 - \frac{(\theta a^2)^2}{24} + \frac{13\,(\theta a^2)^4}{4320} - \dots\right),$$

where we have used $\qquad \theta = \dfrac{p\sigma}{c^2}; \qquad R_0 = \dfrac{1}{\sigma\pi a^2}.$

The effective resistance only departs slightly from the value for a uniform current.

If the period of the oscillation is very big or the radius of the wire large then we must use the asymptotic representation of the Bessel function by semi-convergent series. We then get the so-called Rayleigh-Stefan formula

$$R = \frac{2}{b}\sqrt{\frac{\nu}{2\sigma c^2}} + \frac{R_0}{4},$$

$$L_i = \frac{2}{b}\sqrt{\frac{c^2}{2\sigma\nu}}.$$

In this case the current only very slightly penetrates into the interior of the conductor so that the resistance and self-induction are both small. This is of course merely a particular example of a general principle established in connection with more general types of alternating fields in a previous chapter.

634. On the flux of energy in radiation fields. The physical significance of the energy flux or Poynting vector may be further illustrated by another application of practical importance. When energy travels by radiation the direction of the flux is along the ray, so that this vector gives not only the direction but also the intensity of the ray (the intensity of a ray being measured by the energy that passes along it per unit time). In the ordinary propagation of plane optical waves in an isotropic medium the ray of light is perpendicular to the front of the waves, because the electric and magnetic force vectors are both in the wave front and the energy flux is normal to both. The energy in this case travels normal to the wave front along the direction of propagation. In crystalline media on the other hand it is the two stream vectors, the electric and magnetic fluxes, that are in the wave front; but the electric and magnetic force vectors are not coincident in direction with the corresponding fluxes and do not therefore in general lie in the wave front. The ray in this case is therefore not normal to the wave front and the energy which flows along it thus crosses the front obliquely. It may be regarded as providing the relation between the ray (i.e. the energy flux) and the wave front. We define the ray in the general case as the path of the energy and its direction at any point will therefore be that of the Poynting energy flux.

635. Returning however to the case of isotropic media let us consider in further details the circumstances involved in the propagation of a train of plane waves travelling parallel to the axis of x in an absorbing medium, the waves being polarised so that the magnetic force is parallel to the axis of z, and the electric force parallel to the axis of y in a system of rectangular coordinates. The equations of propagation are then just as before

$$\frac{4\pi}{c}\left(\sigma\mathbf{E}_y + \frac{\epsilon}{4\pi}\frac{d\mathbf{E}_y}{dt}\right) = -\frac{\partial\mathbf{H}_z}{\partial x}, \qquad -\frac{1}{c}\frac{d\mathbf{H}_z}{dt} = \frac{\partial\mathbf{E}_y}{\partial x}.$$

It follows that $$\frac{4\pi\sigma}{c^2}\frac{d\mathbf{E}_y}{dt} + \frac{\epsilon}{c^2}\frac{d^2\mathbf{E}_y}{dt^2} = \frac{\partial^2\mathbf{E}_y}{\partial x^2},$$

which is the equation of propagation. Considering the case of radiation of period $\frac{2\pi}{p}$ so that

$$\mathbf{E}_y = E_0 e^{ipct-(a+ib)x},$$

this gives $$\frac{4\pi ip\sigma}{c} - \epsilon p^2 = (a+ib)^2,$$

$$a + ib = + ip\sqrt{\epsilon}\left(1 - \frac{4\pi\sigma i}{pc}\right)^{\frac{1}{2}}.$$

On separating the real parts we have

$$\mathbf{E}_y = E_0 e^{-ax}\cos(pct - bx),$$

corresponding to

$$\mathbf{H}_z = \frac{E_0}{p}(a^2+b^2)^{\frac{1}{2}} e^{-ax}\sin(pct - bx + \theta),$$

where $$\tan\theta = \frac{b}{a},$$

and $$\mathbf{C}_y = \frac{E_0}{p}(a^2+b^2)^{\frac{1}{2}} e^{-ax}\sin(pct - bx + 2\theta).$$

Thus as we have already seen the magnetic flux is in a different phase from the electric force, involving a diminution in their vector product which determines the energy transmitted across any plane.

636. The energy per unit volume of the radiation at any part of the wave consists of an electric part $\frac{1}{8\pi}\mathbf{E}_y\mathbf{D}_y$, or $\frac{1}{8\pi}\epsilon\mathbf{E}_y{}^2$ and a magnetic part $\frac{\mathbf{H}^2}{8\pi}$: and the ratio of the time averages of these is exactly as before

$$\frac{\epsilon}{p^2(a^2+b^2)} \quad \text{or} \quad \left(1 + \frac{4\pi^2\sigma^2}{p^2c^2\epsilon^2}\right)^{-\frac{1}{2}},$$

which is constant, but not unity except for transparent media. The time rate of propagation of energy is, by Poynting's theorem,

$$\frac{dE}{dt} = \frac{[\mathbf{HE}]}{4\pi c} = \frac{E_0{}^2}{4\pi pc^2} e^{-2ax}\left(\tfrac{1}{2}b^2 + \text{periodic term}\right).$$

Across the plane $x = 0$ it is therefore on the average

$$\frac{E_0{}^2 b^2}{8\pi pc^2},$$

which corresponds to a density of energy equal to the mean square of electric force travelling with the speed $\frac{pc}{a}$ of the waves. This involves the result* that only the fraction

$$\frac{2}{\epsilon}\Big/\left\{1 + \left(1 + \frac{4^2\pi^2\sigma^2}{p^2c^2\epsilon^2}\right)^{\frac{1}{2}}\right\}$$

* Cf. Larmor, *Aether and Matter*, p. 135.

of the total energy of the wave system can be considered as propagated; in the case of an undamped wave train this is only the purely aethereal part. The aethereal wave train, passing across the material medium, sets its molecules into sympathetic independent vibration: the energy of these vibrations constitutes a part of the total energy per unit volume, but that part is not propagated. This remark applies equally to all optical theories in which change of velocity of propagation is traced to the influence of sympathetic vibrations of the molecules, in fact it applies to all cases in which velocity depends upon the wave length.

637. We must however leave these considerations and return to the discussion of further aspects of the general flux of energy in radiation fields. We first consider the flux of energy in the field surrounding the ideally simple type of vibrator discussed in § 538 of chapter XII. It was there shown that the field of a small vibrating electric doublet of moment $f(ct)$ at time t and placed at the origin and along the axis of a system of spherical polar coordinates reduces at a large distance from the vibrator to the simple radiation field in which the electric and magnetic forces are simply

$$\mathbf{E}_\theta = \mathbf{H}_\phi = \frac{\sin\theta f''(ct-r)}{r},$$

and the wave front surfaces are the spheres

$$r = \text{const.}$$

It follows therefore by direct application of Poynting's theorem that the flux of energy at the point (r, θ, ϕ) is radially outwards, i.e. in the direction of propagation, and of density per unit area

$$\frac{\sin^2\theta}{4\pi c}\frac{(f''(ct-r))^2}{r^2}.$$

The total flux over the sphere of radius r is thus

$$\frac{(f''(ct-r))^2}{4\pi c}\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\, d\theta\, d\phi = \frac{2}{3c}(f''(ct-r))^2.$$

638. In the particular case when the vibrations are periodic so that we may take

$$f(ct-r) = A\sin p(ct-r),$$

it is on the time average

$$\frac{A^2p^4}{3c},$$

or if λ is the wave length this is

$$\frac{16\pi^4 A^2}{3c\lambda^4}.$$

This energy which is radiated outwards from the vibrator is of course lost to the system, and it must have been drawn from the store of the energy

in the original vibrations. Thus unless the oscillations of such a system can be maintained by external agency they will gradually decay owing to the dissipation of their energy by radiation. It is important to notice that the rate of dissipation increases rapidly as the wave length is decreased.

We must therefore conclude that however ideal the conditions may be there must necessarily be dissipation accompanying any electrical vibrations and that therefore it seems absolutely essential to take such dissipation into account in the complete theory of such vibrations: and then of course it is necessary to go still further and treat the problem along the lines suggested by Prof. Love, as one which involves a finite time extent, so that the conditions on the initiation of the field require specification. In this way other aspects of the dissipation process arising from radiation are brought out: we can illustrate them by two simple cases* analysed in detail in chapter XII.

639. In the case of the charge oscillating on the perfectly conducting sphere the field inside the sphere $r = ct + a$ is specified by

$$\mathbf{E}_r = \frac{4E}{\sqrt{3}} \frac{\cos\theta}{r^3} \sqrt{1 - \frac{r}{a} + \frac{r^2}{a^2}}\, e^{-\Theta} \sin(\sqrt{3}\,\Theta + \beta_r),$$

$$\mathbf{E}_\theta = \frac{2E}{\sqrt{3}} \frac{\sin\theta}{r^3} \sqrt{1 - \frac{r}{a} + \frac{r^2}{a^2}}\, e^{-\Theta} \sin(\sqrt{3}\,\Theta + \beta_\theta),$$

$$\mathbf{H}_\phi = -\frac{2E}{\sqrt{3}} \frac{\sin\theta}{ar^2} \sqrt{1 - \frac{r}{a} + \frac{r^2}{a^2}}\, e^{-\Theta} \sin(\sqrt{3}\,\Theta + \beta_\phi),$$

the notation being exactly as before

$$\Theta = \frac{ct - r + a}{2a}.$$

In the initial state the aether in the region between the spheres r and $r + dr$ possesses electric energy of amount $\frac{1}{8\pi} 4\pi r^2 dr E^2 \frac{2}{r^6}$ or $r^{-4}E^2 dr$ and the total energy of the field is $\frac{1}{3}E^2 a^{-3}$. In the subsequent state of wave disturbance the same portion of the medium possesses magnetic energy of amount

$$\frac{1}{8\pi} 4\pi r^2 dr \cdot \frac{2}{3} r^2 \cdot \frac{4E^2}{3r^4 a^2} \left\{\sin\sqrt{3}\,\Theta - \frac{r}{a} \sin\left(\sqrt{3}\,\Theta - \frac{\pi}{3}\right)\right\}^2 e^{-2\Theta}.$$

Thus as soon as the wave front has travelled to a distance from the conducting surface which is at all large compared with the radius of this surface the factor $r^{-2\Theta}$ will be small except in the immediate neighbourhood of the wave front: the energy of the wave motion will be accumulated near the wave front. Also when r is large compared with a the above expression may approximately be replaced by

$$\frac{2}{9} \frac{E^2}{a^4} e^{-2\Theta} \left\{1 - \cos\left(2\sqrt{3}\,\Theta - \frac{2\pi}{3}\right)\right\} dr.$$

* Both cases are worked out in detail by Prof. Love in the papers there cited.

We may calculate the energy between the wave front and a spherical surface within it and not far from it by integrating this expression. Consider the case where the inner of these surfaces is at a distance of half a wave length behind the front, i.e. at a distance $2\pi a/\sqrt{3}$. The magnetic energy between the surfaces is approximately

$$\frac{2}{9}\frac{E^2}{a^4}\int_0^{\pi/\sqrt{3}} e^{-2\Theta}\left\{1+\frac{1}{2}\cos 2\sqrt{3}\,\Theta-\frac{\sqrt{3}}{2}\sin 2\sqrt{3}\,\Theta\right\} 2ad\Theta,$$

which is $$\frac{1}{6}E^2a^{-3}\left(1-e^{-\frac{2\pi}{\sqrt{3}}}\right).$$

640. If we had taken the first wave length of the advancing wave instead of the first half wave length, we should have found

$$\frac{1}{6}E^2a^{-3}\left(1-e^{-\frac{4\pi}{\sqrt{3}}}\right)$$

as the approximate value of the magnetic energy between the surfaces. If we calculate the electric energy in the same way and to the same order of approximation, we find the same values, so that the total energy between the two surfaces is approximately equal to

$$\frac{1}{3}\frac{E^2}{a^3}\left(1-e^{-\frac{2\pi}{\sqrt{3}}}\right),$$

when the surfaces are half a wave length apart and

$$\frac{1}{3}\frac{E^2}{a^3}\left(1-e^{-\frac{4\pi}{\sqrt{3}}}\right)$$

when they are a wave length apart. The terms omitted in the calculations are small compared with those retained in the order $\frac{a}{r}$ and higher powers of $\frac{a}{r}$, r denoting the radius of the wave front. It appears therefore that the energy of the electrostatic field is propagated outwards with the wave in such a way that the energy that was initially within a spherical surface surrounding the conductor is the energy of the wave motion when that surface is the wave front and it is gathered up close behind the wave front. When the wave front is at a great distance from the conductor the accumulation of energy at the front is so great that all but about $\frac{2}{75}$ of the total initial energy of the field is gathered up in the first half wave length and all but $\frac{1}{1400}$ of it is gathered up in the first wave length. Thus as the wave advances it transforms into electromagnetic energy the excess of the statical energy of the initial field over that of a free charge distribution of the same total amount (in this case zero) on the same conductor; and this electromagnetic energy is transferred continually towards the front of the advancing wave in such a way that at a distance from the conductor the wave practically

passes as a pulse or in other words as the wave front of the disturbance travels out over the field the energy in that field is picked up almost entirely by the first portions of the disturbance which travel across it.

Similar considerations of course apply to the case of a Hertzian oscillator and they show very vividly how unsuitable such an arrangement really must be for the generation of a continuous train of waves unless it is possible in some way or other to maintain by forcing the vibrations of which it is capable.

641. The results obtained above concerning the loss of energy by radiation from a Hertzian oscillator can be applied to deduce the loss experienced by an electron moving with an acceleration. The method is obvious and need not be given in full; it is found that the amount of energy per unit time that is lost to the electron moving with a comparatively small velocity is

$$\frac{2e^2}{3c^3}\left[\frac{d\mathbf{v}}{dt}\right]^2,$$

so that the total amount lost during the time between the effective instants t_1 and t_2 is

$$\frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2}\left[\frac{d\mathbf{v}}{dt}\right]^2 dt.$$

This constant draining of energy from the electron in accelerated motion is often interpreted as implying the existence of a resistance to its accelerated motion and in many cases this is a convenient way of describing the dissipation action of the radiation. The idea is obtained from the fact that the total loss in the effective interval stated may be written in the form

$$\frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2}\left(\frac{d\mathbf{v}}{dt}\right)^2 dt = \frac{2e^2}{3c^3}\Big|(\mathbf{v}\,.\,\dot{\mathbf{v}})\Big|_{t_1}^{t_2} - \frac{2e^2}{3c^3}\int_{t_1}^{t_2}(\mathbf{v}\,.\,\ddot{\mathbf{v}})\,dt.$$

Now the first term on the right disappears, if, in the case of a periodic motion the integration is extended to a full period; also, if at the instants t_1 and t_2 either the velocity or the acceleration is zero. In either of these cases

$$\frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2}\left(\frac{d\mathbf{v}}{dt}\right)^2 dt = -\frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2}(\mathbf{v}\,.\,\ddot{\mathbf{v}})\,dt,$$

so that the energy dissipated is exactly the same as if the force

$$\frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{v}},$$

acted on the electron for the period under consideration.

This conception of a retarding force on the electron has been used by Planck to account for the dissipation of the energy of the electronic vibrations* inside a material atom and the consequent absorption of energy from incident radiation fields which is observed: on this view of the matter the energy of the incident radiation is merely absorbed by the vibrating electrons to be immediately re-emitted as radiation, of a dispersed or scattered type. It

* *Berlin Ber.* 1902.

would appear* however that this representation of the matter is not effective in accounting for the absorption that is really observed in many cases; but as no satisfactory explanation yet appears to be forthcoming concerning the mechanism of this inter-molecular absorption the suggestion offered by Planck is at least illustrative of the possibilities of a strict theory.

642. General electrodynamic theory†. The tendency of the physical investigations outlined in the previous chapters has been towards the construction of a dynamical theory which shall give a consistent account of electrodynamic phenomena, i.e. to answer the question as to the possibility of obtaining a complete parallel to the processes in any electromagnetic field from those observed in some imaginary system of masses moving according to the ordinary laws of mechanics. To reply completely to such a question it is not necessary to make any definite assumptions as to the mechanism underlying the phenomena; all we have to do is to show that they can be described by means of the general equations of mechanics.

The most general dynamical principle which determines the motion of every material system is the law of Least Action, expressible in the usual form

$$\delta \int (T - W)\, dt = 0,$$

wherein T denotes the kinetic energy and W the potential energy of the system in any configuration and formulated in terms of any coordinates that are sufficient to specify the configuration and motion in accordance with its known properties and connections; and where the variation refers to a fixed time of passage of the system from the initial to the final configuration considered. The power of this formula lies in the fact that once the energy function is obtained in terms of any measurements of the system that are convenient and sufficient for the purposes in view, the remainder of the investigation involves only the exact processes of mathematical analysis.

643. Now we have in the first section succeeded in obtaining expressions for the potential and kinetic energies associated with any electromagnetic field and it thus only remains to interpret these functions in terms of suitable coordinates, before applying the general laws of dynamics to determine the sequence of events in any such system. But whatever view we may take as to the constitution of the aether and the electrons it is quite obvious from the whole of the preceding discussion that all electrical effects must be explicable on the hypothesis of the aether with the electrons or discrete atomic charges moving about in it freely or grouped into material atoms: thus as far as we are at present concerned the only difference between the

* Cf. Lorentz, *The Theory of Electrons*, p. 140.

† Cf. Larmor, *Aether and Matter*, Ch. VI; Lorentz, *La theorie electromagnetique*, §§ 55–61; Helmholtz, *Ann. Phys. Chem.* XLVII. (1892), p. 1; Sommerfeld, *Ann. d. Phys.* XLVI. (1892), p. 139; Reiff, *Elastizität und Elektrizität* (Leipzig, 1893).

aether and any material medium must simply be due to the presence of convection currents of electrons; that is, wherever there is matter there are these convection currents. Thus in a mechanical theory the electrodynamic state of any system will be completely known if we can specify the positions and motions of all the electrons in it together with the displacement, in Maxwell's sense, in the aether, and herein we have sufficient data for our present dynamical analysis. Of course for the purposes of electrodynamic phenomena of material which we can only test by observation and experiment on matter in bulk a complete atomic analysis of this kind is almost useless; for we are unable to take cognizance of the single electrons to which this analysis has regard. The development of the theory which is to be in line with experience ought instead to concern itself with an effective differential element of volume, containing a crowd of molecules numerous enough to be expressible continuously as regards their average relations, as a volume density of matter. As regards the actual distribution in the element of volume of the really discrete electrons, all that we can usually take cognizance of is an excess of one kind, positive or negative, which constitutes a volume density of electrification, or else an average polarisation in the arrangement of the groups of electrons in the molecules which must be specified as a vector by its intensity per unit volume: while the movements of the electrons, free and paired, in such elements of volume must be combined into statistical aggregates of translational fluxes and molecular whirls of electrification. With anything else than mean aggregates of the various types that can be thus separated out, each extended over the effective element of volume mechanical science, which has for its object matter in bulk as it presents itself to our observation and experiment, is not directly concerned. Nevertheless it is convenient on account of simplicity to formulate the problem in terms of the separate electrons and to reserve the details of the process of averaging, which must in reality be implied throughout, until the mechanical relations of the system have been formulated in full.

644. Let us therefore proceed directly to the formulation of the mechanical relations of a system of discrete electrons in a field of aether, the potential or electrostatic energy of this system being expressed by the integral

$$W = \frac{1}{8\pi}\int \mathbf{E}^2 dv,$$

extended throughout the entire volume of the electrodynamic field, whilst the kinetic energy* is expressed by

$$T = \frac{1}{8\pi}\int \mathbf{B}^2\, dv.$$

* The principle of Least Action was employed in the manner here adopted by Prof. Larmor but with the kinetic energy expressed in terms of the vector potential. The present deduction was given by the author, *Phil. Mag.* XXXII. (1916), p. 195.

The complete Lagrangian function for the system is therefore

$$L = L_0 + \frac{1}{8\pi}\int(\mathbf{B}^2 - \dot{\mathbf{E}}^2)\, dv,$$

L_0 being that part which does not depend on the aethereal configuration as specified in the displacement $\frac{\mathbf{E}}{4\pi}$, but which does depend essentially on the size, constitution and motion of the electronic nucleus, as well as on the forces, not of electric origin, exerted on it from the material atoms, or otherwise, if such are presumed to exist.

645. We could now conduct the variation directly were it not for the circumstance that our variables are not wholly independent: in fact the variations of **E** and **B** are subject to the conditions as

$$\int \operatorname{div} \mathbf{E}\, dv - 4\pi\Sigma q = 0,$$

and
$$\int\left(\operatorname{curl} \mathbf{B} - \frac{1}{c}\frac{d\mathbf{E}}{dt}\right) dv - \frac{4\pi}{c}\Sigma q\dot{\mathbf{r}} = 0.$$

In these expressions Σ denotes a sum relative to all the electrons in the volume considered, each with its proper charge q and velocity $\dot{\mathbf{r}}$, $\mathbf{r}$ being the position vector of the typical electron. The second relation is a vector one and is therefore equivalent to three independent equations.

Hence we must now introduce into the variational equation four Lagrangian undetermined functions of position ϕ, $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{A}_z$, the last three of which may be regarded as the rectangular components of a vector **A**. It is thus the variation of

$$\int L dt + \int dt\left[\int\frac{dv}{4\pi}\left\{\phi \operatorname{div} \mathbf{E} - \left(\mathbf{A},\ \operatorname{curl}\ \mathbf{B} - \frac{1}{c}\frac{d\mathbf{E}}{dt}\right)\right\} - \Sigma q\phi + \Sigma\frac{q}{c}(\mathbf{A}\dot{\mathbf{r}})\right]$$

that is to be made zero; afterwards determining the form of ϕ and **A** to satisfy the restrictions which necessitated their introduction. In conducting the variation we can now treat the electric force, magnetic induction and the position coordinates of the electrons as all independent.

646. As regards the electrons q the variation gives

$$\int dt\left\{\delta L_0 - \Sigma q\,(\delta\mathbf{r}\nabla)\,\phi + \Sigma\frac{q}{c}(\dot{\mathbf{r}},\,(\delta\mathbf{r},\,\nabla)\,\mathbf{A}) + \Sigma\frac{q}{c}(\delta\dot{\mathbf{r}},\ \mathbf{A})\right\}$$

where **A** must now be regarded as assuming the succession of values it takes as the point whose position is defined by the vector **r** moves through the aether, not the succession of values it takes at a fixed point. Integrating by parts we get that this part of the variation is equal to terms at the time limits together with

$$\int dt\left\{\delta L_0 - \Sigma q\,(\delta\mathbf{r}\nabla)\,\phi + \Sigma\frac{q}{c}(\dot{\mathbf{r}},\,(\delta\mathbf{r}\nabla)\,\mathbf{A}) - \Sigma\frac{q}{c}\left(\delta\mathbf{r}\,\frac{\delta\mathbf{A}}{dt}\right)\right\}$$

where the symbol $\frac{\delta \mathbf{A}}{dt}$ is used to denote the time rate of variation of $\mathbf{A}$ relative to the moving electron: thus

$$\frac{\delta \mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\dot{\mathbf{r}}\nabla)\,\mathbf{A}.$$

Now
$$(\dot{\mathbf{r}}, (\delta\mathbf{r}\nabla)\,\mathbf{A}) - (\delta\mathbf{r}\,(\dot{\mathbf{r}}\nabla)\,\mathbf{A}) = ([\dot{\mathbf{r}}\ \mathrm{curl}\,\mathbf{A}]\,\delta\mathbf{r})$$

so that the main part of the variation of the integral due to the electrons is

$$\int dt\left\{\delta L_0 + \Sigma q\left(\delta\mathbf{r}, -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}[\dot{\mathbf{r}}, \mathrm{curl}\,\mathbf{A}]\right)\right\}.$$

As regards the variation of the state of the free aether defined by the vectors $\mathbf{E}$ and $\mathbf{B}$ we have the terms

$$\frac{1}{4\pi}\int dt\int dv\left\{(\mathbf{B}\delta\mathbf{B}) - (\mathbf{E}\delta\dot{\mathbf{E}}) + \phi\ \mathrm{div}\,\delta\mathbf{E} - (\mathbf{A}\ \mathrm{curl}\,\delta\mathbf{B}) + \left(\mathbf{A}\,\frac{d\delta\mathbf{E}}{dt}\right)\right\}.$$

On integrating the last three terms by parts we get that this part of the variation is equal to terms at the time limits together with

$$\int dt\int\frac{dv}{4\pi}\left[(\mathbf{B} - \mathrm{curl}\,\mathbf{A}, \delta\mathbf{B}) - \left(\mathbf{E} + \nabla\phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \delta\mathbf{E}\right)\right]$$
$$- \int dt\int\frac{df}{4\pi}\{\phi\delta\mathbf{E}_n - [\mathbf{A}\delta\mathbf{B}]_n\}$$

wherein the last integral is taken over the infinite bounding surface of the field.

As usual in such problems we are not concerned with the terms at the time limits because they may be as a rule suitably chosen. Also if we impose the natural condition on ϕ that it should be continuous everywhere and vanish at the infinitely distant boundary, the surface integrals introduced also vanish and we are left with the complete variation of our generalised Lagrangian function in the form

$$\int\delta L_0 - \Sigma\int_{t_0}^{t} dt\,q\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c}[\dot{\mathbf{r}}\ \mathrm{curl}\,\mathbf{A}] + \mathrm{grad}\,\phi, \delta\mathbf{r}\right)$$
$$- \int_{t_0}^{t} dt\int\left(\frac{1}{c}\dot{\mathbf{A}} + \frac{1}{c}\mathbf{E} + \mathrm{grad}\,\phi, \frac{\delta\mathbf{E}}{4\pi}\right)dv + \int_{t_1}^{t_2} dt\int dv\,(\mathbf{B} - \mathrm{curl}\,\mathbf{A}, \delta\mathbf{B}).$$

647. Now the variation $\delta\mathbf{r}$ which determines the virtual displacement of the electron q and the variations $\delta\mathbf{E}$ and $\delta\mathbf{B}$ which specify the electric displacement of a point in the free aether, can now be considered as independent and arbitrary: hence the coefficient of each must vanish separately in the dynamical variational equation. We thus obtain three sets of equations of types

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \mathrm{grad}\,\phi = 0,$$

$$\mathbf{B} - \mathrm{curl}\,\mathbf{A} = 0$$

and
$$\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{\mathbf{r}}}\right) - \frac{\partial L_0}{\partial \mathbf{r}} + \frac{1}{c}\dot{\mathbf{A}} + \mathrm{grad}\,\phi - \frac{1}{c}[\dot{\mathbf{r}}\ \mathrm{curl}\,\mathbf{A}] = 0,$$

where for simplicity L_0 has been assumed to depend only on the coordinates and velocities of the electronic charges. These equations are the same as

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} - \operatorname{grad}\phi,$$

$$\mathbf{B} = \operatorname{curl}\mathbf{A},$$

$$\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{\mathbf{r}}}\right) - \frac{\partial L_0}{\partial \mathbf{r}} = -\frac{1}{c}\dot{\mathbf{A}} - \operatorname{grad}\phi + \frac{1}{c}[\dot{\mathbf{r}}\mathbf{B}],$$

These are the differential equations which determine the sequence of events in the system. The first two show that the functions ϕ and A, introduced as undetermined multipliers, are respectively the scalar and vector potentials of the theory. The third, expressed in the ordinary language of electrodynamics which avails itself of the conception of force, shows that

$$-\frac{1}{c}\frac{d\mathbf{A}}{dt} - \operatorname{grad}\phi + \frac{1}{c}[\dot{\mathbf{r}}\mathbf{B}] = \mathbf{E} + \frac{1}{c}[\dot{\mathbf{r}}\mathbf{B}]$$

is the electric force which tends to accelerate the motion of the electrons q, each electron being presumed to have a constitution which enables it to offer a kinetic reaction of an electric nature to the action of this force. We here speak of the electric force acting on the single electron which in strictness is really more than our analysis gives us. The equation thus interpreted should really have a Σ, a sign of summation in front of it to show that it is an aggregate equation for all the electrons in the volume element, with which we are in reality dealing. Certain considerations will however be offered which point to the conclusion that the result is correct if interpreted for the single electron separately, and we shall therefore often make use of the result in this form.

648. Thus the whole mechanics of the electromagnetic system is summed up in terms of these forces of ordinary type acting in the aggregate on the individual electrons. Therefore for a complete specification of such a system it is merely necessary to know in addition to the ordinary dynamical relations of the masses moving in it, also the aggregate of the 'applied' forces acting on the individual electrons which they contain; the force of electrodynamic origin acting on the matter in bulk is the aggregate of the forces acting on its electrons, and it is only in these impressed forces that the electrical conditions manifest themselves. The total force of electrodynamic origin on any body is thus*

$$\Sigma q\left(\mathbf{E} + \frac{1}{c}[\mathbf{uB}]\right),$$

* The occurrence of the magnetic induction instead of the magnetic force in this expression is important and must be emphasised. It points once again to the conclusion that the induction is the fundamental vector of the theory, as in fact is obvious from our previous discussions of the energy relations of the magnetic field; in fact from one point of view the only essential point where our treatment of these relations differs from that usually given, lies in the choice of

which makes up in all a static part $\Sigma q\mathbf{E}$ and a kinetic part

$$\Sigma q \frac{1}{c} [\mathbf{uB}].$$

We shall return to a more detailed examination of these forces in a later paragraph, but it is perhaps worth while considering at the present stage the results of an experiment made by H. A. Wilson to distinguish between the electromotive force acting on the electrons and the electric force in the aether, by examining the effect on a dielectric body of motion through the aether.

Wilson rotated a hollow dielectric cylinder in a uniform magnetic field parallel to its axis, thus bringing in a force of electrodynamic origin which for all the electrons in the dielectric acts radially outwards from the axis of the cylinder, i.e. perpendicular to the direction of their motion and to the lines of force of the magnetic field and of amount at any point equal to

$$\frac{1}{c} (\mathbf{Bv}),$$

$\mathbf{v}$ being the velocity. This force gives rise to an electric displacement across the shell of dielectric from the inner to the outer surface and in consequence there will be a difference of potential established between these surfaces; by coating them metallically and connecting by sliding contacts with the quadrants of a galvanometer a measurement of the potential difference was easily made.

The force producing the displacement being merely of kinetic origin there will be no aethereal part in the displacement which will therefore be of intensity per unit volume equal to

$$\mathbf{D} = \frac{\epsilon - 1}{4\pi} \mathbf{Bv},$$

ϵ being the dielectric constant. Wilson verified that the potential difference between the surfaces was proportional to

$$\frac{\epsilon - 1}{4\pi} \mathbf{Bv},$$

with sufficient exactness to justify the basis of the explanation here offered. The result of this experiment also has another important bearing which will be mentioned later.

649. The Hall effect. In 1879 Hall* found that the lines of flow of an electric current through a metallic conductor are distorted when the conductor is placed in a magnetic field, the distortion being of the character of that produced by a slight additional electromotive force directed at right angles

the magnetic induction instead of the more usual magnetic force, as the 'aethereal vector.' Again the conclusion that the induction is the true magnetomotive force would appear to invalidate the argument of Kempken (*Ann. d. Phys.* XX. (1906)) whereby he derives the fact that I/μ is constant in permanent magnets, instead of I, which is usually regarded as remaining constant in these cases. Cf. also Gans, *Ann. d. Phys.* (4), XVI. (1905), p. 178.

* *Phil. Mag.* IX. (1880), pp. 225, 301.

to the current and to the direction of the magnetic force. Thus, if a metal bar, along which a current is flowing, be placed in a magnetic field perpendicular to the direction of the current, the current will tend to be deflected from its course and to move towards the one or the other of the edges of the bar. This cross flux of electricity will not however be permanent since it will give rise to a slight accumulation of charge on the outer edges of the bar, the additional electric field thus set up tending to oppose the transverse current. A final steady state of equilibrium will be attained in which there is no cross flux of electricity, the additional electric field being just sufficient to balance the transverse electromotive force arising from the magnetic field. But in this state of equilibrium there will be a difference of potential between the opposite edges of the bar which may be measured by connecting them up to the opposite terminals of an electrometer. This is the measurement made by Hall.

The simple explanation of this phenomenon is almost obvious if we adopt our former view that the current in the metal consists simply of a diffusion of negative electrons through the spaces between the atoms: the action of the electric driving field is then to impose on the swarm of free electrons thus imagined a finite average velocity in the direction opposite to that of the electric force. If we suppose for the moment that this average drift velocity is $\mathbf{v}$ then the average action of the magnetic field of force of intensity $\mathbf{H}$ may be represented as a force on each electron equal to $\frac{1}{c}[\mathbf{vB}]$, if $\mathbf{B}$ is the induction due to the force $\mathbf{H}$. The statistical effect of all these forces on all the electrons comprising the current is exactly the same as that of an applied electromotive force in the same direction This is the gist of the explanation which has been offered by Riecke*, Drude† and Thomson‡, but on account of the great importance of the matter it seems worth while at least indicating the more exact analysis§, if only as an example of the application of previous principles to such problems.

650. The general basis of the present explanation involves the usual assumption that the whole of the electrical properties of the metals arises entirely in the average motion impressed by the external circumstances on the swarm of (negative) electrons which are otherwise moving about quite irregularly and freely in the spaces between the atoms or atom-complexes. In the absence of actions from any external agency the electrons are presumed to be moving about in such a manner that the distribution of velocities among them at any instant is precisely that specified by Maxwell's law so that if N is the total number of free electrons per unit volume the number

* *Wied. Ann.* XLVI. (1898). † *Ann. d. Phys.* I. (1900), p. 566; III. (1900), p. 369.

‡ *Rapp. Congr. Phys.* III. p. 143 (Paris, 1900)

§ This was first given by Gans, *Ann. d. Phys.* XX. (1906). The present mode of treatment is due to the author (*Phil. Mag.* XXX. (1915), p. 526). Cf. also N. Bohr, *l.c.* p. 313.

in the same volume with their velocity components between (ξ, η, ζ) and $(\xi + d\xi, \eta + d\eta, \zeta + d\zeta)$ is given as usual by the formula

$$\delta N = N \sqrt{\frac{q^3}{\pi^3}}\, e^{-qu^2} d\xi\, d\eta\, d\zeta,$$

wherein $u^2 \equiv \xi^2 + \eta^2 + \zeta^2$ and q is a constant connected with the mean square of u, viz. $u_m{}^2$, for all the N electrons by the relation

$$q = \frac{3}{2u_m{}^2}.$$

When however external electric and magnetic fields are in action throughout the interior of the metal all this alters because then the velocity of each electron while on its free path is modified by the forces in the external fields. The main problem is now to determine the new steady velocity distribution. If we follow the assumptions of our previous investigations and regard the electrons and atoms as perfectly elastic spheres, the latter being of such comparatively large mass that their energy and motion are unaffected by the collision with them of the electrons, the problem is comparatively simple, for if the new velocity distribution is one in which

$$\delta N = f(\xi, \eta, \zeta, x, y, z, t)\, d\xi\, d\eta\, d\zeta,$$

then we know from our previous investigation that the function f must satisfy the differential equation

$$\mathbf{F}_x \frac{\partial f}{\partial \xi} + \mathbf{F}_y \frac{\partial f}{\partial \eta} + \mathbf{F}_z \frac{\partial f}{\partial \zeta} + \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} + \frac{f}{\tau_m} = \frac{f_0}{\tau_m},$$

where f_0 is used, as there, for

$$N \sqrt{\frac{q^3}{\pi^3}}\, e^{-qu^2}.$$

Here $(\mathbf{F}_x, \mathbf{F}_y, \mathbf{F}_z)$ are the components of the applied forces on the typical electron resulting from the action of the field; and all that is now necessary is to find these forces.

651. For the sake of simplicity we shall assume that the magnetic induction is directed along the x-axis of the coordinates chosen and is of uniform intensity $\mathbf{B}$ throughout the metal. The electric field is of uniform intensity $\mathbf{E}$ but for the sake of generality we need not specify its direction beyond stating that its components in the three principal directions are $\mathbf{E}_x$, $\mathbf{E}_y$, $\mathbf{E}_z$. The motion of a typical electron while traversing a free path is thus given by the following equations

$$m \frac{d\xi}{dt} = e\mathbf{E}_x,$$

$$m \frac{d\eta}{dt} = e\mathbf{E}_y + \frac{e\mathbf{B}}{c} \zeta,$$

$$m \frac{d\zeta}{dt} = e\mathbf{E}_z - \frac{e\mathbf{B}}{c} \eta.$$

The last two equations may be written in the form

$$\frac{d^2\eta}{dt^2} = -\frac{e^2\mathbf{B}^2}{m^2c^2}\eta + \frac{e\mathbf{E}_z}{m}\frac{e\mathbf{B}}{mc},$$

$$\frac{d^2\zeta}{dt^2} = -\frac{e^2\mathbf{B}^2}{m^2c^2}\zeta - \frac{e\mathbf{E}_y}{m}\cdot\frac{e\mathbf{B}}{mc},$$

so that if we use

$$n = \frac{e\mathbf{B}}{mc};$$

we shall have

$$\xi_1 = \xi + \frac{e\mathbf{E}_x}{m}t_1,$$

$$\eta_1 = \eta + \left(\frac{e\mathbf{E}_y}{mn} + \zeta\right)\sin nt_1 + \left(\frac{e\mathbf{E}_z}{mn} - \eta\right)(1 - \cos nt_1),$$

$$\zeta_1 = \zeta + \left(\frac{e\mathbf{E}_z}{mn} - \eta\right)\sin nt_1 - \left(\frac{e\mathbf{E}_y}{mn} + \zeta\right)(1 - \cos nt_1),$$

where (ξ_1, η_1, ζ_1) denote the values of (ξ, η, ζ) at a time t_1 after the instant t at which the latter are measured. Now in any real case n is always small and since the formulae are to be applied with t_1 of the order of magnitude of the time of description of a free path, the product nt_1 will always be very small so that we may approximate to these formulae and use

$$\xi_1 = \xi + \frac{e\mathbf{E}_x}{m}t_1,$$

$$\eta_1 = \eta + \left(\frac{e\mathbf{E}_y}{m} + n\zeta\right)t_1 + \frac{e\mathbf{E}_z n}{2m}t_1^2,$$

$$\zeta_1 = \zeta + \left(\frac{e\mathbf{E}_z}{m} - n\eta\right)t_1 - \frac{e\mathbf{E}_y n}{2m}t_1^2.$$

The accelerations of the typical electron may therefore be expressed as functions of the time by

$$(\mathbf{F}_x, \mathbf{F}_y, \mathbf{F}_z) = \left(\frac{e\mathbf{E}_x}{m}, \frac{e\mathbf{E}_y}{m} + n\zeta + \frac{e\mathbf{E}_z n}{m}t_1, \frac{e\mathbf{E}_z}{m} - n\eta - \frac{e\mathbf{E}_y n}{m}t_1\right),$$

and the above differential formula is to be used with these values.

652. We shall now suppose that the physical conditions are the same at all points of the metal so that the form of the velocity distribution function f will not vary from point to point: in this simple steady case the equation for f becomes

$$\mathbf{F}_x\frac{\partial f}{\partial \xi} + \mathbf{F}_y\frac{\partial f}{\partial \eta} + \mathbf{F}_z\frac{\partial f}{\partial \zeta} + \frac{f}{\tau_m} = \frac{f_0}{\tau_m},$$

of which the solution just as previously may be written in the form

$$f = f_0 - \int_0^\infty e^{-\frac{\tau}{\tau_m}}\frac{d\tau}{\tau_m}\int_{t_1=t-\tau}^{t_1=t}\left(\mathbf{F}_x\frac{\partial f_0}{\partial \xi} + \mathbf{F}_y\frac{\partial f_0}{\partial \eta} + \mathbf{F}_z\frac{\partial f_0}{\partial \zeta}\right)_1 dt_1,$$

the index 1 denoting that the values are to be properly expressed as functions of the auxiliary time variable t_1 used to express the integration.

Now $$\mathbf{F}_x \frac{\partial f_0}{\partial \xi} + \mathbf{F}_y \frac{\partial f_0}{\partial \eta} + \mathbf{F}_z \frac{\partial f_0}{\partial \zeta} = -2q\,(\xi \mathbf{F}_x + \eta \mathbf{F}_y + \zeta \mathbf{F}_z) f_0,$$

so that to the same order of approximation we may write

$$f = f_0 \left(1 + 2q \int_0^\infty e^{-\frac{\tau}{\tau_m}} \frac{d\tau}{\tau_m} \int_{t_1=t-\tau}^{t_1=t} (\mathbf{F}_x \xi + \mathbf{F}_y \eta + \mathbf{F}_z \zeta)\right)_1 dt,$$

and on inserting the values of (X, Y, Z), and (ξ, η, ζ), this is easily found to be

$$f = N \sqrt{\frac{q^3}{\pi^3}}\, e^{-qu^2} \left[1 + \frac{2qe\tau_m}{m} (\mathbf{E}_x \xi + \mathbf{E}_y \eta + \mathbf{E}_z \zeta) - \frac{2qe\tau_m{}^2 n}{m} (\mathbf{E}_z \eta - \mathbf{E}_y \zeta)\right],$$

and again we may write

$$\tau_m = \frac{l_m}{u},$$

where l_m is the length of the mean free path, which can be taken to be independent of the velocity: thus the expression for the general law of distribution of velocities which is correct to the order of approximation adopted is obtained with the function

$$f = N \sqrt{\frac{q^3}{\pi^3}}\, e^{-qu^2} \left\{1 + \frac{2qel_m}{mu} (\mathbf{E}_x \xi + \mathbf{E}_y \eta + \mathbf{E}_z \zeta) - \frac{2qel_m{}^2 n}{mu^2} (\mathbf{E}_z \eta - \mathbf{E}_y \zeta)\right\},$$

or if we adopt the notations of vector analysis we can write this in the more concise form

$$f = N \sqrt{\frac{q^3}{\pi^3}}\, e^{-qu^2} \left\{1 + \frac{2qel_m}{mu} (\mathbf{uE}) - \frac{el_m{}^2}{mcu^2} (\mathbf{B} \,.\, [\mathbf{uE}])\right\}.$$

The various constituents of the complete flux of electricity are now easily calculated on the same lines as adopted above in the simpler example of these same principles; they are obtained as the components of the vector

$$\frac{4\pi N \sqrt{\frac{q^3}{\pi^3}}\, e^2 l_m}{3mq} \left\{\mathbf{E} - \frac{el_m \sqrt{\pi q}}{2mc} [\mathbf{BE}]\right\},$$

or since $$\frac{4\pi N e^2 l_m}{3mq} \sqrt{\frac{q^3}{\pi^3}}.$$

is the ordinary conductivity this may be written in the form

$$\sigma \mathbf{E} + \frac{\sigma e l_m \sqrt{\pi q}}{2mc} [\mathbf{BE}].$$

653. Now in Hall's experiments if the current was sent along the metal in the direction of the y-axis and the magnetic field was directed along the x-axis as above the induced potential gradient in the direction of the z-axis (i.e. perpendicular to the lines of the magnetic field and the direction of the average electronic flow) necessary to maintain the current in its undisturbed path was measured. The conditions parallel to the z-axis are therefore such that there is no flow along that axis: this implies that

$$\mathbf{E}_z + \frac{el_m \sqrt{\pi q}}{2mc} \mathbf{B}\mathbf{E}_y = 0.$$

If J_y denotes the current in the direction of the y-axis then we know that to a first approximation

$$J_y = \sigma \mathbf{E}_y,$$

so that

$$\mathbf{E}_z = -\frac{3\pi}{8Nec}\mathbf{B}J_y,$$

which is exactly the law usually adopted to express empirically the magnitude of this effect. It shows that the potential increases in the positive or negative direction of the z-axis according as e is positive. Now since all the other evidence points to the fact that it is only the negative electrons that are freely moveable we must conclude that the observed potential difference will always be in the direction of decreasing z and proportional both to the strength of the current and to the magnetic induction. In non-magnetic media the magnetic induction is equal to the force so that the effect in such media would be proportional to the force, but in the ferro-magnetic media it would practically be proportional to the polarisation.

Although these theoretical deductions are satisfactorily verified in a large number of cases there are very big discrepancies in many other, and equally prominent cases; for instance the effects observed in iron and many other metals are exactly in the opposite direction to that predicted by theory. These discrepancies are probably however due to secondary constitutional characteristics of the metals concerned and are in no way detrimental to the general theory just developed.

654. There are other actions analogous to that discovered by Hall, which we have however not thought it necessary to explain in detail, although they are theoretically important as tending to confirm the general characteristic basis of the present form of theory. We assumed above in discussing the phenomenon of electrical and thermal conduction, that the thermal conduction current was produced by the convection of the same electrons as give rise to the electric current. Now suppose the ends of a metal bar are kept at different temperatures so that there is a flow of heat along the bar: there will on the average be no drift of the electrons from one end of the bar to the other; but the electrons which are travelling from the hot end to the cold end possess a greater kinetic energy and greater velocity than those which are travelling in the opposite direction. And since the force on an electron travelling in a magnetic field tending to deflect it at right angles to its motion and to the direction of the field is proportional to the velocity of the electron, the force tending to deflect the electrons travelling from the hot end to the cold will be greater than that tending to deflect (in the opposite direction) those travelling from the cold end to the hot. The electrons will thus tend to move on the whole in the direction in which those moving from the hot end to the cold are deflected by the magnetic field. This tendency to cross flow of electricity will, in the steady state, be balanced by an initially

established potential gradient which can be measured as in Hall's experiments. This potential difference which is an essential consequence of the present explanation of these phenomena has in fact been observed, first by Nernst and von Ettinghausen in 1886.

It is not only in metals that the current of conduction is carried by freely moving ions, but also in liquid electrolytes and in gases. The uses to which Thomson has put the phenomenon analogous to the Hall effect in gaseous conductors has already been dealt with in detail: it merely remains to add that the same phenomenon has been often observed and measured in electrolytes with results in full agreement with the theoretical predictions as far as it is possible to follow them.

655. The Faraday and Zeeman effects. The next phenomenon which is fundamental in this theory is concerned with an optical application of the same principle. In 1845* Faraday discovered that on passing a plane polarised ray of light through a piece of glass in the direction of the lines of force of an imposed magnetic field, the plane of polarisation was rotated by an amount proportional to the thickness of glass traversed and the strength of the field: this discovery remained for a long time the only instance of an optical effect brought about by a magnetic field, and the connexion between optical and electromagnetic phenomena which is suggested by it could not be further substantiated. In 1877, however, Kerr† showed that the state of polarisation of the rays reflected by an iron mirror is altered by a magnetisation of the metal; but it was not until 1896‡ when Zeeman discovered the well-known phenomenon of the magnetic separation of spectral lines which now bears his name that a clue to the connection was obtained although previously it had been formulated theoretically by Lorentz.

656. Now we have already explained how the emission and absorption of light may be regarded as due to the vibratory motions of electric charges or electrons contained in the atoms of ponderable bodies. The distribution of these charges and their vibrations may be very complicated, but if we wish only to explain the production of a single spectral line, we may be content with the hypothesis already introduced that there is a single vibrating electron in each molecule with the necessary positive charge. If this electron executes simple harmonic vibrations it may be regarded as executing these about a definite position of equilibrium to which it is bound by a force which

* On the magnetisation of light and the illumination of magnetic lines of force, *Phil. Trans.* 1846, I. p. 1; *Experimental Researches*, 1855, III. p. 1.

† On rotation of the plane of polarisation by reflection from the pole of a magnet, *Phil. Mag.* (5), III. (1877), p. 321; (5), V. (1878), p. 161.

‡ *Phil. Mag.* (5), XLIII. (1897), p. 226. Zeeman has published a connected account of his epoch-making investigations in book form in *Magneto-Optics* (1914, Macmillan), and the reader is referred to this work for further details of the subject.

is proportional to its displacement from that position. The equations of its simple vibratory motion will then be

$$m\ddot{x} = -kx, \qquad m\ddot{y} = -ky, \qquad m\ddot{z} = -kz,$$

of which the general solution is

$$(x, y, z) = (x_0, y_0, z_0) \cos (pt + \epsilon_0),$$

where
$$p_0{}^2 = \frac{k}{m},$$

so that the frequency of the free oscillation is $\dfrac{2\pi}{p_0}$.

657. Let us next consider the influence of an external magnetic field **H** giving rise to a magnetic induction **B**; this will modify the motion of the electron by introducing a new force

$$\frac{e}{c}[\mathbf{v} \,.\, \mathbf{B}],$$

where e denotes the charge of the electron and **v** its velocity. If the magnetic force **H** is parallel to the axis of z, the components of this force are

$$\frac{e\mathbf{B}_z}{c}\frac{dy}{dt}, \qquad -\frac{e\mathbf{B}_z}{c}\frac{dx}{dt}, \qquad 0.$$

Hence the equations of motion of the typical electron become*

$$m\frac{d^2x}{dt^2} = -kx + \frac{e\mathbf{B}_z}{c}\frac{dy}{dt},$$

$$m\frac{d^2y}{dt^2} = -ky - \frac{e\mathbf{B}_z}{c}\frac{dx}{dt},$$

$$m\frac{d^2z}{dt^2} = -kz.$$

The last equation shows that the component vibrations in the direction of the x-axis are not affected by the magnetic field. The particular solution given therefore still holds for this component of the vibration, which is therefore still simply periodic with period $\dfrac{2\pi}{p_0}$. The first two equations are equivalent to the two equations

$$m\frac{d^2(x \pm iy)}{dt^2} = -k(x \pm iy) \mp i\frac{e\mathbf{B}_z}{c}\frac{d}{dt}(x \pm iy),$$

of which the solutions are

$$x + iy = A_1 e^{ip_1 t},$$

and
$$x - iy = A_2 e^{ip_2 t},$$

* This theory is due to Lorentz, *Ann. Phys. Chem.* LXIII. (1897), p. 278. Cf. also Larmor, *Phil. Mag.* (5), XLIV. (1897), p. 503.

wherein
$$p_1^2 + \frac{e\mathbf{B}_z}{mc} p_1 = p_0^2,$$
$$p_2^2 - \frac{e\mathbf{B}_z}{mc} p_2 = p_0^2.$$

The corresponding real solutions are of the type

(i) $x = a_1 \cos(p_1 t + \epsilon_1)$, $y = + a_1 \sin(p_1 t + \epsilon_1)$,

(ii) $x = a_2 \cos(p_2 t + \epsilon_2)$, $y = - a_2 \sin(p_2 t + \epsilon_2)$,

respectively, the constants a_1, a_2, ϵ_1, ϵ_2 being arbitrary.

These last two solutions represent circular vibration in a plane perpendicular to the magnetic field and taking place in opposite directions. The frequency of one is higher and that of the other lower than the original frequency p_0. In all real cases the difference between the frequencies is found to be very small compared with the frequency itself so that we may put
$$p_1 = p_0 - \frac{e\mathbf{B}_z}{2mc}, \qquad p_2 = p_0 + \frac{e\mathbf{B}_z}{2mc},$$
and therefore
$$p_1 - p_2 = \frac{e\mathbf{B}_z}{mc}.$$

658. We have now to consider the nature of the light emitted by the vibrating electron. The total radiation is made up of several parts, corresponding to the particular solutions we have obtained.

Our previous discussion of the radiation from a moving electron shows that, if such a particle has a vibration about a point O the vibration curve of the electric force in the field due to it at a distant point P is similar to the projection of the hodograph of the orbit of the electron on the wave front surface through P. It follows that the radiation emitted by the molecule in a direction parallel to the lines of force in the magnetic field will be composed of two circularly polarised constituent rays, one vibrating in a right-handed direction and the other in a left-handed direction, our conventions being such that the former has a period p_1 and a period p_2.

In a direction perpendicular to the lines of the field the radiation will be composed of three linearly polarised constituents of frequencies p_1, p_0, p_2; the polarisation of the inner component (p_0) being perpendicular to that of the other two.

These theoretical predictions have been fully verified by Zeeman's observations, who separated the various constituents in the radiation emitted in any direction by passing it through a prism or grating, from which also he was able to draw two very remarkable conclusions.

In the first place, it was found that, for light emitted in a direction coinciding with that of the magnetic force the polarisation of the component

of the doublet for which the frequency is lowest is right-handed. This proves that the frequency p_1 is smaller than p_2 or that e is negative: this agrees with the general result of other lines of research that the vibrations are those of negative electrons, these having much greater mobility than the positive charges.

The other result relates to the ratio between the numerical values of the electric charge and mass of the electron. This ratio can be calculated as soon as the distance between the components, from which we can find $p_1 - p_2$, and the strength of the magnetic field have been measured. The number deduced by Zeeman from the distance between the components of of the D lines of sodium, or rather from the broadening of these lines, whose components partly overlap, was one of the first values of e/m that have been published and agrees remarkably well, considering the errors to which its determination is subject, with the numbers that have been found for the negative electrons of the cathode rays.

Unfortunately the satisfaction caused by this success of the theory of electrons in explaining new phenomena, could not last long. It was soon found that many spectral lines are split up into more than three components; up to the present no very satisfactory explanation of these complications have been offered*.

659. The magnetic rotation of the plane of polarisation of light transmitted through a dispersive medium parallel to the lines of force of an imposed field is closely connected with this phenomenon of the Zeeman effect and is in fact due to a similar action of the magnetic field on the resonance vibrations of the contained electrons which are effective in modifying the propagation of light. This will be obvious when we realise that if in any propagation the relation between the inducing field intensity and the dielectric displacement current in the medium is of the form

$$\dot{\mathbf{D}} = \frac{\epsilon}{4\pi}\dot{\mathbf{E}},$$

then the propagation is with the velocity $\frac{c}{\sqrt{\epsilon}}$. Now let us consider the circumstances in a simple case where the magnetic force is parallel to the z-axis of coordinates and the propagation is that of a homogeneous wave train also in this direction, so that the vectors of the theory depend only on the time by the factor e^{ipt}. If the material dielectric has molecules containing

* Cf. however W. Voigt, *Magneto and elektro-optik*, Ch. IV. In this work will also be found full references to all the theoretical and experimental work bearing on this subject up to 1908. Reference may also be made to the article 'Theorie der magneto-optischen phänomene,' by H. A. Lorentz, in *Encyklop. d. math. Wissensch.* Bd. V.

a system of vibrating electrons of the type we have already assumed the equations of motion of the typical electron will be of the form

$$m\ddot{x} + kx - \frac{e\mathbf{B}_z}{c}\dot{y} = e(\mathbf{E}_x + a\mathbf{P}_x),$$

$$m\ddot{y} + kx + \frac{e\mathbf{B}_z}{c}\dot{x} = e(\mathbf{E}_y + a\mathbf{P}_y),$$

$$m\ddot{z} + kz \qquad\qquad = e(\mathbf{E}_z + a\mathbf{P}_z),$$

where the notation is exactly as previously. In dealing with imposed vibrations of period 2π the first two of the above equations reduce to

$$(k - mp^2)\,x - \frac{ie\mathbf{B}_z p}{c}\,y = e(\mathbf{E}_x + a\mathbf{P}_x),$$

$$(k - mp^2)\,y + \frac{ipe\mathbf{B}_z}{c}\,x = e(\mathbf{E}_y + a\mathbf{P}_y),$$

whence we see that the two equations

$$\left(k - mp^2 \mp \frac{e\mathbf{B}_z p}{c}\right)(x \pm iy) = e(\mathbf{E}_x \pm i\mathbf{E}_y) + ae(\mathbf{P}_x \pm i\mathbf{P}_y)$$

must be satisfied.

660. Also since $\qquad \Sigma ex = \mathbf{P}_x, \qquad \Sigma ey = \mathbf{P}_y,$

we have

$$\mathbf{P}_x \pm i\mathbf{P}_y = \left(\Sigma \frac{e^2/m}{p_0{}^2 - p^2 \mp \dfrac{e\mathbf{B}_z}{mc}p}\right)(\mathbf{E}_x \pm i\mathbf{E}_y + a\mathbf{P}_x \pm i\mathbf{P}_y),$$

where the sums Σ are taken per unit volume over all the optically excitable electrons and $p_0{}^2 = \dfrac{k}{m}$.

Now if also $\qquad \mathbf{D} = \epsilon\mathbf{E},$

we have $\qquad \mathbf{P} = \dfrac{\mathbf{D}}{4\pi} - \dfrac{\mathbf{E}}{4\pi} = \dfrac{(\epsilon - 1)\,\mathbf{E}}{4\pi},$

so that the two equations are equivalent to

$$(\mathbf{E}_x \pm i\mathbf{E}_y)\left[\epsilon - 1 - \frac{\dfrac{\Sigma e^2/m}{p_0{}^2 - p^2 \mp \dfrac{e\mathbf{B}_z}{mc}}}{1 - \Sigma \dfrac{ae^2/m}{p_0{}^2 - p^2 \mp \dfrac{e\mathbf{B}_z}{mc}}}\right] = 0.$$

These can be satisfied in either of two ways.

Either by*
$$\mathbf{E}_x + i\mathbf{E}_y = 0,$$

and
$$\epsilon = 1 + \frac{\Sigma \dfrac{e^2/m}{p_0{}^2 - p^2 + \dfrac{e\mathbf{B}_z}{mc}}}{1 - \Sigma \dfrac{ae^2/m}{p_0{}^2 - p^2 + \dfrac{e\mathbf{B}_z}{mc}}},$$

or by
$$\mathbf{E}_x - i\mathbf{E}_y = 0,$$

and
$$\epsilon = 1 + \frac{\Sigma \dfrac{e^2/m}{p_0{}^2 - p^2 - \dfrac{e\mathbf{B}_z}{mc}}}{1 - \Sigma \dfrac{ae^2/m}{p_0{}^2 - p^2 - \dfrac{e\mathbf{B}_z}{mc}}}.$$

661. Now let us examine what this means: let ϵ_1 and ϵ_2 be the respective values of the constants ϵ determined by these equations and let us consider the propagation of a beam of light which starts with the electric force polarised in the plane of the axis of y and with an amplitude
$$\mathbf{E}_x = \mathbf{E}e^{ipt}, \qquad \mathbf{E}_y = 0.$$
The medium effectively splits this beam into two oppositely circularly polarised beams, in the one of which
$$\mathbf{E}_{x_1} = \tfrac{1}{2}\mathbf{E}e^{ipt}, \qquad \mathbf{E}_{y_1} = -\frac{i}{2}\mathbf{E}e^{ipt},$$
initially, and in the other
$$\mathbf{E}_{x_2} = \tfrac{1}{2}\mathbf{E}e^{ipt}, \qquad \mathbf{E}_{y_2} = +\frac{i}{2}\mathbf{E}e^{ipt}.$$

The first of these beams, in which $\mathbf{E}_{x_1} + i\mathbf{E}_{y_1} = 0$ is propagated through the medium with a velocity $\dfrac{c}{\sqrt{\epsilon_1}}$; while the second in which $\mathbf{E}_{x_2} - i\mathbf{E}_{y_2} = 0$ is propagated with a velocity equal to $\dfrac{c}{\sqrt{\epsilon_2}}$. The result is that at a depth z into the medium from the place of entry of this beam the amplitudes in the respective components are
$$\mathbf{E}_{x_1} = \tfrac{1}{2}\mathbf{E}e^{ip\left(t - \frac{z\sqrt{\epsilon_1}}{c}\right)},$$
$$\mathbf{E}_{y_1} = -\frac{i\mathbf{E}}{2}e^{ip\left(t - \frac{z\sqrt{\epsilon_1}}{c}\right)},$$
and
$$(\mathbf{E}_{x_2}, \mathbf{E}_{y_2}) = \left(\frac{1}{2}, \frac{i}{2}\right)\mathbf{E}e^{ip\left(t - \frac{z\sqrt{\epsilon_2}}{c}\right)},$$

* These forms were first given by the author [*Phil. Mag.* 26 (1913), p. 362]. Those usually given have $a=0$ but seem to be inconsistent with certain experimental results on the constitutive character of the phenomena concerned.

so that in all

$$\mathbf{E}_x = \frac{\mathbf{E}}{2}\left\{e^{ip\left(t-\frac{z\sqrt{\epsilon_1}}{c}\right)} + e^{ip\left(t-\frac{z\sqrt{\epsilon_2}}{c}\right)}\right\},$$

$$\mathbf{E}_y = -\frac{i\mathbf{E}}{2}\left\{e^{ip\left(t-\frac{z\sqrt{\epsilon_1}}{c}\right)} - e^{ip\left(t-\frac{z\sqrt{\epsilon_2}}{c}\right)}\right\},$$

and therefore
$$\frac{\mathbf{E}_y}{\mathbf{E}_x} = i\frac{e^{-ip\frac{\sqrt{\epsilon_2}z}{c}} + e^{-ip\frac{\sqrt{\epsilon_1}z}{c}}}{e^{-ip\frac{\sqrt{\epsilon_2}z}{c}} - e^{-ip\frac{\sqrt{\epsilon_1}z}{c}}}$$

$$= \tan\left(\frac{\sqrt{\epsilon_1}-\sqrt{\epsilon_2}}{2c}pz\right).$$

Thus the beam of light is still plane polarised even at the depth z, but the plane of polarisation has been turned through the angle

$$pz\frac{\sqrt{\epsilon_1}-\sqrt{\epsilon_2}}{2c},$$

and this is the result determined by Faraday.

This simple explanation, although probably fundamentally correct in its essence, has nevertheless proved to be inadequate to describe many of the observed complications of the phenomenon; this is of course hardly surprising when we consider the result of the explanation based on the same principles of the more obvious Zeeman phenomenon described above. The discussion of the necessary modifications would however take us beyond the scope of the present work: reference may be made to Voigt, *Magneto and Electro Optik,* for full details of the theoretical side of the question and to Zeeman's book already mentioned for a complete account of the experimental details and results.

662. The phenomena just examined, which depend on the modifications of existing electronic orbits by the magnetic field, are closely connected with the phenomena of magnetic induction in diamagnetic bodies, for it is precisely these same modifications which give rise to the observed diamagnetic polarity under similar circumstances.

Let us* examine the motions of the electrons in any small element of a continuous piece of matter which give rise to its magnetic polarity. If $\mathbf{r}$ denote the vectorial displacement of the typical electron, it is a condition involved in the general permanency of the motions, that

$$\Sigma q\mathbf{r}_x{}^2,\ \Sigma q\mathbf{r}_y{}^2,\ \Sigma q\mathbf{r}_z{}^2,\ \Sigma q\mathbf{r}_x\mathbf{r}_y, \ldots$$

are all independent of the time. If the distribution of electrons in the

* This analysis is due to Langevin, *Ann. de chim. et phys.* VIII. t. 5, p. 70 (1905); but the physical basis is due to Larmor and Lorentz.

element is symmetrical the first three of these sums will be equal and the last three zero. We assume that this is the case and write $\frac{1}{2}a$ for the general value of the first three.

The motion of the typical electron is determined by an equation of the type

$$m\ddot{\mathbf{r}} = \mathbf{F}_e + e\mathbf{E}_e + \frac{e}{c}[\dot{\mathbf{r}}\mathbf{B}_e]$$

where $\mathbf{F}_e$ represents generally the resultant internal force of reaction on the displaced electron and the suffix e is used to denote the values of functions at the position of this electron.

It follows then since

$$\mathbf{I}dv = \Sigma \frac{e}{2c}[\mathbf{r}\dot{\mathbf{r}}]$$

that

$$\frac{d\mathbf{I}}{dt} = \Sigma \frac{e}{2c}[\mathbf{r}\ddot{\mathbf{r}}]$$

$$= \Sigma \frac{e}{2mc}[\mathbf{r}\mathbf{F}_e] + \Sigma \frac{e^2}{2mc}[\mathbf{r}\mathbf{E}_e] + \Sigma \frac{e^2}{2mc^2}[\mathbf{r}[\mathbf{r}\mathbf{B}_e]]$$

where Σ is now used to denote a sum per unit volume. If the element possesses no resultant magnetisation before the application of the field, then

$$\Sigma \frac{e}{m}[\mathbf{r}\mathbf{F}_e] = 0$$

the position of the electron not being appreciably altered. Again we may write

$$\mathbf{E}_e = \mathbf{E} + (\mathbf{r}\nabla)\mathbf{E}$$

$$\mathbf{B}_e = \mathbf{B} + (\mathbf{r}\nabla)\mathbf{B}$$

where the values without suffices denote those appropriate to the mean position of the electrons.

Substituting these values in the expression for the rate of change of **I** and using the conditions involved in the magnetic distribution we find that

$$\frac{d\mathbf{I}}{dt} = \frac{e}{4mc}\left[a \operatorname{curl} \mathbf{E} - \frac{\mathbf{B}}{c}\frac{da}{dt}\right].$$

But since

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{d\mathbf{B}}{dt}$$

this is the same as

$$\frac{d\mathbf{I}}{dt} = -\frac{e^2}{4mc}\frac{d}{dt}(a\mathbf{B}).$$

The aggregate change of **I** on the establishment of a field of strength **B** is therefore

$$\mathbf{I} = -\frac{ae^2}{4mc}\mathbf{B}$$

so that the susceptibility of diamagnetic polarisation is practically

$$\mu' = -\frac{ae^2}{4mc}.$$

The fact that this constant is dependent entirely on the conditions in the atom is consistent with our experience of these things, as the diamagnetic property is unaffected by the physical circumstance under which the molecules exist*.

663. The transmission of force in the electromagnetic field. We now turn to the discussion of the mode of transmission of force in the electromagnetic field. The discussion here is far more complicated than that of the first paragraph; energy is a scalar thing whereas force is a vector quantity and requires much more precise specification. We know from experience that forces are exerted by one body on another as a result of the existence of an electromagnetic field in the space surrounding them, or even as the result of an interchange of radiation and we want to get a theory of the matter. According to the ideas which we have developed the mechanical actions on the parts of the material system resulting from the established electromagnetic field are to be regarded merely as the terminal aspects of a state of stress in the medium (the aether) between the bodies, the action of one body on another being transmitted through and by this medium. We should therefore be able to represent these forces as an imposed geometrical stress system applied in the medium between the bodies. The forces acting on the part of the system enclosed in any surface drawn in the field would then be expressed as statically equivalent to a system of tractions over the surface (statically equivalent meaning that the resultants are the same as if we imagined the forces to be applied to rigid systems).

The problem of determining this stress admits of an infinite number of solutions owing to our indefinite knowledge of the actual properties of the aether, which is the ultimate seat of the strain condition. A rough mechanical analogy would be obtained by the consideration of two oppositely electrified bodies set in an insulating jelly; they will tend to come together and will thus create a state of stress in the jelly around them, and this stress will balance their attractions. This balancing stress reversed would thus completely represent the actions between the bodies. But different kinds of jelly would give different stress-representations, and the problem of determining this representation is indefinite until the jelly is specified. This indefiniteness does not however trouble us much at the present stage. We only want a physical solution of the problem which can be expressed in general terms independent of the particular nature of the problem, and this can be

* Cf. however A. E. Oxley 'On the influence of molecular constitution and temperature on magnetic susceptibility,' *Phil. Trans.* A, 214, pp. 109–146.

obtained with certain limitations. The present discussion however leads to one of the points where electric theory has not yet been probed right to the bottom. The solution obtained is useful and suggestive but it cannot yet be linked with our general physical theories.

We must here emphasise that it is the mechanical forces on the material bodies that we are going to deal with. If there is no matter in the small volume any system of tractions over its surface must balance among themselves.

664. We examine quite generally the forcive on the portion of any electrical system enclosed by an arbitrarily chosen surface f, assuming that it is ultimately the same as the resultant of the forces on the separate electrons associated with the matter of the system and constituting in their average relations its free charge and dielectric and magnetic polarisation. In estimating these forces account must however be taken of *all* the electrons properly associated with the matter, even if they are displaced across to the outside of the surface f, but not of those electrons temporarily inside f but really belonging to the matter outside. The force exerted by the field on any 'bound' electron is in reality applied to the matter at the point of it to which the electron is bound and not at the point where the electron may be found. For the purposes of the calculation we may therefore conveniently regard the portion of the material system under consideration as abruptly terminated by the surface f and therefore isolated from any portion outside this surface. In all other respects it will be assumed to be perfectly continuous throughout.

665. If $\mathbf{v}$ denote the vector velocity of the typical electron with charge e moving in a field at a point where the electric force and magnetic induction vectors are $\mathbf{E}$, $\mathbf{B}$ respectively, the force on it is

$$e\left\{\mathbf{E} + \frac{1}{c}[\mathbf{vB}]\right\}.$$

The force on the element δv of the matter inside the surface f is therefore

$$\left\{\rho'\mathbf{E} + \frac{1}{c}[\mathbf{C}'\mathbf{B}]\right\}\delta v$$

where ρ' denotes the average charge density in the element and $\mathbf{C}'$ the average current density of the electric flux.

The average charge on a small element δv inside the surface is

$$(\rho - \operatorname{div}\mathbf{P})\,\delta v$$

if ρ is the density of the free charge and $\mathbf{P}$ the intensity of the polarisation at the point. In addition to this distribution there is however a surface charge of density $\mathbf{P}_n$ at the abrupt outer boundary of the portion of the system under consideration, that is the surface f itself.

Again the average current density in the interior of the medium is

$$\mathbf{C}_1 + \rho\mathbf{u} + \frac{d\mathbf{P}}{dt} + c\,\mathrm{curl}\,\mathbf{I}_1$$

where $\mathbf{C}_1$ is the true current of conduction; $\mathbf{u}$ the average velocity of the matter at the point, and

$$\mathbf{I}_1 = \mathbf{I} + \frac{1}{c}[\mathbf{Pu}]$$

is the magnetic polarisation intensity, including the quasi-magnetism arising from the convection of electrically polarised molecules.

This current distribution has also to be adjusted at the boundary of f by the inclusion of a current sheet on that surface of density

$$c\,[\mathbf{I}_1\mathbf{n}_1]$$

where $\mathbf{n}_1$ is the unit normal vector at the point.

666. The electric part of the total forcive on the enclosed system will therefore be given as regards its linear component by

$$\int(\rho - \mathrm{div}\,\mathbf{P})\,\mathbf{E}\,dv + \int\mathbf{P}_n\mathbf{E}\,df$$

the volume integral being taken throughout the space inside f and the surface integral over this surface itself. A reduction of the latter integral by Green's lemma shows that this forcive is the same as

$$\int\{\rho\mathbf{E} + (\mathbf{P}\nabla)\,\mathbf{E}\}\,dv.$$

Remembering now that the surface f was arbitrarily chosen we may interpret this as implying that there is a forcive per unit volume on the system of intensity

$$\mathbf{F}_e = \rho\mathbf{E} + (\mathbf{P}\nabla)\,\mathbf{E}.$$

This is the whole of the average linear electric force on the medium. In addition there is a torque per unit volume of intensity

$$\mathbf{C}_e = [\mathbf{PE}]$$

as is easily seen on analogy with the statical case.

667. The force on the portion of the medium under review due to the magnetic field is similarly

$$\frac{1}{c}\int\left[\mathbf{C}_1 + \rho\mathbf{u} + \frac{d\mathbf{P}}{dt} + c\,\mathrm{curl}\,\mathbf{I}_1,\ \mathbf{B}\right]dv + \int[[\mathbf{I}_1\mathbf{n}_1]\,\mathbf{B}]\,df.$$

The second integral transforms similarly by Green's lemma to the volume integral of

$$-[\mathrm{curl}\,\mathbf{I}_1,\ \mathbf{B}] + \mathrm{grad}\,(\mathbf{I}_1\mathbf{B}) - \mathbf{I}\,\mathrm{div}\,\mathbf{B}$$

where the differential operator in the second term affects only the $\mathbf{B}$ function.

Thus since

$$\operatorname{div}\mathbf{B} = 0$$

we may take this part of the forcive as distributed throughout the medium with intensity

$$\mathbf{F}_m = \frac{1}{c}\left[\mathbf{C}_1 + \rho\mathbf{u} + \frac{d\mathbf{P}}{dt}, \mathbf{B}\right] + \operatorname{grad}(\mathbf{I}_1\mathbf{B})$$

per unit volume at any place. We shall henceforth use $\mathbf{C}_1$ to include the convection and polarisation currents as well as the conduction currents, so that the expression for this part of the forcive reduces to

$$\mathbf{F}_m = \frac{1}{c}[\mathbf{C}_1\mathbf{B}] + \operatorname{grad}(\mathbf{I}_1\mathbf{B}),$$

where the restriction is still implied in the operator in the last term.

This is the complete expression for the magnetic part of the total forcive per unit volume on the medium.

668. But of the magnetic induction $\mathbf{B}$ at any place a part, viz. $\mathbf{H}$ the magnetic force arises from the system in general, and the remainder $4\pi\mathbf{I}$ is the expression of the local influence arising in the element of volume itself. The question then arises whether the latter part is to be rejected in effecting the summation over the element of volume, as being compensated by reaction exerted by the elements of charge under consideration on the magnetism existing in the same element. If, as at present, it is a question of finding the mechanical force acting on the complete element of volume this compensation will certainly subsist; the action of the magnetism in the element on the charges will just cancel the action of the charges on the magnetism. In other conceivable cases this compensation may not occur. Thus in a mechanical theory which considers only elements of volume the magnetic part of the total force must be replaced by*

$$\frac{1}{c}[\mathbf{C}_1\mathbf{B}] + \operatorname{grad}(\mathbf{I}_1\mathbf{H}).$$

669. Following our previous theory we now try and express these forces on the electrical system or any part of it by surface integrals over the boundary of the volume containing it. To do this we must first express them in such a form that

$$\mathbf{F}_x = \frac{\partial \mathbf{T}_{xx}}{\partial x} + \frac{\partial \mathbf{T}_{xy}}{\partial y} + \frac{\partial \mathbf{T}_{xz}}{\partial z},$$

* We have retained the induction vector in the first part of this expression becau complete form is required in the subsequent analysis: the difference is a purely lo representing forces between the magnetism in an element and the free electrons th presence is a matter of indifference in a mechanical theory. It is however only f if the induction vector is retained throughout a reduction of the forces to a r a stress system is still possible but it is slightly different to that given in th pressure constituent contains the additional term $2\pi I_1^2$.

for then $\int_v \mathbf{F}_x dv$ can be transformed to

$$\int_f \mathbf{T}_{xn} df,$$

taken over the surface of the volume. This determines the **T**'s as the components of the stress system in the usual way.

670. We consider the electric and magnetic forces separately. As regards the electric part we follow Maxwell's hint developed in our previous discussions* and try as before

$$\mathbf{F}_{e_x} = \frac{\partial}{\partial x}\left(\mathbf{E}_x\mathbf{D}_x - \frac{1}{8\pi}\mathbf{E}^2\right) + \frac{\partial}{\partial y}(\mathbf{E}_x\mathbf{D}_y) + \frac{\partial}{\partial z}(\mathbf{E}_x\mathbf{D}_z).$$

In this case however the relationship does not hold: there is an outstanding term which cannot be included in the differentials. To see this easily we notice on differentiating this expression out that it is

$$(\mathbf{D}_x\nabla)\,\mathbf{E}_x - \frac{1}{8\pi}\frac{\partial}{\partial x}(\mathbf{E}^2) + \mathbf{E}_x\rho,$$

which differs from the above value by

$$\mathbf{D}_y\frac{\partial \mathbf{E}_x}{\partial y} + \mathbf{D}_z\frac{\partial \mathbf{E}_x}{\partial z} - \frac{1}{4\pi}\left(\mathbf{E}_y\frac{\partial \mathbf{E}_y}{\partial x} + \mathbf{E}_z\frac{\partial \mathbf{E}_z}{\partial z}\right) - \mathbf{P}_y\frac{\partial \mathbf{E}_x}{\partial y} - \mathbf{P}_z\frac{\partial \mathbf{E}_x}{\partial z},$$

which is $$\frac{1}{4\pi}\left(\mathbf{E}_y\frac{\partial \mathbf{E}_x}{\partial y} - \frac{\partial \mathbf{E}_y}{\partial x}\right) - \frac{1}{4\pi}\mathbf{E}_z\left(\frac{\partial \mathbf{E}_z}{\partial x} - \frac{\partial \mathbf{E}_x}{\partial z}\right),$$

or in vector notation $$\frac{1}{4\pi}(\mathbf{E}\,\mathrm{curl}\,\mathbf{E})_x,$$

but $$\mathrm{curl}\,\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{B}}{dt},$$

so that this is $$-\frac{1}{4\pi c}\left[\mathbf{E}\frac{d\mathbf{B}}{dt}\right]_x.$$

The difference between this case and that discussed in chapter IV is obvious. In the statical case we were able to put

$$\frac{\partial \mathbf{E}_x}{\partial y} = \frac{\partial \mathbf{E}_y}{\partial x}, \qquad \frac{\partial \mathbf{E}_z}{\partial x} = \frac{\partial \mathbf{E}_x}{\partial z},$$

so that there was no discrepancy. These equalities are however no longer ·atisfied because they imply the existence of an electric potential.

·71. Similar results apply of course to the other components of the stress Thus if we leave out for the present this outstanding part of the ·e see that the main part of the electric force acting on the matter · as a stress system which can be specified by the matrix

* See page 181.

$$\left|\begin{array}{ccc} \mathbf{E}_x\mathbf{D}_x - \frac{1}{8\pi}\mathbf{E}^2, & \mathbf{E}_x\mathbf{D}_y, & \mathbf{E}_x\mathbf{D}_z \\ \mathbf{E}_y\mathbf{D}_x, & \mathbf{E}_y\mathbf{D}_y - \frac{1}{8\pi}\mathbf{E}^2, & \mathbf{E}_y\mathbf{D}_z \\ \mathbf{E}_z\mathbf{D}_x, & \mathbf{E}_z\mathbf{D}_y, & \mathbf{E}_z\mathbf{D}_z - \frac{1}{8\pi}\mathbf{E}^2 \end{array}\right|.$$

This is identical with that obtained in the statical theory and is therefore amenable to the simpler specification there developed. It can in fact be dissected into parts.

(i) A simple hydrostatic pressure $\frac{1}{8\pi}\mathbf{E}^2$ throughout the medium. The negative sign shows that it is a pressure.

(ii) A tension along the internal bisector of the angle between **D** and **E** equal to

$$\mathbf{E}\,.\,\mathbf{D}\cos^2\widehat{\mathbf{ED}}.$$

(iii) A pressure along the external bisect… …l to

$$\mathbf{E}\,.\,\mathbf{D}\sin^2\widehat{\mathbf{E}\Gamma}$$

(iv) And the torque per unit volum…

$$[\mathbf{E}\,.\,\mathbf{D}$$

This couple or torque is represente… , part of the stress system and so the specification is complete, there is … discrepancy.

This is the general result; if the … lium is isotropic it reduces to a pull along the lines of force equal to (…) with a hydrostatic pressure $\frac{1}{8\pi}\mathbf{E}^2$ and this is Maxwell's system.

672. A similar argument ap… to the magnetic part of the forcive on the element. The x-componen… this force is given by

$$\mathbf{F}_{m_x} = \quad \frac{\partial\mathbf{H}}{\partial x}\Big) + \frac{1}{c}\,[\mathbf{C}_1\,.\,\mathbf{B}]_x.$$

Now the usual analysis show… at

$$4\pi\left(\mathbf{I}\frac{\partial\mathbf{H}}{\partial x}\right) = \frac{\partial}{\partial x}\left[\mathbf{B}_x\mathbf{H}_x - \frac{\quad}{2}\ \cdot\right] + \frac{\partial}{\partial y}(\mathbf{B}_y\mathbf{H}_x) + \frac{\partial}{\partial z}(\mathbf{B}_z\mathbf{H}_z) + [\mathbf{B},\,\text{curl}\,\mathbf{H}]_x,$$

and since $$\text{curl}\,\mathbf{H} = \frac{4\pi\mathbf{C}}{c} = \frac{1}{c}\left(4\pi\mathbf{C}_1 + \frac{d\mathbf{E}}{dt}\right)$$

the last term in this expression is equal to

$$[\mathbf{B}\,.\,\text{curl}\,\mathbf{H}] = \frac{4\pi}{c}[\mathbf{BC}] = -\frac{4\pi}{c}[\mathbf{CB}] = -\frac{4\pi}{c}[\mathbf{C}_1\mathbf{B}] - \frac{1}{c}\left[\frac{d\mathbf{E}}{dt},\mathbf{B}\right],$$

whence

$$4\pi\mathbf{F}_{m_x} = \frac{\partial}{\partial x}\left[\mathbf{B}_x\mathbf{H}_x - \frac{1}{2}\mathbf{H}^2\right] + \frac{\partial}{\partial y}(\mathbf{B}_y\mathbf{H}_x) + \frac{\partial}{\partial z}(\mathbf{B}_z\mathbf{H}_x) - \frac{1}{c}\left[\frac{d\mathbf{E}}{dt}\,\mathbf{B}\right]_x,$$

and similar results apply to the other components. The main part of the forcive can thus be specified as a stress system whose components are given in the matrix

$$\frac{1}{4\pi}\left|\begin{array}{ccc} \mathbf{H}_x\mathbf{B}_x - \frac{1}{2}\mathbf{H}^2, & \mathbf{B}_y\mathbf{H}_x, & \mathbf{B}_z\mathbf{H}_x \\ \mathbf{B}_x\mathbf{H}_y, & \mathbf{H}_y\mathbf{B}_y - \frac{1}{2}\mathbf{H}^2, & \mathbf{H}_y\mathbf{B}_z \\ \mathbf{B}_x\mathbf{H}_z, & \mathbf{H}_z\mathbf{B}_y, & \mathbf{H}_z\mathbf{B}_z - \frac{1}{2}\mathbf{H}^2 \end{array}\right|,$$

but this leaves out a part

$$-\frac{1}{4\pi c}\left[\frac{d\mathbf{E}}{dt}\,.\,\mathbf{B}\right],$$

which cannot be expressed by a surface integral.

Thus of the whole electrodynamic forcive per unit volume on the medium there is a total outstanding part

$$-\frac{1}{[illegible]}\left[[illegible]\,\frac{d\mathbf{B}}{dt}\right]-\frac{1}{4\pi c}\left[\frac{d\mathbf{E}}{dt}\,.\,\mathbf{B}\right]$$

which cannot be inclu[illegible] the stress specification.

673. The result is that i[illegible] have a lot of bodies in an electromagnetic field attracting one another, th[illegible]ultant of the electrodynamic forces acting on the matter inside any given c[illegible]d surface drawn in the field may in the main be expressed as the resulta[illegible]f the attractions across the surface as specified above and which are to [illegible]treated according to the ordinary procedure in the theory of elasticity. [illegible]ddition to this part there is however a forcive per unit volume of amou[illegible]

$$-\frac{1}{4\pi c}\,[illegible]\mathbf{B}],$$

which cannot be so reduced.

What is the meaning of this outstandi[illegible]rm? It is a complete differential with respect to the time and thus follow[illegible] an ordinary dynamical analogy there is a very strong temptation to say [illegible]t it represents a rate of change of some kind of momentum. This implies [illegible]istribution of 'electromagnetic momentum' throughout the field with a [illegible]ity at each point equal to

$$\frac{1}{4\pi c}\,[\mathbf{E}\,.\,\mathbf{B}].$$

Such a tentative hypothesis would provide a convenient representation for many purposes, but there are difficulties involved in it. In any case it is a pure assumption, the only justification for it lying in the fact that it is a complete differential with respect to the time.

From this, the modern, point of view the actual force distribution on the matter enclosed in any surface would be expressible partly as a static stress distribution over its surface and partly as a kinetic distribution throughout

the interior. Or to reverse the argument the actual forces between the bodies partly bear a statical stress and are partly used up in a change of momentum of some kind; the part which does not appear as a stress being capable of consideration as the kinetic reaction to a rate of change of momentum. The forces of this latter type are of the nature of 'motional forces,' to use Kelvin's phrase.

Part at least of this distribution of momentum would have to be ascribed to the aether, as it would exist if there were no matter present. Thus even if there is no matter inside the closed surface above considered there would still be no balance between the stresses over the surface, which would act to change the 'aethereal' electromagnetic momentum inside.

Even if we presume the existence of this momentum, it appears that it is certainly not the main part of the momentum in the field: it is for instance not that part of the momentum which is involved primarily in the propagation of electromagnetic or light waves. The actual working forces which effect the propagation of radiation depend on the first power of the field vectors concerned whereas the momentum here discussed depends on the second power of these vectors. It is therefore a second order effect and does not change sign with the reversal of the field. This means that it would be too difficult to detect even if it existed.

There is however after all no substantial reason for adopting this point of view of the matter as anything more than a convenient mode of expression. There is as yet no physical explanation to show why the forces act in this way.

674. If we consider an electromagnetic field like that in the propagation of electromagnetic radiation of any sort the field is oscillating and so the time-average value of

$$\frac{1}{c}\frac{d}{dt}[\mathbf{B}\,.\,\mathbf{E}]$$

is zero. Thus in all cases with alternating fields of this kind we can neglect this part when determining average forces, because it occurs as much positively as negatively in any time average.

675. The mechanical pressure of radiation. An important application of the foregoing principles is to the explanation of the pressure of radiation on absorbing and reflecting bodies. In this case the average of the momentum te m in the complete expression of the forcive is zero and the simpler Maxwellian specification is applicable.

Consider the case of a train of plane polarised waves advancing through an isotropic medium in the direction of the axis of x, so that all the quantities are functions of x only, the electric force being $(0, \mathbf{E}_y, 0)$ and the magnetic

force $(0, 0, \mathbf{H}_z)$*. We assume for simplicity that the permeability is unity. The equations of propagation

$$\frac{4\pi}{c}\mathbf{C}_y = -\frac{d\mathbf{H}_z}{dx}, \qquad \frac{d\mathbf{E}_y}{dx} = -\frac{1}{c}\frac{d\mathbf{H}_z}{dt},$$

wherein

$$\mathbf{C}_y = \sigma\mathbf{E}_y + \frac{\epsilon}{4\pi}\frac{d\mathbf{E}_y}{dt}$$

are satisfied by

$$\mathbf{E}_y = A \, . \, e^{-ax}\cos(nt - bx),$$

$$\mathbf{H}_z = \frac{cA}{n}\sqrt{a^2 + b^2}\, e^{-ax}\cos(nt - bx + \theta),$$

wherein

$$-\epsilon n^2 + i4\pi n\sigma = c^2(a + ib)^2,$$

or

$$c^2(a^2 - b^2) = -\epsilon n^2 \quad \text{and} \quad 4\pi n\sigma = 2c^2ab.$$

The mechanical forcive per unit volume is as before given by its single component†

$$\mathbf{F}_x = \frac{1}{c}\left(\mathbf{C}_y - \frac{1}{4\pi}\frac{d\mathbf{E}_y}{dt}\right)\mathbf{H}_z.$$

Now

$$\int_{x_1}^{x_2}\mathbf{C}_y\mathbf{H}_z\,dx = \int_{x_1}^{x_2} -\frac{c\mathbf{H}_z}{4\pi}\frac{d\mathbf{H}_z}{dx}\,dx = -\frac{c}{8\pi}\Big|\,\mathbf{H}_z^2\,\Big|_{x_1}^{x_2},$$

so long as $\mathbf{H}_z$ is continuous between the limits of integration. Also

$$-\int_{x_1}^{x_2}\mathbf{H}_z\frac{d\mathbf{E}_y}{dt}\,dx = -\frac{c}{n^2}\int_{x_1}^{x_2}\frac{d\mathbf{E}_y}{dt}\cdot\frac{d^2\mathbf{E}_y}{dt\,dx}\,dx$$

since for the harmonic oscillatory motion assumed

$$-n^2\mathbf{H}_z = \frac{d^2\mathbf{H}_z}{dt^2} = -c\frac{d^2\mathbf{E}_y}{dt\,dx},$$

and thus this integral is

$$-\frac{c}{2n^2}\Big|\left(\frac{d\mathbf{E}_y}{dt}\right)^2\Big|_{x_1}^{x_2},$$

provided $\dfrac{d\mathbf{E}_y}{dt}$ is continuous throughout the range of integration as is always the case, though ϵ may change gradually or abruptly. We have thus

$$\int_{x_1}^{x_2}\mathbf{F}_x\,dx = -\frac{1}{8\pi}\Big|\,\mathbf{H}_z^2 + \frac{1}{n^2}\left(\frac{d\mathbf{E}_y}{dt}\right)^2\Big|_{x_1}^{x_2},$$

which gives the aggregate mechanical forcive on the stretch of the medium

* Throughout this section we have used the magnetic force instead of the magnetic induction, in order to conform to the usual practice. The two are identical in the present case of non-magnetic media.

† We have followed Larmor in deducing the expression for the forcive directly from first principles. The same result can however be readily obtained by an application of Maxwell's stress formulae, but the present procedure deduces it without resort to these formulae, the validity of which may be doubted.

between x_1 and x_2 in the form of pressures on its ends. Thus for the simple forms of $\mathbf{E}_y$ and $\mathbf{H}_z$ assumed the time average of the pressure on either end is

$$\frac{1}{6\pi}(A_e^2 + A_m^2),$$

A_e and A_m representing the amplitudes of the magnetic and electric vibrations. This is however the sum of the mean kinetic and potential energies per unit volume of the radiation, less that involved in the electric polarisation in the molecules; hence on any portion of the medium there is a mechanical force, directed along the waves equal per unit cross section to the difference of these densities of energy at its ends.

676. In a transparent medium

$$\left(\frac{d\mathbf{E}_y}{dt}\right)^2 = \frac{c^2}{\epsilon}\left(\frac{d\mathbf{E}_y}{dx}\right)^2 = \frac{c^2}{\epsilon}\left(\frac{d\mathbf{H}_z}{dt}\right)^2,$$

so that the above internal pressure may be expressed in the form

$$\frac{1}{8\pi}\left(\mathbf{H}_z^2 + \frac{1}{n^2\epsilon}\left(\frac{d\mathbf{H}_z}{dt}\right)^2\right).$$

If there is in the medium a directly incident wave whose vibration at the interface is $A_{m_i}\cos nt$ and also a reflected wave $A_{m_r}\cos(nt-\epsilon)$ and also a refracted wave, this result may be applied to a layer of the medium containing the interface; thus there will be a mechanical traction on the interface represented by a difference of pressure on its two sides, that on the incident side being

$$\frac{1}{8\pi}\left[\{A_{m_i}\cos nt + A_{m_r}\cos\overline{nt-\epsilon}\}^2 + \frac{1}{\epsilon}\{A_{m_i}\sin nt + A_{m_r}\sin\overline{nt-\epsilon}\}^2\right].$$

In air or vacuum this is

$$\frac{1}{8\pi}(A_{m_i}^2 + A_{m_r}^2 + 2A_{m_i}A_{m_r}\cos\epsilon),$$

or

$$\frac{1}{8\pi}A_m^2,$$

where A_m is the amplitude of the resultant magnetic vibration on that side.

When the radiation is directly incident on an opaque medium $\mathbf{H}_z$ and $\frac{d\mathbf{E}_y}{dt}$ are null in its interior; so that, when the surrounding medium is air or vacuum, its surface sustains in all a mechanical inward normal traction of intensity $\frac{1}{2}A_m^2$, that is, equal to the mean energy per unit volume of the radiation just outside it, in agreement with Maxwell's original statement*.

This is the famous pressure of radiation which has now been experimentally examined and the theoretical conclusions as to its intensity verified to within

* Treatise II. § 792.

one per cent. by the measurements of Nicholls and Hull*. We can in a case of this kind thus say that radiation exerts a force just as if it carried momentum. This leaves open the question of the actual existence of the momentum spoken of.

677. To illustrate these matters† further and to bring out another aspect of the subject let us consider the opposite action, viz. the reaction or back pressure exerted by the radiation emitted by a perfectly black body into free space. Similar reasoning to the above will show that the back pressure is of a similar amount to that there calculated. To exhibit the argument in a simpler manner let us examine the component of the radiation emitted from a small patch of the surface as a plane beam travelling out into space.

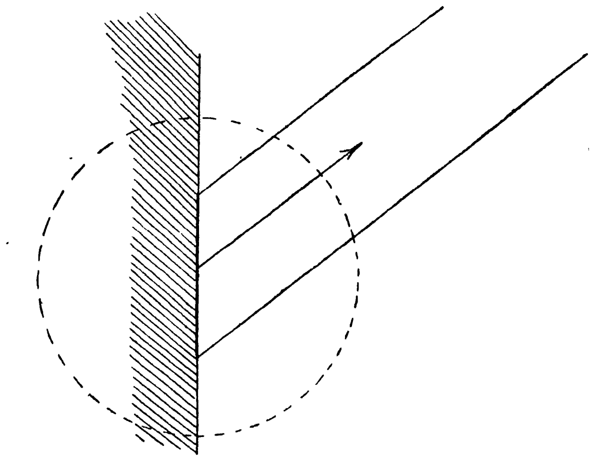


Fig. 97

Surround the patch by a small closed surface cutting across the ray on the one side. Our previous general theory then shows that the static resultant of the forces on everything inside this surface is represented by a stress system over the single patch of the surface where the ray cuts through it, this being the only part on the surface where the field in this ray is different from zero. If the ray cuts through this part of the surface normally it appears that the normal stress at that point is a normal pull of amount

$$\tfrac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2)\, df_1,$$

* *Phys. Rev.* XIII (1901), p. 293. Cf. also Lebedew, *Arch. des Sciences Phys. et Nat.* (4), VIII. (1899), p. 184; Poynting, *Phil. Mag.* IX. (1905), pp. 169, 475.

† The treatment here followed is given by Larmor in his lectures: certain aspects of it are dealt with in his address on 'The dynamics of radiation' to the Mathematical Congress in 1912. (Cf. *Proceedings*, vol. I. p. 206.)

and thus there is a total pull normally on the element of the emitting surface equal to

$$\frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{H}^2)\,\mathbf{n}_1 df_1,$$

$\mathbf{n}_1$ being the normal vector direction to the elementary patch of area df_1.

As it does not matter how big we draw this bounding sphere this stress is the representative of a real force on the patch of the surface emitting the beam of radiation; and thus this beam exerts a back pressure on the body emitting it which is equal per unit area to the density of the energy in the field just outside it.

This back pressure is equal to the forward pressure on the body at the other end receiving and absorbing the radiation. The action and reaction are equal and opposite at the two ends of the beam, so that on this view of things the ray behaves as if it were a carrier of momentum.

678. The question naturally arises as to what happens before the ray reaches the second body. Where is then the corresponding reaction to the

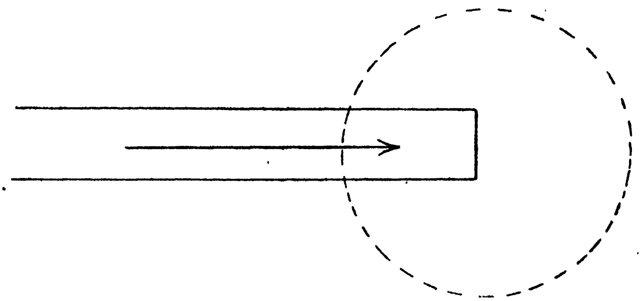


Fig. 98

back pressure on the body giving out the radiation? To answer this question completely we must return to our original scheme. According to the general theory the stresses acting over any geometrical surface drawn in the field which does not contain any matter are balanced by the kinetic reaction to the rate of change of the quasi-momentum in the aether inside the surface. Now consider a small parallel beam of light advancing into free space. The plane perpendicular front of the beam is advancing with the velocity c of radiation. Draw a surface in the field much as that shown in the figure. The stresses over the boundary of this surface are represented by the pressure of radiation over the small patch of the surface where the beam cuts through it: and this pressure must account for and just balance the rate of change of the quasi-momentum of the aether included inside the surface. Now where does this change of momentum come in? The propagation by waves

is an alternating affair and so on the average the momentum in any part of the beam remains constant; the beam is however getting longer; a new region is being added in which there is momentum and so the change due to this added momentum per unit time must be equal to the pressure. The quasi-momentum per unit volume in the electromagnetic field in the general case is of amount

$$\frac{1}{4\pi c} \cdot [\mathbf{H} \cdot \mathbf{E}],$$

and is directed perpendicular to both vectors. In our case of plane propagation of wave motion in the aether this is

$$\frac{1}{4\pi c} \cdot \mathbf{H}_z \mathbf{E}_y,$$

and the general equations give

$$\mathbf{H}_z = \mathbf{E}_y,$$

so that the quasi-momentum per unit volume is

$$\frac{1}{4\pi c} \cdot \mathbf{H}_z^2,$$

or in the mean it is

$$\frac{1}{8\pi c} \mathbf{H}^2.$$

Thus on the average the momentum added in a time δt is

$$\frac{1}{8\pi c} \mathbf{H}^2 c\delta t,$$

reckoned per unit area of cross section in the beam. The rate of change of this is equal to

$$\frac{1}{8\pi} \mathbf{H}^2.$$

But the energy per unit volume in our wave is on the average equal to

$$\frac{1}{16\pi} \mathbf{E}^2 + \frac{1}{16\pi} \mathbf{H}^2 = \frac{1}{8\pi} \mathbf{H}^2.$$

That is the average momentum per unit length in the beam is equal to the energy transmitted per unit time across any cross section and this shows that it is balanced exactly by the pressure of radiation on the patch of the surface aforesaid.

679. The whole theory is thus consistent. It must however be noticed that the stresses and momentum involved in this discussion, about whose actual existence there may still be some doubts, are excessively small. The stress for example which gives rise to the phenomena of radiation pressure depends on the square of the vectors defining the field and is therefore nearly always smothered by the stress which propagates the wave and depends only on

the first power of the vectors. A rough analogy is provided by the attraction of small objects by a vibrator like a tuning fork in air. Very near such a vibrator in air the atmospheric pressure is less than that at a distance and so any object placed near the vibrator would have a greater pressure on its surface farthest from the vibrator and would therefore be impelled towards that body. But this resultant pressure depends on the square of the average pressure of the air whereas the sound propagation depends on the first order things. Thus in a body emitting light the reactions of the pressure of radiation would hardly ever be appreciable, being almost entirely swamped by the reaction to the setting up of the vibrations. This fact renders it almost impossible experimentally to test the existence of the 'momentum' force in free aether by testing for the reaction on a radiator, before the radiation from it has reached an absorber.

In these considerations the radiator has been considered as at rest, we must now calculate the effects due to motion.

680. In order to investigate whether the back pressure depends on the motion we proceed as before; and examine the reaction of an oblique beam emitted by a plane radiator travelling normally to itself with a velocity v. We

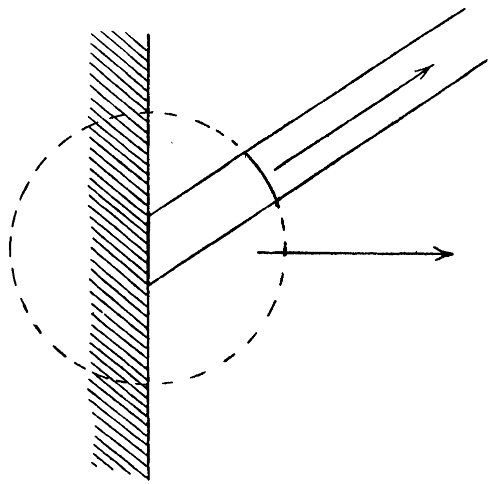


Fig. 99

consider any boundary drawn in the field as shown. The force acting on the path of the radiating surface which is inside this volume would then be balanced by the radiation pressure on the single patch of the geometrical surface were it not for the fact that the average 'momentum' of the field inside the surface is changing, owing to the fact that the beam is becoming

shorter at a rate $v\cos\theta$, θ being the angle between the beam and normal to the surface. This rate of change of momentum together with the pressure along the ray from the radiator are balanced by the radiation pressure on the patch of the geometrical surface. Thus if E' is the energy density in the beam the back push or radiation pressure is easily seen to be

$$p = \left(1 - \frac{v\cos\theta}{c}\right) E'.$$

681. But now we want E'. The question is whether the energy per unit volume in the radiation from a moving body is different from that of the same body at rest or does the nature of the radiation from a perfect radiator depend on its velocity?

A simple thermodynamic argument can be adopted* to prove that the periods and amplitudes of the motions of the molecules or electrons in a body moving with uniform velocity do not depend on the velocity, so that the amplitude of the oscillation in the emitted wave train is the same but the wave length is necessarily altered by the motion according to the Doppler principle being shortened by the factor

$$\left(1 - \frac{v\cos\theta}{c}\right),$$

because if the radiator moves forward the waves are crowded up or shortened. The period of the wave is therefore shortened by a similar factor and thus the average energy per unit length is increased by the factor

$$\left(1 - \frac{v\cos\theta}{c}\right)^{-2}.$$

Thus if we use E for the energy of the statical radiation we have

$$E' = \left(1 - \frac{v\cos\theta}{c}\right)^{-2},$$

and thus

$$p = \left(1 - \frac{v\cos\theta}{c}\right)^{-1} E.$$

Similarly for a perfect black body absorbing radiation and moving with a velocity $v'\cos\theta'$ in the same direction we should have the pressure of radiation on its surface equal per unit area to

$$p' = \left(1 + \frac{v'\cos\theta}{c}\right)^{-1} E.$$

682. The thermodynamics of radiation†. According to the views developed in detail in the previous chapter all radiation whether of thermal,

* Cf. Larmor, *Phil. Trans.* A, 185 (1894), p. 781.

† The discussion of the following paragraph follows closely that given by Larmor in the article 'Radiation,' in the *Encyclopedia Britannica.* Reference may also be made to Planck, 'Die Theorie der Wärmestrahlung.'

optical or electrical type consists essentially in vibrational waves of fundamentally identical types in the aether of space. A molecule or in fact any piece of matter is to be regarded as a kinetic system compounded of simpler systems so that its energy may be classified into constitutive energy essential to its constitution and vibratory energy which it can receive from or radiate away into the aether. A piece of matter isolated in free space would in time lose all energy of the latter type by radiation; but the former will remain so long as the matter persists, along with the energy of uniform translatory motion to which it is ultimately reduced. Thus all matter is in continual exchange of vibratory energy with the aether and it is with the laws of this exchange of energy that the general theory of the present title deals.

683. The foundation of this subject is the principle arrived at independently by Balfour Stewart and Kirchhoff* about the year 1858 that the constitution of the radiation which pervades an enclosure surrounded by bodies in a steady thermal state must be a function of the temperature of these bodies, and of nothing else. Their reasoning rests on the dynamical principle that by no process of ordinary reflexion or transmission can the period and therefore the wave-length of any harmonic constituent of the radiation be changed; each constituent remains of the same wave-length from the time it is emitted until the time it is absorbed again. If we imagine a field of radiation to be enclosed within perfectly reflecting walls, then, provided there is no material substance in the field which can radiate and absorb, the constitution of the radiation in it may be any whatever and it will remain permanent. It is only the presence of material bodies that can transform the surrounding radiation towards the unique constitution which corresponds to their temperature. We can define the temperature of an isolated field of radiation, of this definite ultimate constitution, to be the same as that of the material bodies with which it would thus be in equilibrium. Further the mutual independence of the various constituents of any field of radiation enclosed by perfect reflectors allows us to assign a temperature to each constituent, such as the part involving wave lengths lying between λ and $\lambda + \delta\lambda$, that will be the temperature of a material system with which this constituent by itself is in equilibrium of emission and absorption. But to reason about the temperature in this way we must be sure that it completely pervades the space and has no special direction; this is ensured by the continual reflections from the walls of the enclosure. The temperature of each constituent in a region of undirected radiation is thus a function of its wave length and its intensity alone.

It is the fundamental principle of thermodynamics that temperatures tend to become uniform. In the present case of a field of radiation, this equalisation cannot take place directly between the various constituents of

* *Ann. Phys. Chem.* 109 (1860), p. 275; *Ges. Abhdl.* (1882), p. 578.

the radiation that occupy the same space, but only through the intervention of the emission and absorption of material bodies; the constituent radiations are virtually partitioned off adiabatically from direct interchange. Thus in discussing the transformations of temperatures of the constituent elements of radiation, we are really reasoning about the activity of material bodies that are in thermal equilibrium with those constituents; and the theoretical basis of the idea of temperature as depending on the fortuitous residue of the energy of molecular motions is preserved.

684. Let us consider a spherical enclosure filled with radiation and having walls of ideal perfectly reflecting quality, so that none of the radiation can escape. If there is no material body inside it, any arbitrarily assigned constitution of this radiation will be permanent. Let us suppose the radius a of the enclosure is shrinking with extremely small velocity v. A ray inside it incident at an angle i, will always be incident on the walls in its successive reflexions at the same angle, except as regards a negligible change due to the motion of the reflector; and the length of its path between successive reflexions is $2a \cos i$. Each undulation in this ray will thus undergo reflexion at intervals of time equal to $\frac{2a \cos i}{c}$, where c is the velocity of light and it is easily verified that on each reflexion it is shortened by the fraction $\frac{2v \cos i}{c}$ of itself: thus in the very long time T required to complete the shrinkage it is shortened by the fraction vT/a, which is $\frac{\delta a}{a}$, where δa is the total shrinkage in the radius, and is independent of i. The wave length of each undulation in the radiation inside the enclosure is therefore reduced in the same ratio as the radius. Now suppose the constitution of the enclosed radiation correspond initially to a definite temperature. During the shrinkage thermal equilibrium must be maintained among its constituents; otherwise there would be a running down of their energies towards uniformity of temperature, if material radiating bodies were present, which would be superposed on the mechanical operations belonging to the shrinkage and the process could not be reversible. Such a state of affairs is not possible for it would land us in processes of the following type. Expand the enclosure gaining the mechanical work of the radiant pressure against its walls, whatever that may be. Then equalise the intensities of the constituent radiations to those corresponding to a common temperature, by taking advantage of the absorptions of material bodies at the actual temperatures of these radiations: when this is done, as it may actually be to some extent by aid of sifting produced by partitions which transmit some kinds of radiation more rapidly than others, a further gain of work can be obtained at the expense of the radiant energy. Now contract the remaining radiant energy to its previous volume, which requires

an expenditure of less work on the walls of the enclosure than the expansion of the greater amount of radiation originally afforded: and finally gain still more work by equalising the temperature of its constituents. The energy now remaining being of smaller amount and under similar conditions must have a temperature lower than the initial one. This process might be repeated indefinitely and would constitute an engine without an extraneous refrigerator, violating Carnot's principle by deriving an unlimited supply of mechanical work for thermal sources at a uniform temperature. Thus independently of any knowledge of the intensity of the mechanical pressure of radiation, or indeed of whether such a pressure exists at all, it is established that the shrinkage of the enclosure must directly transform the contained radiation to the constitution which corresponds to some definite new temperature.

685. Let us next consider the enclosure filled with radiation of energy density E at volume V, of any constitution but devoid of special direction and let it be shrunk to volume $V - \delta V$ against its own pressure; if the density thereby becomes $(E - \delta E)$ the conservation of energy requires that

$$EV - (E - \delta E)(V - \delta V),$$

is equal to the work done against the pressure, viz. $p\delta V$. To obtain the pressure p we have to average the direct pressures for the different values of the angle of incidence θ, there being no sideway pressure: the foreshortening of the area pressed gives a factor $\cos\theta$, and resolving of the pressure normally gives another $\cos\theta$, so that the resultant pressure on the interface is equal to one-third of the total density of radiant energy in the enclosure

$$p = \tfrac{1}{3}E,$$

and thus
$$EV + \tfrac{1}{3}E\delta V = (E - \delta E)(V - \delta V),$$

so that
$$\tfrac{4}{3}E\delta V + V\delta E = 0,$$

or
$$E \sim V^{-\frac{4}{3}}.$$

686. Again, but now with the restriction to radiation with its energy distributed as regards wave length so as to be of uniform temperature, the performance of the mechanical work $\frac{1}{3}E\delta V$ has changed the energy of radiation EV from the state that is in equilibrium of absorption and emission with a thermal source at temperature ϑ to the state in equilibrium with an absorber of some other temperature $\vartheta - \delta\vartheta$ and that in a reversible manner, thus by Carnot's principle

$$\frac{1}{3}\frac{E\delta V}{EV} = -\frac{\delta\vartheta}{\vartheta},$$

so that ϑ varies as $V^{-\frac{1}{3}}$, or inversely as the linear dimensions when the enclosure is shrunk.

Combining these results it appears that E varies as ϑ^4; this is Stefan's* empirical law for the complete radiation corresponding to the temperature, first established on these lines by Boltzmann†. Starting from the principle that this radiation must be a function of the temperature alone, this adiabatic process has in fact given us the form of the function. These results cannot however be extended without modification to each separate constituent of the complete radiation, because the shrinkage of the enclosure alters its wave length and so transforms it into a different constituent. The effect of compressing the complete radiation is thus to change it to the constitution belonging to a certain higher temperature, by shortening all its wave lengths by the proportion of one-third of the compression of volume, the temperature being in fact raised by the same proportion at the same time increasing in a uniform ratio the amounts corresponding to each interval $\delta\lambda$, so as to get the correct total amount of energy for the new temperature. In the compression each constituent alters so that $\vartheta\lambda$ remains constant and the energy $E_\lambda \delta\lambda$ in the range $\delta\lambda$ in other respects changes as a function of T alone. Hence generally $E_\lambda \delta\lambda$ must be of the form

$$F(\vartheta) f(\lambda\vartheta) \delta\lambda.$$

But for each temperature $\int_0^\infty E_\lambda d\lambda$ is equal to E and so varies as ϑ^4, by Stefan's law; that is

$$\vartheta^{-1} F(\vartheta) \int_0^\infty f(\vartheta\lambda)\, d(\vartheta\lambda) \sim \vartheta^4,$$

so that

$$\vartheta^{-1} F(\vartheta) \sim \vartheta^4.$$

Thus finally $E_\lambda \delta\lambda$ is of the form

$$A\vartheta^{-5} f(\vartheta\lambda)\, \delta\lambda,$$

or

$$A\lambda^{-5} \phi(\vartheta\lambda)\, \delta\lambda,$$

which is Wien's‡ general result known as the *displacement law*: it states that the distribution of the energy among the various wave lengths are, at any two temperatures homologous, in the sense that the intensity curves after the wave lengths in one of them have been reduced in a ratio depending definitely on the two temperatures differ only in the absolute scale of magnitude of the ordinates.

* *Wiener Bericht.* 79 (1879), p. 39. This proof of Stefan's law is apparently all right, but there are assumptions involved in it which ought not really to be allowed to stand without further examination. An amended form of the proof is given by Larmor (*Aether and Matter*, p. 137), but it can be shown to lead to the additional result that the energy density varies as the temperature. A critical examination of the subject would require too much space and cannot be attempted in the present work, but it may be stated that there is ample experimental evidence of the truth of the law as stated.

† *Ann. Phys. Chem.* 22 (1884), p. 29.

‡ *Berlin. Ber.* (1893), p. 55. *Ann. Phys. Chem.* 52 (1884), p. 132.

687. It is of interest to follow out this adiabatic process for each constituent of the radiation as a verification and also to ascertain whether anything new is thereby gained. To this end let $E(\lambda, \vartheta)\,\delta\lambda$ represent the intensity of the radiation between λ and $\lambda + \delta\lambda$ which corresponds to the temperature ϑ. The pressure of this radiation when it is without special direction, is in intensity one-third of this; thus the application of Carnot's principle shows, as before, that in adiabatic compression $\vartheta \sim V^{-\frac{1}{3}}$ so that a small linear shrinkage in the ratio $1 - x$ raises T in the ratio $1 + x$. We have still to express the equation of energy. The vibratory energy $E(\lambda, \vartheta)\,\delta\lambda V$ in volume V, together with the mechanical work $\frac{1}{3}E(\lambda, \vartheta)\,\delta\lambda \,.\, 3xV$ yields the vibratory energy

$$E(\lambda(1 - x), \vartheta(1 + x))\,\delta\lambda\,(1 - x)\,V(1 - 3x),$$

thus writing E for $E(\lambda, \vartheta)$ we have, neglecting x^2

$$E(1 + x) = E - x\lambda\frac{\partial E}{\partial\lambda} + x\vartheta\frac{\partial E}{\partial\vartheta}(1 - 4x),$$

so that

$$5E + \lambda\frac{\partial E}{\partial\lambda} - \vartheta\frac{\partial E}{\partial\vartheta} = 0,$$

a partial differential equation of which the integral is

$$E = A\lambda^{-5}\phi(\vartheta\lambda),$$

the same formula as was before obtained.

This method treating each constituent of the radiation separately, has in one respect some advantage, in that it is necessary only to postulate an enclosure which totally reflects that constituent, this being a more restricted hypothesis than an absolutely complete reflector.

688. To determine theoretically the form of the function ϕ we must have some means of transforming one type of radiation into another, different in essence from the adiabatic compression already utilised. The condition that the entropy of the independent radiations in an enclosure is a maximum when they are all transformed to the same temperature with total energy unaltered is already implicitly fulfilled; it would thus appear that any further advance must involve the dynamics of the radiation and absorption of material bodies.

We cannot here discuss all the attempts* that have been made to obtain an insight into the form of this fundamental function; but it is of interest to follow the reasoning employed by Lorentz† to deduce it in a particular

* Cf. Wien's article, 'Theorie der Strahlung' in *Encyklop. der math. Wiss.* Bd. v. 23, where a short account of the subject is given with full references to the more important original work bearing on the subject.

† *Amsterdam Proc.* (1903). Cf. also *Theory of Electrons*, ch. II. The theory has been subsequently generalised along the lines suggested for the theory of metallic conduction in ch. VII.

case, as it depends merely on an application of principles already expounded in previous chapters.

In the electron theory of metals as expounded in various places above we deal with substances in which the circumstances of the emission and absorption of radiation are completely known, and it would appear then that if a calculation of these quantities can be made in a particular case some knowledge of the density of the radiation which would be in equilibrium with the metal at a given temperature could be obtained.

689. We consider, with Lorentz, a thin metallic plate in which a large number of free electrons are moving about in a perfectly irregular manner, consistent with the general laws of the conservation of their total energy and momentum. We know that an electron can be the centre of an emission of energy when its velocity is changing, consequently, as a result principally of the numerous collisions of the electrons with the atoms, which produce alterations of the directions and magnitudes of the velocities of the electrons, a part of the heat energy of the irregular motion of the electrons will be radiated away from the metal. This radiant energy, which is subsequently to be the subject of a detailed examination is, however, presumed to be so small compared with the energy of motion of the electrons that it can be neglected in any dynamical considerations respecting those motions extended over a finite time. To this extent the analysis offered is only a first approximation to the actual state of affairs.

690. Now let f and f' be two infinitely small parallel surface elements, f being on the plate itself and f' at a distance r outside it on the normal to the plate through the centre of f. Then of the whole radiation emitted by the metal plate, a certain portion will travel outwards through f and f'. Suppose we decompose this radiation into rays of different wave lengths and each ray again into its plane polarised constituents in two planes at right angles through the chosen normal to the plate (these two planes and the plane of the plate being parallel to a system of properly chosen rectangular coordinate planes in which $z = 0$ is the plane of the plate). Now consider in particular those of the rays in this beam whose wave length lies between the two infinitely near limits λ and $\lambda + d\lambda$ and which are polarised in the plane $y = 0$; the amount of energy emitted by the plate per unit time through both elements f and f' so far as it belongs to these rays, must be directly proportional to f, f' and $d\lambda$ and inversely proportional to r^2, so that it can be represented by an expression of the form

$$E\frac{ff'd\lambda}{r^2}.$$

The coefficient E is called the *emissivity* of the plate and is a function not only of the positions of f, f' and λ but also of the conditions and type of the metal composing the plate.

We have already seen how, as a result of the collisions between the electrons and atoms, part at least of any regular or organised energy acquired by the electrons during their free motion between the atoms can be dissipated into heat energy of the irregular motion of the same electrons. In this way it is possible for a metal to absorb a portion of the energy from an incident beam of radiation, because the electric force in the electromagnetic field associated with the radiation will pull the electrons about during their otherwise free motion between collisions, imparting kinetic energy to them, which is then dissipated by collision at the end of each path into irregular heat motion. Suppose then that a plane polarised beam such as that specified above is incident through the small surface f', on the patch f of the plate: then we know that a certain portion of the energy of this beam will be absorbed in the metal and converted into heat energy, instead of being reemitted as a portion of the reflected or transmitted beams. The fraction expressing the proportion of the energy absorbed is called the *coefficient of absorption* of the plate under the conditions specified and may be denoted by A.

691. If however the metal plate is in exchange of steady thermal radiation either with itself or other bodies at the same uniform temperature in a perfectly reflecting enclosure the amount of radiant energy of a particular type absorbed by it must be equal to the amount of the same type which is emitted by it: this means that the ratio

$$\frac{E}{A}$$

determines the density of the energy in the particular constituent of the radiation under consideration and this from the principles discussed above must be dependent only on the temperature and wave-length of this constituent.

But in the present case both E and A are directly calculated by known principles. If we consider that the thickness Δ of the metallic plate is so small that the absorption may be considered as proportional to it, we shall find by an obvious calculation

$$A = \frac{\sigma}{c}\Delta^*,$$

c being the usual radiation constant and σ the conductivity of the metal. Now in all applications involving steady or only slowly varying currents the conductivity σ is given by

$$\sigma = \sqrt{\frac{8}{3\pi}}\frac{Ne^2 l_m}{mu_m},$$

where N denotes the number of free electrons per unit volume in the metal, each of mass m and charge e, moving with velocities the average square of

* See Ex. 334 in the Appendix.

which is u_m^2; l_m is the length of the mean free path; but in applications involving more rapid alternations in the current it is necessary to use the more complete formula obtained in the previous chapter, viz.

$$\sigma = \sqrt{\frac{8}{3\pi}} \frac{Ne^2 l_m}{m u_m} \int_0^\infty \frac{z e^{-z} dz}{1 + \frac{4\pi^2 c^2 l_m^2 q}{z\lambda^2}}.$$

The coefficient of absorption of the metal plate is thus completely determined.

692. Now let us consider the radiation from the plate. We need only consider the radiation normally from the small volume $f\Delta$ of the plate, as this is the only part of all the radiation through f from the whole plate that gets to f'. Now according to a formula established above, a single electron moving with a velocity $\mathbf{v}$ in the part of the plate under consideration will produce at the position of f' an electromagnetic field in which the x-component of the electric force is given by

$$-\frac{e}{c^2 r}\frac{d\mathbf{v}_x}{dt},$$

if we take the value of the differential quotient at the proper instant. But on account of the assumption as to the thickness of the plate, this instant may be represented for all the electrons in the portion $f\Delta$ by $\left(t - \frac{r}{c}\right)$, if t is the time for which we wish to determine the state of things at the distant surface f'. We may therefore with the same notation as previously employed write for the x-component of the electric force in the total field at f'

$$\mathbf{E}_x = -\frac{1}{rc^2}\left[\Sigma e \frac{d\mathbf{v}_x}{dt}\right],$$

and then the flow of energy through f' per unit of time will be

$$\frac{c\mathbf{E}_x^2 f'}{4\pi},$$

as far as this one component is concerned.

Since however the motion of the electrons between the metallic atoms is highly irregular and of such a nature that it is impossible to follow it in detail, we must rather content ourselves with mean values of the variable quantities calculated for a sufficiently long interval of time. We shall therefore always consider only the mean values of our quantities taken over the large time between the instants $t = 0$ and $t = \vartheta$. For example, the flow of energy through f' is, on the average, equal to

$$\frac{cf'}{4\pi}\frac{1}{\vartheta}\int_0^\vartheta \mathbf{E}_x^2 dt = \frac{cf'}{4\pi}\overline{\mathbf{E}_x^2},$$

say.

693. Now whatever be the way in which $\mathbf{E}_x$ changes from one instant to the next we can always expand it in a series by the formula

$$\mathbf{E}_x = \sum_{s=1}^{\infty} a_s \sin \frac{s\pi t}{\vartheta},$$

where s is a positive integer and

$$a_s = \frac{2}{\vartheta} \int_0^{\vartheta} \sin \frac{s\pi t}{\vartheta} \mathbf{E}_x \, dt.$$

The frequency of the sth term of this series is $\frac{s\pi}{\vartheta}$, so that the wave length of the vibration represented on it is

$$\lambda = \frac{2c\vartheta}{s}.$$

If ϑ is very large the small part of the spectrum corresponding to the small interval of length $d\lambda$ between the wave lengths λ and $\lambda + d\lambda$ will contain a large number $\frac{2c\vartheta}{\lambda^2} d\lambda$ of spectral lines represented by terms of this series.

If now we substitute the Fourier series for $\mathbf{E}_x$ into the expression for the mean energy of flux through f', we shall find in the usual manner that it is equal to

$$\frac{c\overline{\mathbf{E}_x{}^2}}{4\pi} f' = \frac{1}{8\pi} cf' \sum_{s=1}^{\infty} a_s{}^2,$$

the product terms when averaged up giving each separately zero. To obtain the portion of this flux corresponding to wave-lengths between λ and $\lambda + d\lambda$ we have only to observe that the $\frac{2c\vartheta}{\lambda^2} d\lambda$ spectral lines lying within that interval, may be considered to have equal intensities. In other words the value of a_s may be regarded as equal for each of them, so that they contribute to the sum Σ in the last equation an amount

$$\frac{2c\vartheta a_s{}^2 d\lambda}{\lambda^2}.$$

Consequently the energy flux through f' belonging to the interval of wave length $d\lambda$ is given by

$$\frac{c^2\vartheta f' a_s{}^2 d\lambda}{4\pi\lambda^2},$$

and we now want to find a_s.

694. From the value of $\mathbf{E}_x$ given above we find that

$$a_s = -\frac{2}{\vartheta c^2 r} \Sigma \left\{ e \int_0^{\vartheta} \sin \frac{s\pi t}{\vartheta} \frac{d\,[\mathbf{v}_x]}{dt} dt \right\},$$

where the square bracket round $\mathbf{v}_x$ serves to indicate the value of this quantity at the time $t - r/c$.

The sign Σ now refers to a sum taken over all the electrons in the part $f\Delta$ of the plate.

On integration by parts we find

$$a_s = \frac{2\pi s e}{\vartheta^2 c^2 r} \Sigma \int_0^{\vartheta} [\mathbf{v}_x] \cos \frac{s\pi t}{\vartheta} dt,$$

or what is the same thing

$$a_s = \frac{2\pi s e}{\vartheta^2 c^2 r} \Sigma \int_{-\frac{r}{c}}^{\vartheta - \frac{r}{c}} \mathbf{v}_x \cos \frac{s\pi}{\vartheta} \left(t + \frac{r}{c}\right) dt.$$

Now the integral on the right is made up of two parts, arising respectively from the intervals between the consecutive impacts of the electrons and from the intervals during these impacts. If we can suppose, as we shall do, that the duration of an encounter of an electron with an atom is much smaller than the time between two successive encounters of the same electron, we may neglect altogether the part that corresponds to the collisions and confine ourselves entirely to the part corresponding to the free paths between the collisions. But while an electron travels over one of these free paths, its velocity $\mathbf{v}_x$ is constant. Thus the part of the integral in a_s which corresponds to one electron and to the time during which it traverses one of its free paths is therefore

$$\mathbf{v}_x \int_t^{t+\tau} \cos \frac{s\pi}{\vartheta} \left(t + \frac{r}{c}\right) dt,$$

where t is now the instant at which this free path is commenced and τ the duration of the journey along it; but this is equal to

$$\frac{2\vartheta \mathbf{v}_x}{s\pi} \sin \frac{s\pi\tau}{2\vartheta} \cos \frac{s\pi}{\vartheta} \left(t + \frac{r}{c} + \frac{\tau}{2}\right).$$

We can now fix our attention on all the paths described by all the electrons under consideration during the time ϑ, and we use the symbol $\mathbf{S}$ to denote a sum relating to all these paths. We have then

$$a_s = \frac{2\pi s e}{\vartheta^2 c^2 r} \mathbf{S} \frac{2\vartheta \mathbf{v}_x}{s\pi} \sin \frac{s\pi\tau}{2\vartheta} \cos \frac{s\pi}{\vartheta} \left(t + \frac{r}{c} + \frac{\tau}{2}\right).$$

695. We now want to determine the square of the sum $\mathbf{S}$. This may be done rather easily because the product of two terms of the sum whether they correspond to different free paths of one and the same electron, or to two paths described by different electrons, will give O if all taken together. Indeed the velocities of two electrons are wholly independent of one another, and the same may be said of the velocities of one definite electron at two instants separated by at least one encounter. Therefore positive and negative values of $\mathbf{v}_x$ being distributed quite indiscriminately between the terms of the series $\mathbf{S}$, positive and negative signs will be equally probable for the

product of two terms. We have therefore only to calculate the sum of the squares of the terms in $\mathbf{S}$ or simply

$$\mathbf{S}\,\frac{4\vartheta^2\mathbf{v}_x^2}{s^2\pi^2}\sin^2\frac{s\pi\tau}{2\vartheta}\cos^2\frac{s\pi}{\vartheta}\left(t+\frac{r}{c}+\frac{\tau}{2}\right).$$

Now since the irregular motion of the electrons takes place with the same intensity in all directions, we may replace $\mathbf{v}_x^2$ by $\frac{1}{3}\mathbf{v}^2$. Also in the immense number of terms included in the sum the quantities τ and $\mathbf{v}$ are very different, and in order to effect the summation we may begin by considering only those terms for which the product $\left(\mathbf{v}\sin\frac{s\pi\tau}{2\vartheta}\right)$ has a certain value. In these terms which are still very numerous, the angle $\frac{s\pi}{\vartheta}\left(t+\frac{r}{c}+\frac{\tau}{2}\right)$ has values that are distributed at random over an interval ranging from 0 to $s\pi$. The square of the cosine may therefore be replaced by its mean value $\frac{1}{2}$, so that

$$a_s^2=\frac{2}{3}\frac{\pi^2 s^2 e^2}{\vartheta^4 c^4 r^2}\,\mathbf{S}\left(\frac{\sin\frac{s\pi\tau}{2\vartheta}}{\frac{s\pi}{2v\vartheta}}\right)^2$$

or introducing after Lorentz, the length of the path l instead of the time τ in it, this may be written

$$a_s^2=\frac{2}{3}\frac{\pi^2 s^2 e^2}{\vartheta^4 c^4 r^2}\,\mathbf{S}\left(\frac{\sin\frac{s\pi l}{2v\vartheta}}{\frac{s\pi}{2v\vartheta}}\right)^2$$

696. The metallic atoms being considered as practically immovable, the velocity of an electron will not be altered by a collision. Let us, therefore, now fix our attention on a certain group of electrons moving along their zig-zag lines with the definite velocity u. During the time ϑ, one of these particles describes a large number of free paths, this number being given by

$$\frac{u\vartheta}{l_m},$$

if l_m is the mean length of the paths; the number of these paths whose length lies between l and $l+dl$, is

$$\frac{u\vartheta}{l_m^2}e^{-\frac{l}{l_m}}dl,$$

so that

$$u\vartheta e^{-\frac{l}{l_m}}\left\{\frac{\sin\left(\frac{s\pi l_m}{2\vartheta u}\cdot\frac{l}{l_m}\right)}{\frac{s\pi l_m}{2\vartheta u}}\right\}^2 dl$$

is the part of the sum contributed by these paths. On integration of this

expression from $l = 0$ to $l = \infty$ we find the part of the sum due to one of the typical electrons, which is therefore

$$\frac{u\vartheta}{\left(\frac{s\pi l_m}{2\vartheta u}\right)^2}\int_0^\infty \sin^2\left(\frac{s\pi l_m}{2\vartheta u}\cdot\frac{l}{l_m}\right) e^{-\frac{l}{l_m}}\,dl = \frac{2u\vartheta l_m}{1+\frac{s^2\pi^2 l_m^2}{\vartheta^2 u^2}}.$$

Now the total number of electrons in the part of the metallic plate under consideration is $Nf\Delta$, and by Maxwell's law, among these

$$4\pi Nf\Delta\,.\sqrt{\frac{q^3}{\pi^3}}\,e^{-qu^2}u^2\,du,$$

have velocities between u and $u+du$; the constant q is related to the mean velocity u_m already mentioned above by the formula

$$q = \frac{3}{2u_m^2}.$$

Thus the total value of the sum $\mathbf{S}$ in a_s^2 is given by

$$4\pi Nfl_m\Delta\sqrt{\frac{q^3}{\pi^3}}\int_0^\infty \frac{2u\vartheta l_m}{1+\frac{s^2\pi^2 l_m^2}{\vartheta^2 u^2}}\,e^{-qu^2}u^2\,du,$$

or, using $z = qu^2$ and $\frac{2\pi c}{\lambda} = \frac{s\pi}{\vartheta}$, by

$$4Nf\Delta\sqrt{\frac{1}{\pi q}}\int_0^\infty \frac{ze^{-z}\,dz}{1+\frac{4\pi^2c^2l_m^2 q}{\lambda^2 z}}.$$

Thus we have

$$a_s^2 = \frac{32}{3}\sqrt{\frac{2}{3\pi}}\,\frac{\pi^2 Ne^2 l_m u_m}{\vartheta c^2 r^2\lambda^2}\,f\Delta\int_0^\infty \frac{ze^{-z}\,dz}{1+\frac{4\pi^2c^2l_m^2q}{\lambda^2 z}},$$

or introducing the expression for the conductivity

$$a_s^2 = \frac{8\pi^2 f\Delta\sigma q}{\vartheta c^2 r^2\lambda^2},$$

and the expression for the partial energy flux through the element f' thus takes the form

$$\frac{Eff'd\lambda}{r^2} = \frac{2\pi ff'\Delta\sigma m}{\lambda^4 q r^2},$$

and thus the emissivity of the plate under the conditions specified is

$$E = \frac{4\pi\Delta\sigma m u_m^2}{3\lambda^4}$$

$$= \frac{4\pi m u_m^2 A}{3\lambda^4},$$

so that

$$\frac{E}{A} = \frac{4\pi m u_m^2}{3\lambda^4}.$$

697. Now as we have already mentioned the mean kinetic energy of an electron for which we may write $\frac{1}{2}mu_m^2$ is presumed to be equal to the mean kinetic energy of a gaseous molecule at the same temperature; and the latter is proportional to the absolute temperature θ so that we may put

$$\tfrac{1}{2}mu_m^2 = a\theta,$$

and then

$$\frac{E}{A} = \frac{8\pi a\theta}{3\lambda^4},$$

so that Kirchhoff's principle which requires this ratio to be absolutely independent of the nature of the metal is exactly verified: its form as a function of λ and θ is also consistent with the general principles laid down above and appears in fact to be perfectly consistent with the experimental facts.

It must however be remembered that the deduction here given restricts the present law for application only to long-wave radiation as it is only in that case that the condition implied in the duration of a collision is satisfactorily verified. The rigorous deduction of a law which shall apply to all wave lengths has not yet been accomplished although one or two successful formulae have been arrived at by distinctly artificial means: a proper discussion of these would however take us too far beyond the purpose which we have at present in view.

CHAPTER XV

THE ELECTRODYNAMICS OF MOVING MEDIA

698. The general equations of electrodynamic theory. We have up to the present confined our considerations mainly to the electromagnetic and electrodynamic phenomena of systems in which the ponderable matter is either actually at rest or is at least in a state of such slow motion that it may at any instant be regarded as at rest relative to the instantaneous electromagnetic field. We must now discuss certain aspects of the more general case of rapidly moving electromagnetic systems: such a discussion appears to be necessary not only because of its intrinsic theoretical interest but because all electrodynamic phenomena are concerned with the more or less rapid motion of electrically charged bodies. Absolute rest is of course unknown by the human intelligence, and for instance all electrostatic fields created on this earth necessarily partake of the motion of the earth through space, so that they are of a type more general than that discussed in the earlier chapters of this book.

We shall begin by formulating the general equations of electrodynamic theory; these have already been set out in full on a previous occasion, but with a view to emphasising the point we may here briefly indicate their deduction on a purely dynamical basis.

699. It has been shown in the previous chapter that on the tentative assumption of appropriate forms for the potential and kinetic energies of an electrical system presumed to comprise merely a group of electrons or electrically charged particles in motion in the aether, the complete circumstances of the configuration and motion of the system can be described by means of the ordinary equations of dynamical theory: in such a mode of formulation of the theory the only effect of the interaction between the electromagnetic condition of the aether and the charge on the moving electron is completely specified as a force of ordinary mechanical type on the typical electron of vector amount

$$e\mathbf{F} = e\left(\mathbf{E} + \frac{1}{c}[\mathbf{vB}]\right),$$

where e is the charge on the electron and $\mathbf{v}$ its velocity; $\mathbf{E}$ is the aethereal

electric force, defined in terms of the vector and scalar potentials by the relation

$$\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{A}}{dt} - \operatorname{grad}\phi :$$

$\mathbf{B}$ is the magnetic induction vector and

$$\mathbf{B} = \operatorname{curl} \mathbf{A}.$$

On such a theory the total effective current is

$$\mathbf{C} = \mathbf{C}_1 + \frac{1}{4\pi}\frac{d\mathbf{E}}{dt} + c \operatorname{curl} \mathbf{I}_1 + \mathbf{v}\rho + \frac{d\mathbf{P}}{dt},$$

where $\mathbf{C}_1$ is the true current of conduction; $\frac{1}{4\pi}\frac{d\mathbf{E}}{dt}$ the fictitious current of aethereal displacement; $\frac{d\mathbf{P}}{dt}$ the true current of material polarisation; $c \operatorname{curl} \mathbf{I}_1$ the current which in its magnetic aspects is the effective equivalent of the distribution of magnetic polarisation, including both the true magnetisation $\mathbf{I}$ and the quasi-magnetisation due to the convection of the polarised medium with velocity $\mathbf{v}$

$$\mathbf{I}_1 = \mathbf{I} + \frac{1}{c}[\mathbf{P}\mathbf{v}];$$

finally $\mathbf{v}\rho$ is the current due to the convection of the material medium charged to density ρ at any point.

700. All of these relations can be regarded either in the light of definitions or as relations of a purely dynamical nature. It follows from them that

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \operatorname{curl} \mathbf{E} + \frac{1}{c}\operatorname{curl}[\mathbf{v}\mathbf{B}]\\ &= -\frac{1}{c}\frac{d}{dt}(\operatorname{curl} \mathbf{A}) + \frac{1}{c}\operatorname{curl}[\mathbf{v}\mathbf{B}]\\ &= -\frac{1}{c}\frac{d\mathbf{B}}{dt} + \frac{1}{c}\operatorname{curl}[\mathbf{v}\mathbf{B}].\end{aligned}$$

Now on reference to the general theorem established in the introduction, § 19, we see that the right-hand side of this equation when multiplied by $-c$ and integrated as regards its normal component over any surface, expresses the time rate of change of the magnetic induction through the surface regarded as moving at each point with the charge system with velocity $\mathbf{v}$. Our equation is thus the analytical expression of Faraday's circuital relation which states that the line integral of the *electromotive force* $\mathbf{F}$ round any circuit which is *carried along with the matter* is equal to the time rate of diminution of the magnetic induction through it multiplied by $1/c$.

701. We have also, of course,

$$\text{curl}\,\mathbf{B} = +\frac{4\pi\mathbf{C}}{c},$$

or if it is preferred not to include the magnetism as molecular current whirls so that the total current is only

$$\mathbf{C} - 4\pi c\mathbf{I}_1,$$

this relation becomes

$$\text{curl}\,\mathbf{H} = \text{curl}\,(\mathbf{B} - 4\pi\mathbf{I}_1)$$

$$= \frac{4\pi}{c}\,(\text{total current}),$$

where the magnetic force vector **H** is defined by the relation

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{I}_1.$$

This is the expression of Ampère's circuital relation that the line integral of the magnetic force round any circuit, fixed or moving*, is at each instant equal to the flow of the Maxwellian total current through it multiplied by the factor $4\pi/c$. It is important to notice that as the magnetic force is here introduced into the theory it is a subsidiary quantity defined in terms of **B** and the magnetisation.

702. When the material medium, however heterogeneous, is at rest in the aether, these electrodynamic equations reduce precisely to Maxwell's original scheme

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{B}}{dt},$$

$$\text{curl}\,\mathbf{H} = \frac{4\pi\mathbf{C}}{c},$$

with

$$\mathbf{C} = \mathbf{C}_1 + \frac{1}{4\pi}\frac{d\mathbf{D}}{dt},$$

$\mathbf{C}_1$ being the true conduction current.

When the material medium is in motion these equations are modified in the following respects; there is a term arising from convection of electric polarisation added to the magnetism, which changes **I** to $\mathbf{I}_1$ and there is the current arising from the convection of electric charge which supplies the term $\rho\mathbf{v}$, a term which Maxwell in some connections temporarily overlooked but which has been fully restored by Fitzgerald and others.

703. The existence of a magnetic field due to the convection of electrically charged bodies and of polarised dielectrics has been experimentally verified by Roentgen† and Rowland‡; doubts were subsequently thrown on the interpretation of their results by Cremieu§ but the experiments have

* Time differentials are not involved.

† *Berlin. Ber.* (1885), p. 198. ‡ *American Jour. of Sc.* (3) 15 (1878), p. 30.

§ *Paris C. R.* 130 (1900), p. 1544, 131 (1900), pp 575, 797.

been repeated with much greater precision by Eichenwald*, with results which completely substantiate the verification of the theoretical predictions.

The arrangement finally adopted by Eichenwald consisted mainly of a parallel circular plate condenser with a uniform dielectric slab. The rapid motion was produced by rotating the whole condenser round an axis of

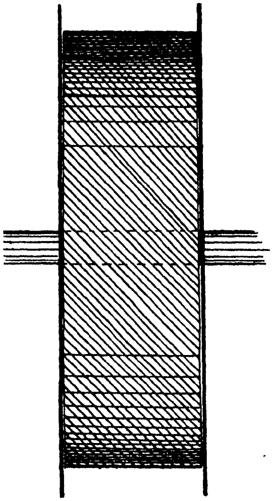


Fig. 100

symmetry perpendicular to the plates. If the charge on the plates of the condenser is of density σ at any point there will be a convection current due to its being dragged on with the system which will be of amount

$$\sigma \mathbf{v},$$

$\mathbf{v}$ denoting the velocity of the system at that point.

In addition the dielectric medium will be polarised to intensity $\mathbf{P}$ and the convection of this will also be equivalent to a current of intensity curl $[\mathbf{vP}]$. If we neglect the irregularity of the edges this field between the plates will be uniform right across and thus $\mathbf{P}$ will be constant throughout the interior of the slab and there will be no volume distribution of current of this latter type; but it exists as a surface distribution on the abrupt interfaces of the dielectric where the density is

$$[\mathbf{n}_1 \,.\, [\mathbf{Pv}]],$$

if $\mathbf{n}_1$ is the unit normal vector whose direction is in the positive direction of $\mathbf{P}$, i.e. straight across between the plates in the present instance. This current thus appears as of magnitude $|\mathbf{P} \,.\, \mathbf{v}|$ and is directed parallel to the direction of $\mathbf{v}$ at each place, but in the opposite sense.

* *Ann. d. Phys.* 11 (1904), p. 421.

The total effective current in this arrangement is a surface current on the plates of the condenser and of density

$$(\sigma - \mathbf{P})\,\mathbf{v}.$$

But if $\mathbf{D}$ is the total electric displacement across the dielectric

$$\sigma = \frac{\mathbf{D}}{4\pi} = \frac{\mathbf{E}}{4\pi} + \mathbf{P},$$

and thus the surface current density is simply

$$\frac{|\mathbf{E}\mathbf{v}|}{4\pi}$$

in the direction of $\mathbf{v}$, i.e. directed in circles round the axes of rotation.

The important point to notice is that this current and therefore also the magnetic field associated with it does not in any way depend on the dielectric material, but only on the potential difference between the plates: and this was exactly verified by Eichenwald.

704. The importance of this experiment is the confirmation which it provides for the fundamental hypothesis on which the present theory is based. The modern theory of electromagnetism is built on the idea of an aether permeated by a large number of electrons or electric point charges, either free or grouped together in material atoms, and it is with these charges and their general configuration and motion that we are alone concerned. The motion of a material medium is thus effectively accounted for in the motion of its constituent electrons. But what about the aether? Can this medium move also; and is it dragged along with the matter which is in motion through it? We have in our discussions tacitly neglected the possibility of any such motion of the fundamental medium and this course appears not only the simplest one but it is found to be more consistent with experimental facts. This is the original view of Fresnel, Lorentz and Larmor; but the opposite view has been strongly advocated by Stokes in optical theory and Hertz in electrical theory. According to their views the motion of a piece of matter through the aether necessarily produces by a sort of mechanical dragging action a convective motion of the aether itself in the neighbourhood of the piece of moving matter. That such a view is inconsistent with the result of Eichenwald's experiment is however easily seen, for according to it no distinction need be made between the separate parts of the total displacement current, and the whole effect summed up in the term $\mathbf{D}$ is presumed to be convected with the matter: the Roentgen current would—in such a theory—have, as is easily seen, a density

$$\frac{1}{4\pi}\operatorname{curl}\,[\mathbf{D}\mathbf{v}],$$

and this adopted into the theory of Eichenwald's experiment would lead to

the result that there should be no resultant current at all, the current due to the convection of the polarised dielectric and aether just balancing that due to the convection of the charges on the plates.

The experiment carried out by H. A. Wilson and described above*, p. 572, also in some respects affords another result in favour of the theory of a stationary aether. It was there shown that the effects of the rotation of a dielectric substance in a magnetic field can be fully and accurately explained on the assumption that it is merely the electrons in the dielectric atoms that are convected with that substance, the aether itself remaining absolutely at rest.

Thus it cannot but be admitted that the course adopted in the above exposition of the theory is at least perfectly consistent with our experience. We shall refer to this point later and mention further and perhaps more exact evidence in its favour, and also some difficulties in the way of its acceptance. We may perhaps here mention a direct attempt made by Lodge to detect an aethereal drag accompanying the mass of a very large rapidly rotating flywheel, but with negative results.

705. On the rotation of a conductor in a magnetic field†. As a first example of the general principles formulated in the previous paragraph we may consider the practically important problem of a conductor in motion in a steady magnetic field. If we consider any closed circuit in the conductor, it is clear that the electromotive force round it depends only on the change produced by its motion in the number of tubes of force that it encloses, and is therefore quite independent of whether the relative motion of the conductor and the field be ascribed to the conductor or to the magnetic field, or to both conjointly. Therefore the currents induced in the body are derived from the same equations whether the axes are fixed or moving *in any manner, uniform or not*. But in the case of an unclosed circuit there is a difference introduced in the value of the electrostatic potential. In fact such an open line which is at rest relatively to the moving axes is displaced across the field owing to the motion of the body: if we suppose the ends of the line in a former position (1) and in a near position (2) at a very short distance from it, to be connected so as to form a closed circuit, the number of tubes of force on the positive side of the line will be diminished by the number which pass through this closed circuit supposed circulating in an anti-clockwise direction. The diminution is therefore equal to the flux of the vector potential along (2) minus its flux along (1), together with its fluxes along the two lines of motion of the ends of the open circuit. Thus when the line has moved from (1) to (2) we must suppose the potential at each end diminished

* *Phil. Trans.* A, 204 (1904), p. 121

† Cf. Larmor, 'Electromagnetic Induction in Conducting Sheets and Solid Bodies,' *Phil. Mag.* (1884), Maxwell, Treatise II. p. 275.

by the flux of the vector potential along the line of motion of that end divided in each case by the usual constant c. Therefore in the equation for the electromotive force we must include terms for the change of the rate of variation of this flux as we pass from point to point of the conductor; that

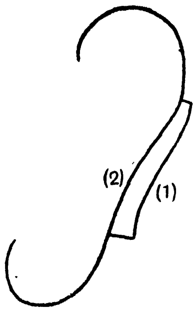


Fig. 101

is instead of the true electrostatic potential ϕ, we shall get from our equations $\phi + \phi'$, where ϕ' is the scalar product of vector potential and velocity of the point supposed connected with the moving system of axes and is therefore

$$-\frac{1}{c}(\mathbf{Av}).$$

This method of statement brings out clearly what it is on which the term ϕ' really depends.

706. The same result here deduced from first principles also follows immediately from the analytical relations between the functions involved. The electromotive force in the field referred to a system of axes at rest is given quite generally by the expression

$$\mathbf{F} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi + \frac{1}{c}[\mathbf{uB}]$$

where $\mathbf{u}$ is the absolute velocity of the moving charge.

Now if $\mathbf{v}$ is any other velocity

$$[\mathbf{vB}] = [\mathbf{v} \,.\, \operatorname{curl} \mathbf{A}] = \nabla(\mathbf{vA}) - (\mathbf{v}\nabla)\,\mathbf{A}$$

so that

$$\mathbf{F} = -\frac{1}{c}\left\{\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{A}\right\} - \nabla\left\{\phi - \frac{1}{c}(\mathbf{Av})\right\} + \frac{1}{c}[\mathbf{u} - \mathbf{v}, \mathbf{B}].$$

Thus we see that if the system is referred to a set of axes moving with a velocity $\mathbf{v}$ the expression for the electromotive force is of exactly the same form as that given above except that the scalar potential ϕ is altered to

$$\phi + \phi' = \phi - \frac{1}{c}(\mathbf{Av})$$

as there explained. This alteration does not of course affect the aggregate force in a closed circuit.

We conclude then that when a *constant* electromagnetic system is moving through the aether the effect produced by the relative motion is an electrostatic charge of the system of such character that its potential is ϕ'. This static charge, however, itself exerts a magnetic effect by virtue of its motion; but it is easy to see that this depends on $\left(\frac{\mathbf{v}}{c}\right)^2$ and is therefore very minute in a real case.

707. In the case of a conductor rotating steadily in a magnetic field a steady distribution of currents in space will ensue when the conductor is symmetrical about the axis of rotation; and the electromotive force along any line will be given by the number of tubes of force of this steady field that are cut through by the line per second. Now when the magnetic field is symmetrical round the axis of rotation the number of tubes enclosed in any closed moving circuit in the conductor will not alter at all, so that there will be no current round any circuit, and therefore no induced currents whatever: the electric force along each open line will accumulate a statical electric charge at one end of it, so that the conductor will become electrified until the induced electromotive force is exactly neutralised by the statical difference of potential. This conclusion holds whatever be the shape of the body.

The state being steady the electromotive force $\mathbf{F}$ given by

$$\mathbf{F} = -\frac{1}{c}\frac{d\mathbf{A}}{dt} - \operatorname{grad}\phi + \frac{1}{c}[\mathbf{vB}],$$

where of course $\dfrac{d\mathbf{A}}{dt}$ is now zero.

Thus, since $\mathbf{F}$ is necessarily derived from a potential Φ,

$$\operatorname{grad}(-\Phi + \phi) = \frac{1}{c}[\mathbf{vB}],$$

so that
$$\phi - \Phi = \frac{1}{c}\int([\mathbf{v}\,.\,\mathbf{B}]\,.\,d\mathbf{s}).$$

708. If the rotation is with steady angular velocity ω about a fixed line which may be chosen as the axis of z in a rectangular frame of reference

$$\mathbf{v}_x = -\omega y, \qquad \mathbf{v}_y = +\omega x,$$

and thus
$$\phi - \Phi = \frac{\omega}{c}\int(\mathbf{B}_x x + \mathbf{B}_y y)\,dz - \mathbf{B}_z(x\,dx + y\,dy),$$

and from this the electrification of the conductor may be determined. For example let us take a uniform field of intensity $\mathbf{H}$ parallel to the axis of rotation

$$\mathbf{B}_x = \mathbf{B}_y = 0, \qquad \mathbf{B}_z = \mathbf{H},$$

we have then

$$\phi - \Phi = C - \frac{1}{2}\frac{\omega}{c}\mathbf{H}(x^2 + y^2),$$

so that

$$\nabla^2(\phi - \Phi) = -\frac{2\mathbf{H}\omega}{c}.$$

Inside the conductor the electromotive potential Φ is constant because electromotive force would induce a compensating charge: thus at internal points

$$\nabla^2\phi = \frac{2\omega\mathbf{H}}{c},$$

thus the electrification there is that belonging to the static potential ϕ and involves a volume density $-\frac{\mathbf{H}\omega}{2\pi c}$ as well as a surface density.

In outside space the circuitality of the aethereal displacement, the force producing which is now identical with the electromotive force, requires that

$$\nabla^2\phi = 0,$$

and ϕ must be itself continuous across the surface because there cannot be discontinuity in the aethereal strain produced in the manner specified; but it is Φ or

$$\phi + \frac{1}{2}\frac{\omega}{c}\mathbf{H}(x^2 + y^2) - C,$$

and not ϕ itself that is constant over the surface of the conductor.

If the conductor is a sphere the outside potential which corresponds to the given value at the surface of the internal potential is

$$\phi = C\frac{a}{r} - \frac{1}{2}\omega\mathbf{H}\frac{a^5}{r^5}\left(\frac{2r^2}{3} - x^2 - y^2\right) + \frac{1}{3}\omega\mathbf{H}\frac{a^3}{r},$$

where r is the radial line. Thus the surface density determined by the difference of the normal gradients of the internal and external electric potentials is

$$\frac{\omega\mathbf{H}a}{8\pi}\left(-\frac{4}{3} + \frac{5(x^2 + y^2)}{r^2}\right).$$

The arbitrary constant C allows us to superpose any free distribution. If the charge on the body is normally zero we may give it such a value that the charge shall remain zero; but if the axis of rotation is uninsulated the condition is that $C = 0$.

709. The case when the axis of rotation is perpendicular to the direction of the field has also a certain amount of practical interest. We consider only the one simple case of a linear conductor consisting of a rigid plane circuit enclosing an area F and rotating with angular velocity ω in a uniform magnetic field of intensity $\mathbf{H}$. If the axis of rotation is the z-axis of the rectangular

coordinate system and the lines of force in the field are parallel to the y-axis then

$$\mathbf{B}_x = \mathbf{B}_x = 0, \quad \mathbf{B}_y = \mathbf{H}$$

and thus

$$\phi - \Phi = \frac{\omega}{c}\int \mathbf{H} y\,dz = \frac{\omega\mathbf{H}}{c}\int y\,dz$$

$$\Phi = \phi - \frac{\omega\mathbf{H}}{c}\int y dz.$$

In the whole circuit there is therefore an additional potential available for driving a current of amount

$$-\frac{\omega\mathbf{H}F\cos\theta}{c}$$

where θ is the angle between the normal to the plane of the current and the direction of the field,

$$\theta = \int \omega\,dt.$$

If the motion is with uniform angular velocity

$$\theta = \omega t + a$$

so that the electromotive force in the circuit induced by its motion is

$$-\frac{\mathbf{H}\omega F\cos(\omega t + a)}{c}$$

and is simply periodic.

710. For the case of a flat disc rotating about an axis perpendicular to its plane in any uniform field, we may divide the force into two components, one parallel to the plane of the disc which produces no induction, on account of the thinness of the sheet, and the other perpendicular to it, whose effect has just been estimated. By connecting one terminal of a wire to the axis and making the other terminal rub along the circumference in Faraday's manner, we utilise the difference of potential to produce a current in the external circuit*.

The well-known phenomenon of uni-polar induction, in which a current is induced when a magnet revolves round its axes of symmetry through its own field, is also explained in a similar manner. We may infer that for a solid magnet of any form, in motion of any type, the induced electromotive force is derived from a potential $-\frac{1}{c}(\mathbf{A}\mathbf{v})$ where $\mathbf{v}$ is the velocity through the aether of the elements of the magnet considered; so that it can at each instant be compensated by the static force due to a minute induced electrification, or it may be used to drive a current in an external circuit.

* Cf. *Exp. Res.* I. § 81 (1831).

The conclusions thus drawn from the theory of the previous paragraph, which have been fully verified by experiments made to substantiate them, have an important bearing on the general theory, as they essentially involve the consideration of electrification as made up of discrete elements surrounded and influenced only by the aether which is the real distinct seat of the electromagnetic field: it is only in such a case that the motion of a conductor independently of the aether in reality involves the transference of electric charges through the electromagnetic field in that aether.

711. Apart however from these theoretical considerations the problem of the rotation of an uncharged conductor in a magnetic field is of practical importance in applications in technology where rotating conducting masses frequently occur*. In these applications however it is the magnetic effects with which we are concerned, so that cases in which the circumstances are as simple as those just discussed are of little importance. It is with the more general case when the steady circumstances involve a distribution of currents in the conductor that we have to deal in actual practice. Confining ourselves entirely to motion in steady fields we see as above that the electromotive force round any circuit depends only on the change produced by its motion in the number of tubes of force that it encloses so that the effects are independent of whether the relative motion of the conductor and the field be ascribed to the conductor or to the field. We can therefore simplify the equations which give the electric currents when the conductor is in motion as we can reduce it to rest and solve the corresponding relative problem, where the motion across the lines of force is replaced by a variation of the field itself. The general considerations of the analogous problems solved in chapter X above will then apply.

712. Let us first consider the phenomenon associated with Arago's disc*. Arago discovered that a magnet placed near a rotating metallic disc experiences a force tending to make it follow the motion of the disc, although when the disc is at rest there is no action between it and the magnet. This action is due to the currents induced in the elements of the plate by their motion across the lines of force in the magnetic field. The distribution of these currents being independent of whether the relative motion is due to the motion of the conductor or magnet, will be the same as that induced in the disc at rest by the same magnetic system to which a rotation about the same axis is imparted in the opposite direction: and this fact, taken in conjunction with the results deduced in the previous problem, shows that

* Cases have been worked out by Larmor, l.c. p. 389; Jochmann, *Crelles Journ.* 63 (1863); Hertz, *Dissertation* (Berlin, 1880); *Ges. Werke*, I. p. 37; Riecke, *Gött. Abhdl.* 21 (1876), p. 1; Gans, *Zeitschr. f. Math. u. Phys.* 48 (1902), p. 1; cf. also S. Valentiner, *Die electromagnet. Rotationen und die Uni-polarinduction* (Karlsruhe, 1904).

† Arago, *Ann. de chim. et phys.* 27 (1824), p. 363, 28 (1825), p. 325.

the magnetic action of the currents in a disc supposed very large so that the irregularities at the edges may be neglected is equivalent to that of a trail of images of the magnetic system in the form of a helix.

If the magnetic system consists of a single magnetic pole of strength unity the helix will lie on the cylinder whose axis is that of the disc and which passes through the magnetic pole. The image will begin at the position of the optical image of the pole in the disc. The distance parallel to the axis between consecutive coils of the helix will be $1/\sigma\omega$. The magnetic effect of the trail will be the same as if this helix had been magnetised everywhere in the direction of a tangent to the cylinder perpendicular to its axis, with an intensity such that the magnetic effect of any small portion is numerically equal to the length of its projection on the disc. The calculation of the force on the magnetic pole would be complicated but it is easy to see that it will consist of (1) a dragging force, parallel to the direction of motion of the disc, (2) a repulsive force acting from the disc, (3) a force towards the axes of the disc. All these were observed by Arago.

713. Let us next consider the case of a thin spherical shell rotating with the angular velocity ω in a uniform field. The conditions being steady the displacement currents in the free aether are non-existent so that the internal and external magnetic fields are derivable from potentials ψ_1 and ψ_2 respectively, which in the most general case satisfy at the surface of the shell the relation

$$\frac{d}{dt}\left(r^2\frac{\partial\psi_1}{\partial r}\right)=\frac{d}{dt}\left(r^2\frac{\partial\psi_2}{\partial r}\right)=\frac{k}{2}\left|\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)\right|_1^2 *.$$

If the sphere is held at rest and the uniform magnetic field rotated so as to produce the same relative motion then the operator d/dt is equivalent to the operator $\omega\,(d/d\phi)$, ϕ denoting the azimuth round the axis of rotation through the centre of the sphere: the condition is therefore

$$\omega\frac{\partial}{\partial\phi}\left(r^2\frac{\partial\psi_1}{\partial r}\right)=\omega\frac{\partial}{\partial\phi}\left(r^2\frac{\partial\psi_2}{\partial r}\right)=\frac{k}{2}\left|\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)\right|_1^2.$$

The inducing field has in the neighbourhood of the sphere the potential

$$\psi=\psi_0-Hr\sin\theta e^{i\phi},$$

or at least the real part of this expression; and we therefore try solutions for the internal and external potentials

$$\psi_1=\psi_0-Hr\sin\theta e^{i\phi}+\frac{A_1a^3}{r^2}\sin\theta e^{i\phi},$$

$$\psi_2=\psi_0-Hr\sin\theta e^{i\phi}+A_2r\sin\theta e^{i\phi},$$

which satisfy all the conditions if

$$i\omega\,(-H+A_2)=i\omega\,(-H-2A_1)=ak\,(A_1-A_2),$$

* $k=\left(\frac{2\pi k\delta}{ac^2}\right)^{-1}$ where a is radius, k conductivity and δ thickness of the shell.

so that if we write $$a = \frac{3ka}{2\omega},$$

we have $$A_2 = \frac{H}{1 - ia} = \frac{1 + ia}{1 + a^2} H.$$

Taking real parts only, we find, if

$$\psi = \psi_0 - Hr \sin\theta \cos\phi,$$

then the external field is determined by its potential

$$\psi_2 = \psi_0 - Hr \sin\theta \cos\phi - Hr \sin\theta \cos\chi \cos(\phi - \chi)$$

$$= \psi_0 - Hr \sin\theta \sin\chi \cos\left(\phi + \frac{\pi}{2} - \chi\right),$$

where $$\tan\chi = a.$$

The internal field thus lags behind the external field by $\left(\frac{\pi}{2} - \chi\right)$, while its intensity is reduced in the ratio $\sin\chi : 1$.

We have also

$$A_1 = -\frac{H}{2(1 + 2ia)},$$

so that the shell has the same outside effect as a simple magnet of moment $\frac{Ha^3}{2\sqrt{1 + a^2}}$, whose axis is inclined to the direction of the original field at an angle $\tan^{-1} a$.

The opposing couple experienced by the rotating shell will therefore be the same as for this magnet, i.e. it will be

$$G = \frac{F^2 a^3 a}{2(1 + a^2)},$$

and the rate of expenditure of power required to keep up the rotation will be $G\omega$.

The case of the solid sphere can be worked out on similar lines to that adopted above and the results are analogous.

714. The steady linear translation of an electrostatic material system*. The general equations of the first paragraph enable us to treat in detail the electrodynamic relations of an electrical system in steady uniform motion through the aether. In order that a steady electric state may be possible without permanent currents of conduction, it is necessary that the configuration of the matter shall be permanent and that its motion shall be the same at all times relative to this configuration and to the aether, and also to the

* Cf. Larmor, *Phil. Trans.* A. 190 (1897) p. 226. The theory is due originally to J. J. Thomson, *Phil. Mag.* (5), 11 (1881), p. 229; *Phil. Mag.* (5), 28 (1889), p. 1; *Phil. Mag.* (5), 31 (1891), p. 149; *Recent Researches*, p. 16. Cf. also Heaviside, *Phil. Mag.* (5), 27 (1889), p. 324; G. F. C. Searle, *Phil. Trans.* 187 A. (1896), p. 675; *Phil. Mag* (5), 44 (1897), p. 329.

extraneous magnetic field, if there is one: this confines it to uniform spiral motion on a definite axis fixed in the aether. We shall here confine our attention to the case when the motion is one of uniform translation and in which there is no extraneous field, electric or magnetic. Under these circumstances the magnetic induction through any circuit moving with the system being constant, the electromotive force **F** is derived from a potential Φ

$$\mathbf{F} = -\operatorname{grad}\Phi,$$

because its line integral round such a circuit vanishes. Inside a conductor the electromotive force **F** must vanish, otherwise electric separation would be going on; therefore Φ must be a constant over and inside any conductor in the system.

Φ is called after Searle the *convection potential* of the field of the moving system*.

715. If we refer the field to axes fixed in and moving with the material system and use **v** as the vector velocity of the system then the total current density at any point in the system is

$$\mathbf{C} = \frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t} + \rho\mathbf{v},$$

ρ being the density of the charge distribution at the point; it is presumed that the system consists entirely of conductors and free aether, no dielectrics being present. But on account of the steadiness of the motion

$$\frac{\partial \mathbf{E}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{E} = 0,$$

so that the current density may be written in the form

$$\mathbf{C} = -\frac{1}{4\pi}(\mathbf{v}\nabla)\,\mathbf{E} + \rho\mathbf{v},$$

or since

$$4\pi\rho = \operatorname{div}\mathbf{E},$$

$$\mathbf{C} = -\frac{1}{4\pi}\{(\mathbf{v}\nabla)\,\mathbf{E} - \mathbf{v}\operatorname{div}\mathbf{E}\}$$

$$= -\frac{1}{4\pi}\operatorname{curl}\,[\mathbf{E}\,.\,\mathbf{v}],$$

the velocity **v** being uniform throughout the system. We have therefore from Ampère's relation

$$\operatorname{curl}\mathbf{H} = \frac{4\pi\mathbf{C}}{c} = -\frac{1}{c}\operatorname{curl}\,[\mathbf{E}\,.\,\mathbf{v}],$$

so that

$$\operatorname{curl}\left\{\mathbf{H} + \frac{1}{c}\,[\mathbf{E}\,.\,\mathbf{v}]\right\} = 0,$$

* Schwarzschild calls it the *electrokinetic potential*. Cf. *Gött. Nachr.* (*math. phys. Kl.*) (1903), p. 125.

which implies that the vector

$$\mathbf{H} - \frac{1}{c}[\mathbf{v} \cdot \mathbf{E}]$$

is the gradient of a potential function: we write

$$\mathbf{H} - \frac{1}{c}[\mathbf{vE}] = -\operatorname{grad}\psi,$$

ψ is an undetermined function which will be continuous as to itself and its gradient except at the surfaces of transition. The most general value of **H** consistent with the circuital relation is thus

$$\mathbf{H} = \frac{1}{c}[\mathbf{vE}] + \operatorname{grad}\psi,$$

the part of it depending on ψ would include the extraneous magnetic field, if there were one, and also the field due to magnets, if any, that belong to the material system itself.

If there is no external applied magnetic field and the moving system itself contains no magnetic matter the magnetic field of the moving charges will be sufficiently defined by the magnetic vector potential **A** so that

$$\mathbf{H} = \frac{1}{c}[\mathbf{vE}],$$

the function ψ being not now necessary, there being no external circumstances to be allowed for.

716. Combining the relation between **E** and **H** (= **B** in free space)* with the direct dynamical relation

$$\mathbf{F} = \mathbf{E} + \frac{1}{c}[\mathbf{vB}] = \mathbf{E} + \frac{1}{c}[\mathbf{v} \cdot \mathbf{H}],$$

we get
$$\mathbf{F} = \mathbf{E} - \frac{1}{c}[\mathbf{v} \cdot \nabla]\psi + \frac{1}{c^2}[[\mathbf{E} \cdot \mathbf{v}] \cdot \mathbf{v}]$$

$$= \mathbf{E}\left(1 - \frac{\mathbf{v}^2}{c^2}\right) + \frac{\mathbf{v}}{c^2}(\mathbf{F} \cdot \mathbf{v}) - \frac{1}{c}[\mathbf{v}\nabla]\psi,$$

wherein as above
$$\mathbf{F} = -\operatorname{grad}\Phi,$$

and
$$(\mathbf{vF}) = (\mathbf{vE}).$$

Again since the total current is always a stream we have

$$\operatorname{div}\mathbf{E} = 4\pi\rho,$$

so that
$$\operatorname{div}\left(\mathbf{F} - \frac{\mathbf{v}}{c^2}(\mathbf{vF})\right) = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)\operatorname{div}\mathbf{E},$$

or
$$\nabla^2\Phi = \frac{1}{c^2}(\mathbf{v}\nabla)^2\Phi - 4\pi\rho\left(1 - \frac{\mathbf{v}^2}{c^2}\right),$$

* Throughout the remainder of this chapter we have conformed to the usual practice of using the magnetic force vector instead of the magnetic induction vector to define the conditions in the free aether. For the present purposes the two vectors are always identical.

where now ψ has disappeared. This is the characteristic equation from which the single independent variable Φ of the problem is to be determined, subject to the condition that it is to be constant over each conductor.

For the interior of a conductor Φ is constant and the electromotive force $\mathbf{F}$ vanishes; but the aethereal displacement $\frac{1}{4\pi}\mathbf{E}$ does not vanish in the conductors, being now given by

$$\left(1 - \frac{\mathbf{v}^2}{c^2}\right)\mathbf{E} = \frac{1}{c}[\mathbf{v}\nabla]\,\psi,$$

which makes it circuital so that there is no volume distribution of electrification.

717. In an investigation in detail of the field produced by the motion, it will conduce to brevity if we take $\mathbf{v}$ to be a velocity parallel to one of the axes of coordinates, say the x-axis. We shall also use the notation

$$\beta \equiv \frac{|\mathbf{v}|}{c} \qquad \kappa^2 = \left(1 - \frac{|\mathbf{v}|^2}{c^2}\right) = 1 - \beta^2.$$

The characteristic equation for the convection potential Φ is then

$$\nabla^2\Phi = \beta^2\frac{\partial^2\Phi}{\partial x^2} - 4\pi\rho\kappa^2,$$

and this has to be solved subject to the condition that Φ is constant over each conductor of the system: as the change in the form of the equation arising from the motion depends on β^2, the differences thereby introduced will all be of the second order of small quantities.

We can restore the above characteristic equation for Φ, the potential of the electromotive force, to an isotropic form by a geometrical strain of the system and the surrounding space represented by

$$(x_0, y_0, z_0) = (\kappa^{-1}x, y, z),$$

where of course $\kappa^2 = 1 - \beta^2$.*

718. Now let us compare our moving system which we may generally describe as S, with the correlative system S_0 obtained by this transformation and supposed at rest: we shall assume that the density ρ_0 of the charge distribution in S_0 is reduced from the corresponding value in S in the ratio $\kappa : 1$ so that corresponding elements of volume contain the same total charges: then if ϕ_0 is the electrostatic potential of these charges on S_0

$$\nabla_0{}^2\phi_0 = -4\pi\rho_0$$

is the characteristic equation satisfied by ϕ_0 in this system. The general type of solution of this equation is, as we had it before,

$$\phi_0 = \int\frac{\rho_0 dv_0}{r_0},$$

* This transformation was suggested by Thomson and Heaviside.

the integral being taken over the entire field and r_0 denoting the distance of the element dv_0 from the point in the field at which the function is calculated.

Now the potential Φ in the moving system satisfies the equation

$$\nabla_0^2 \Phi = -4\pi\kappa^2 \rho = -4\pi\kappa\rho_0,$$

so that
$$\Phi = \kappa\phi_0 = \kappa \int \frac{\rho_0 dv_0}{r_0},$$

and since
$$\rho\, dv = \rho_0 dv_0$$

and
$$r_0^2 = \frac{1}{\kappa^2}(x - x_P)^2 + (y - y_P)^2 + (z - z_P)^2,$$

(x_P, y_P, z_P) denoting the coordinates of the point at which Φ is calculated, and (x, y, z) the coordinates of the position of dv, we may write

$$\Phi = \kappa \int \frac{\rho\, dv}{r_0},$$

and this is the general type of solution for the convection potential in any moving system of the type under consideration.

719. Again comparing the components of the electrostatic force

$$\mathbf{E}_0 = -\nabla\phi_0,$$

in S_0 with the corresponding components of the electromotive force

$$\mathbf{F} = -\nabla\Phi,$$

in S we see that

$$\mathbf{F}_x = -\frac{\partial\Phi}{\partial x} = -\frac{\partial\phi_0}{\partial x_0} = \mathbf{E}_{0x},$$

$$(\mathbf{F}_y, \mathbf{F}_z) = -\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\Phi = -\kappa\left(\frac{\partial}{\partial y_0}, \frac{\partial}{\partial z_0}\right)\phi_0$$
$$= \kappa\,[\mathbf{E}_{0_y}, \mathbf{E}_{0_z}].$$

Thus the forces on corresponding elements of charge in the two systems are equal as regards their components in the direction of motion, but the components in any direction at right angles to this direction are smaller in the moving system in the ratio $\kappa : 1$.

Thus if we have solved the electrostatic problem for any system S_0 at rest, i.e. if we have determined the equilibrium distribution of electricity on the conductors in the system under the influence of the rigid charge distribution, then we can immediately deduce the solution for the equilibrium distribution of electricity on the conductors in a uniformly moving system S obtained from S_0 by a uniform contraction in the direction of the motion in the ratio $\kappa : 1$.

720. Suppose, for example, that the system S_0 is represented by a uniform distribution of electricity of total amount Q_0 throughout the thin homoeoidal ellipsoidal shell between two similar and similarly situated concentric ellipsoids

and that there are no other bodies in the system. If a_{0_1}, a_{0_2}, a_{0_3} are the axes of the mean ellipsoid on which this shell lies then we know from the investigation of the second chapter that the appropriate form of the electrostatic potential ϕ_0 has the constant value

$$\frac{Q_0}{2}\int_0^\infty \frac{dt}{\sqrt{(a_{0_1}{}^2+t)(a_{0_2}{}^2+t)(a_{0_3}{}^2+t)}}$$

throughout the interior of the ellipsoid, whilst at external points the value is

$$\frac{Q_0}{2}\int_\lambda^t \frac{dt}{\sqrt{(a_{0_1}{}^2+t)(a_{0_2}{}^2+t)(a_{0_3}{}^2+t)}},$$

where Q_0 is the total charge on the ellipsoid and in the last expression λ is the positive root of the cubic equation

$$\frac{x^2}{a_{0_1}{}^2+t}+\frac{y^2}{a_{0_2}{}^2+t}+\frac{z^2}{a_{0_3}{}^2+t}=1.$$

Moreover we have seen also that, since the potential ϕ_0 is constant throughout the interior of the ellipsoid, the distribution of charge thus specified is identical in the limit with the surface distribution of charge of the same total amount on the same ellipsoid when composed of conducting material.

Now by uniformly contracting this ellipsoid and its space in the ratio κ parallel to any definite line we obtain another ellipsoid with semi-axes (a_1, a_2, a_3). If this new ellipsoid is moved parallel to the chosen line with the velocity appropriate to the ratio κ the original statical system and its field will exactly correspond in the manner just defined to the system it defines; on it the convection potential will therefore take the constant value

$$\Phi=\frac{\kappa Q}{2}\int_0^\infty \frac{dt}{\sqrt{(a_{0_1}{}^2+t)(a_{0_2}{}^2+t)(a_{0_3}{}^2+t)}},$$

whilst at external points its value is

$$\frac{\kappa Q}{2}\int_\lambda^\infty \frac{dt}{\sqrt{(a_{0_1}{}^2+t)(a_{0_2}{}^2+t)(a_{0_3}{}^2+t)}},$$

λ being the positive root of the equation

$$\frac{x_0{}^2}{a_{0_1}{}^2+t}+\frac{y_0{}^2}{a_{0_2}{}^2+t}+\frac{z_0{}^2}{a_{0_3}{}^2+t}=1.$$

The equilibrium distribution of electricity on a moving conductor is characterised by the fact that the electromotive force in its interior vanishes, i.e. the convection potential Φ is constant there. Thus the distribution on the moving ellipsoid obtained by contraction of the static distribution on the conducting ellipsoid in S_0 is identical with the distribution which would hold if the moving ellipsoid were conducting. But when it is remembered that the electric distribution in S_0 is the limit of a uniform distribution between two concentric, similar and similarly situated ellipsoids and that in the process of uniform contraction these ellipsoids remain concentric similar

and similarly situated, it follows that the new distribution of charge on the moving ellipsoid (a_1, a_2, a_3) would be exactly the same as if it were in equilibrium. Thus the distribution of charge on a conducting ellipsoid is not disturbed by imparting a uniform translatory motion to it*.

Two particular cases of this general theorem have assumed special importance on account of the applications which have been made of them to illustrate the properties of a moving electron, which is nothing more nor less than a charged particle.

721. In the first case the conductor in motion is assumed to be spherical in form, say of radius a†. The conductor in the correlative static system will then be a prolate spheroid with axes $\left(\frac{a}{\kappa}, a, a\right)$, if the motion is along the direction of the x-axis. The appropriate form of the convection potential can then be written in the form

$$\Phi = \frac{\kappa Q}{2}\int_\lambda^\infty \frac{dt}{(a^2+t)\sqrt{\frac{a^2}{\kappa^2}+t}}.$$

where λ is the positive root of the quadratic

$$\frac{x^2}{\kappa^2\left(\frac{a^2}{\kappa^2}+t\right)} + \frac{y^2+z^2}{a^2+t} = 1,$$

and reduces to the constant value

$$\Phi = \frac{\kappa Q}{2}\int_0^\infty \frac{dt}{(a^2+t)\sqrt{\frac{a^2}{\kappa^2}+t}}$$

on the surface of the sphere.

The integrals in these cases can be directly evaluated by the substitution

$$\tau^2 = \frac{a^2}{\kappa^2}+t,$$

so that it becomes

$$\Phi = \kappa Q\int_\lambda^\infty \frac{d\tau}{\tau^2 - a^2\frac{1-\kappa^2}{\kappa^2}}$$

$$= \frac{\kappa^2 Q}{2a\sqrt{1-\kappa^2}}\log\frac{\sqrt{a^2+\lambda\kappa^2}+a\sqrt{1-\kappa^2}}{\sqrt{a^2+\lambda\kappa^2}-a\sqrt{1-\kappa^2}},$$

* Mr H. S. Jones has suggested to me a modification of this proof. If it is assumed that the distribution on the conducting ellipsoid in motion which gives zero force inside it is the equilibrium one, we can argue exactly as in the statical case that the surface density varies as the central perpendicular on the tangent plane at the point, since we have shown that the electric force due to any moving point charge is radial and, *for any given direction*, varies inversely as the radius squared. Thus since $\sigma \propto p$ and the total charge is unaltered, the distribution must remain unaffected by the motion.

† M. Abraham, *Ann. d. Phys.* (IV.) x. p. 105 (1903).

which reduces to the value

$$\Phi = \frac{\kappa^2 Q}{2a\sqrt{1-\kappa^2}} \log \frac{1+\sqrt{1-\kappa^2}}{1-\sqrt{1-\kappa^2}}$$
$$= \frac{Q}{2a} \cdot \frac{1-\beta^2}{\beta} \log \frac{1+\beta}{1-\beta},$$

on the surface of the sphere. The full details of the field can now be determined: the charge on the sphere is uniformly distributed over the surface.

722. In the second case* the moving surface is an oblate spheroid of axes

$$(a\kappa,\ a,\ a),$$

the first being in the direction of motion. The surface in the correlative static system which has axes

$$\left(\frac{a\kappa}{\kappa},\ a,\ a\right),$$

is therefore a sphere of radius a. In this case the electrostatic potential ϕ_0 is

$$\phi_0 = \frac{Q}{r} = \frac{Q}{\sqrt{x^2+y^2+z^2}},$$

so that the convection potential Φ is

$$\Phi = \frac{\kappa^2 Q}{\sqrt{x^2+\kappa^2(y^2+z^2)}},$$

which reduces to the constant value

$$\Phi = \frac{\kappa Q}{a},$$

on the surface of the moving conductor of which the equation is

$$x^2 + \kappa^2(y^2+z^2) = a^2\kappa^2.$$

The field in the moving system is of a much simpler character than that of the previous example and we may therefore examine it in more detail with a view to illustrating some of the general features of these convection fields. The electromotive force $\mathbf{F}$ at any point in the field has components

$$\mathbf{F}_x,\ \mathbf{F}_y,\ \mathbf{F}_z = -\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\Phi$$
$$= \frac{\kappa^2 Q}{\{x^2+\kappa^2(y^2+z^2)\}^{\frac{3}{2}}}\{x,\ \kappa^2 y,\ \kappa^2 z\},$$

the electric force at the same point, which in the most general case of translation along the axis Ox has components

$$(\mathbf{E}_x,\ \mathbf{E}_y,\ \mathbf{E}_z) = \left(\mathbf{F}_x,\ \frac{\mathbf{F}_y}{\kappa^2},\ \frac{\mathbf{F}_z}{\kappa^2}\right),$$

is therefore given by

$$\frac{\kappa^2 Q}{\{x^2+\kappa^2(y^2+z^2)\}^{\frac{3}{2}}}(x,\ y,\ z).$$

* Cf. Lorentz, *The Theory of Electrons*, p. 210. This is also the solution for a point charge.

The magnetic force in the field, given generally by the relation

$$\mathbf{H} = \frac{1}{c}[\mathbf{v} \,.\, \mathbf{E}],$$

has therefore the components

$$(\mathbf{H}_x, \mathbf{H}_y, \mathbf{H}_z) = \frac{\beta Q \kappa^2}{(x^2 + \kappa^2 (y^2 + z^2))^{\frac{3}{2}}} (0, -z, y).$$

723. The formulae thus obtained for the electric and magnetic force vectors in the field are considerably simplified by a simple transformation in which account is taken of the fact that the conditions in the field at any point have really originated from the motion of the sphere not at the instant at which they are examined but at a previous effective instant and that they have been propagated out with the velocity c from the position of the sphere at this instant. We now choose as the origin of a spherical polar coordinate system the centre O (see figure) of the sphere in its effective position for the

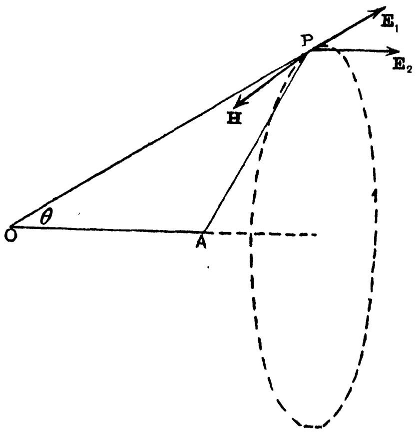


Fig. 102

special point P of the field at which the conditions are investigated: the polar axis is taken along the direction of the motion: then if $OP = r$ and if A is the actual position of the centre of the sphere at the instant t (and this is the origin of the rectangular coordinate system), then

$$OA = \beta r,$$

and thus
$$x^2 = (r \cos\theta - \beta r)^2,$$

$$y^2 + z^2 = r^2 \sin^2\theta.$$

Thus
$$x^2 + \kappa^2 (y^2 + z^2) = r^2 - 2\beta r \cos\theta + \beta^2 r^2 \cos^2\theta$$
$$= r^2 (1 - \beta\cos\theta)^2,$$

and thus the electric force has the components

$$(\mathbf{E}_x,\ \mathbf{E}_y,\ \mathbf{E}_z) = \frac{\kappa^2 Q}{r^2(1-\beta\cos\theta)^3}\{\cos\theta-\beta,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi\},$$

whilst the components of the magnetic force are

$$\frac{-\kappa^2\beta Q\sin\theta}{r^2(1-\beta\cos\theta)^3}(0,\ \sin\phi,\ -\cos\phi).$$

The electric force at any point has therefore a radial component from the effective centre of the sphere of amount

$$\frac{Q}{r^2}\frac{1-\beta^2}{(1-\beta\cos\theta)^3},$$

the remaining part of it being a single component in the meridian plane of amount

$$\frac{Q}{r^2}\frac{(1-\beta^2)\beta\sin\theta}{(1-\beta\cos\theta)^3}$$

parallel to the line of motion.

The magnetic force is in circles round the direction of the motion and at any position $(r,\ \theta)$ its intensity is

$$\frac{Q}{r^2}\frac{(1-\beta^2)\beta\sin\theta}{(1-\beta\cos\theta)^3},$$

which is equal to the latter component of the electric force*.

724. The dynamics of moving electrified systems. In the previous paragraph we have discussed the character of the field in the neighbourhood of an electrical system moving with uniform velocity along a straight line. It is of course assumed that the system thus discussed has been in motion in the manner specified for a sufficiently long time previous to the instant at which it is examined : the conditions implied in this restriction will be fairly obvious when reference is made to the discussions of chapter XII where the mode of establishment of such steady fields is reviewed in detail. It was there seen that in the process of setting up the uniform motion, say from rest, a shell of disturbance is sent out into the surrounding electrostatic field : this shell travels out and away from the system with the velocity c of radiation leaving behind it the new steady field associated with the charges in motion and in which the field vectors are expressed by functions of position decreasing rapidly (like $1/r^2$) as the point is taken more and more distant from the charges themselves. Thus if the shell of disturbance has got to such a distance from the system that the field vectors in the uniform field are negligibly small we may regard the effective conditions for the uniform motion as practically established, and we have then no further concern with the expanding radiation field in which the energy has attained a constant value.

* These formulae are due to Heaviside, *Phil. Mag.* 27 (1889), p. 332; *Electrician*, Dec. 7, 1888, p. 148.

The practical aspect of the results thus obtained lies, of course, in their application to the explanation of the observed mechanical relations of such moving systems of charges and it is with these explanations that we shall now concern ourselves. The considerations of the present paragraph will be confined mainly to the translatory motion of such rigid electrical systems as have been examined in the previous paragraph.

725. In the previous chapter we showed quite generally that the resultant force of electrodynamic origin on any material system contained within any closed surface f can be separated into two parts, the first of which can be represented simply as a stress across the surface f which per unit area is of intensity proportional to the square of the field vectors, whilst the second turns out to be expressible as the result of a distribution of bodily forces throughout the interior field of intensity (vectorial) per unit volume

$$-\frac{1}{4\pi c}\frac{d}{dt}[\mathbf{EH}].$$

This conclusion takes a remarkable form if the system under consideration is of finite dimensions and if the surface f is then extended indefinitely. For then the field vectors at points of the surface are so small that the first part of the total forcive just specified may be neglected. We conclude then that the total force on the whole system in this case can be represented as a wrench which when reduced to the origin of the rectangular coordinates as base point is specified by the linear and angular components $\mathbf{F}$ and $\mathbf{G}$ respectively, where

$$\mathbf{F} = -\frac{1}{4\pi c}\int \frac{d}{dt}[\mathbf{EH}]\,dv$$

$$= -\frac{d\mathbf{M}_e}{dt},$$

and

$$\mathbf{G} = -\frac{1}{4\pi c}\int \left[\mathbf{r}\,.\,\frac{d}{dt}[\mathbf{EH}]\right] dv$$

$$= -\frac{d\mathbf{N}_e}{dt} + [\mathbf{M}_e\,.\,\mathbf{v}],$$

wherein

$$\mathbf{M}_e = \frac{1}{4\pi c}\int [\mathbf{EH}]\,dv$$

and

$$\mathbf{N}_e = \frac{1}{4\pi c}\int [\mathbf{r}\,.\,[\mathbf{E}\,.\,\mathbf{H}]]\,dv,$$

all integrals being extended throughout the entire field: $\mathbf{r}$ denotes the vector coordinate of position of a point whose components are (x, y, z).

The forcive exerted by and through the aether on any electrical system without induced or intrinsic magnetisation is therefore equal and opposite to the change per unit time of the quantity which we have tentatively defined above as the electromagnetic momentum in the aethereal field of the system.

726. If the motion of the charges in the system is one of uniform translation in a fixed direction the field will, as we have just seen, be carried on with the system in a steady configuration. The total electromagnetic momentum in it will therefore be constant. The linear component of the reacting electromagnetic force in the system would then be zero but there will be a couple unless the field is symmetrical about the direction of motion.

In the more general case however whenever the motion of the system is accelerated there will arise a reaction on account of its charges specified as a wrench with linear and angular components

$$-\frac{d\mathbf{M}_e}{dt}, \quad -\frac{d\mathbf{N}_e}{dt} + [\mathbf{M} \cdot \mathbf{v}].$$

There will of course in the general case be a reaction force to the acceleration of the motion of the system on account of its material momentum and this will be a wrench with linear and angular components

$$-\frac{d\mathbf{M}_0}{dt}, \quad -\frac{d\mathbf{N}_0}{dt} + [\mathbf{N}_0\mathbf{v}].$$

Thus to maintain the accelerated motion external action of some kind is necessary; if the external force system reduces to the same base as a wrench with components

$$\mathbf{F}_0, \ \mathbf{G}_0,$$

then we shall have

$$\mathbf{F}_0 + \mathbf{F} = \frac{d\mathbf{M}_0}{dt}, \qquad \mathbf{F}_0 = \frac{d\,(\mathbf{M}_0 + \mathbf{M}_e)}{dt},$$

$$\mathbf{G}_0 + \mathbf{G} = \frac{d\mathbf{N}_0}{dt} + [\mathbf{v} \cdot \mathbf{N}_0], \qquad \mathbf{G}_0 = \frac{d}{dt}(\mathbf{N}_0 + \mathbf{N}_e) + [\mathbf{v} \cdot \mathbf{N}_0 + \mathbf{N}_e].$$

From this point of view it would appear that the idea of an electromagnetic momentum is just as legitimate a conception as that of ordinary material momentum; in any case, of course, the conception, legitimate or not, provides a convenient mode of expressing a definite fact of theory.

727. In the practical application of these principles it is first necessary to determine the vectors $\mathbf{M}_e$ and $\mathbf{N}_e$ for the system under consideration, and this requires a knowledge of the complete electromagnetic field of the moving charges. Now we have already examined the details of the mode of generation and propagation of such electromagnetic fields: we saw that the conditions at any point of the field at a definite instant were made up by superposition of disturbances from all the separate electrons in the system, emitted at the appropriate effective previous time and place for each of them and transmitted thence with the velocity of radiation. The definition of the field at any instant in the most general case is therefore tremendously complicated and it is only in a very few restricted cases that it has been accomplished. But for the majority of the applications we shall have to make of these principles, such generality of procedure, however desirable it might be from the theoretical

point of view, is not at all necessary. In fact it will be seen by reference back to the analysis of the previous chapter that the effective conditions in the field for the determination of the force on the system of electrodynamic origin are in reality merely the conditions which hold in the immediate neighbourhood of the charges in the system. Thus if the system of charges is of finite extent and the changes in its motion are not too rapid the conditions in the field throughout the extent of the field covering the charge distribution will at any instant be practically the same as those which exist in a system moving uniformly with the instantaneous configuration and velocity of the given system. Now the new conditions in the field are smoothed out with the velocity c of radiation so that the condition here implied is that the relative configuration and motion of the system must not change appreciably in the time taken by radiation to cross the system, so that the effective field is at each instant smoothed out to the type appropriate to the motions of the charges before these are appreciably altered. We shall of course in our calculations assume that the uniform field extends to an indefinite distance beyond the system so that we can avail ourselves of the separation of the forcive mentioned above. This procedure is legitimate because, since the force is in reality defined by the conditions of the field in the immediate neighbourhood of the system, the type of field to which this local field continues is irrelevant and may for the purposes of the detailed calculations be taken to be the simple uniform field of the instantaneous motion.

This is the essence of an equilibrium theory and it restricts our analysis to application only in the cases of so-called quasi-stationary motion. The importance in a dynamical theory of the restriction thus implied lies in the fact that the effective conditions in the field are then determined solely by the instantaneous configuration and motion of the charge system giving rise to it, so that instead of having to include in the analysis, in addition to the finite number of the coordinates of the charge system, an infinite number of coordinates to specify the conditions in the aethereal medium surrounding it, we have only to reckon with the former by themselves.

There is a mechanical analogy in the theory of the motion of a solid body through an ordinary elastic medium such as the air. If the motion of the solid is slow enough the conditions of the elastic medium adjust themselves at each instant to the state of motion then existing, and an equilibrium theory applies just as if the medium were incompressible. But in the other extreme case of rapid motion with large accelerations the whole of the circumstances in the surrounding medium are complicated by the continual generation of waves of compression starting out from the solid. In this case the resistance to the motion of the solid is practically all due to the elastic resistance of the surrounding medium to the setting up of waves in it.

728. Thus in the calculation of the quantities involved we may assume that at each instant the field is of the steady type appropriate to that instantaneous motion and position investigated in the previous paragraph; but in this case the motive forces of electromagnetic origin on the elements of charge in the system have a potential Φ, just as the ordinary static forces in electrostatics. We may therefore conclude as in the electrostatic theory that the mechanical forces of electrodynamic origin acting on the material bodies of the system carrying charges have a potential which in the general case is expressed by the integral

$$W = \tfrac{1}{2}\int \Phi\rho\, dv,$$

taken throughout the field, but which in simpler cases will reduce to the forms already discussed at length in the electrostatic theory. From another aspect this quantity may be regarded as potential energy stored up in the system; but it is to be noticed that it also includes a part, due to the magnetic field which we have previously classed as kinetic energy and which therefore when reckoned as potential energy must have its sign changed: it is easy to verify this in the particular case of a charged conductor moving uniformly in a straight line such as discussed in the previous paragraph: for in that case the magnetic or kinetic energy of the system is

$$T = \int \frac{\mathbf{H}^2 dv}{8\pi},$$

and

$$\mathbf{H} = \frac{1}{c}[\mathbf{vE}],$$

so that if the motion is along the x-axis

$$T = \frac{\beta^2}{8\pi}\int dv\, (\mathbf{E}_y{}^2 + \mathbf{E}_z{}^2):$$

whilst the ordinary potential or electric energy of elastic strain in the aether is

$$U = \int \frac{dv}{8\pi}(\mathbf{E}_x{}^2 + \mathbf{E}_y{}^2 + \mathbf{E}_z{}^2),$$

so that

$$\begin{aligned} W = U - T &= -\int \frac{dv}{8\pi}(\mathbf{E}_x{}^2 + \kappa^2\mathbf{E}_y{}^2 + \kappa^2\mathbf{E}_z{}^2) \\ &= -\int \frac{dv}{8\pi}\left(\mathbf{F}_x{}^2 + \frac{\mathbf{F}_y{}^2 + \mathbf{F}_z{}^2}{\kappa^2}\right) \\ &= -\frac{1}{8\pi\kappa^2}\int (\kappa^2\mathbf{F}_x{}^2 + \mathbf{F}_y{}^2 + \mathbf{F}_z{}^2) \\ &= -\frac{1}{8\pi\kappa^2}\int \left\{\kappa^2\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right\} dv, \end{aligned}$$

and this is easily transformed in the usual manner by Green's theorem and gives

$$W = U - T = -\frac{1}{8\pi\kappa^2}\int \phi \left(\kappa^2 \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}\right) dv,$$

$$W = +\tfrac{1}{2}\int \Phi\rho dv$$

as above*.

It must of course be remembered that this is not necessarily all the energy that is in the field, but it represents the variable part in the present case. The motion itself when under examination is of course of the quasi-stationary type so that the radiation field and the energies associated with it are negligible: but the motion must have been started by a non-stationary impulse in some remote past time and in this process a circular shell of disturbance of the wave motion type is sent out into the field and the energy in this part which however tends to a constant value is not necessarily negligible. But the more distant the time of generation, the farther is this wave shell away from the system and the nearer does the field inside it approximate to the actual field belonging to the quasi-stationary motion which the analysis maps out.

729. The above deduction of the form of W shows that the function

$$L = -W$$

also serves as a sort of Lagrangian function. The fact that the force function and Lagrangian function with the sign changed agree in the cases of quasi-stationary motion is not necessarily confined to the present problem: in fact the general equations of motion of any system are of the type

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = \frac{\partial W}{\partial \theta},$$

and if the changes in the motion only take place very slowly, if the accelerations, that is, are small this equation practically reduces to

$$-\frac{\partial L}{\partial \theta} = \frac{\partial W}{\partial \theta},$$

so that

$$W = -L,$$

except perhaps for a constant.

This interpretation of the force function as a Lagrangian function has another significance: in fact we see from the form given for L, viz.

$$L = -\frac{1}{8\pi}\int \{\mathbf{E}_x^2 + \kappa^2(\mathbf{E}_y^2 + \mathbf{E}_z^2)\}\, dv,$$

that

$$\frac{\partial L}{\partial \beta} = \beta \int \frac{dv}{4\pi}(\mathbf{E}_y^2 + \mathbf{E}_z^2) - \int \frac{dv}{4\pi}\left\{\mathbf{E}_x \frac{\partial \mathbf{E}_x}{\partial \beta} + \kappa^2\left(\mathbf{E}_y \frac{\partial \mathbf{E}_y}{\partial \beta} + \mathbf{E}_z \frac{\partial \mathbf{E}_z}{\partial \beta}\right)\right.$$

* Cf. Schwarzschild, Zwei Formen des Prinzips der kleinsten Aktion in der Elektronentheorie, *Gött. Nachr.* (*math. phys. Kl.*), 1903, p. 125.

Now the former of the integrals on the right is

$$= \int \frac{dv}{4\pi} (\mathbf{E}_y \mathbf{H}_z - \mathbf{E}_z \mathbf{H}_y)$$

$$= c\mathbf{M}_e :$$

the second is on the other hand equal to zero for it is

$$- \frac{1}{4\pi} \int \left(\frac{\partial \Phi}{\partial x} \cdot \frac{\partial \mathbf{E}_x}{\partial \beta} + \frac{\partial \Phi}{\partial y} \cdot \frac{\partial \mathbf{E}_y}{\partial \beta} + \frac{\partial \Phi}{\partial z} \cdot \frac{\partial \mathbf{E}_z}{\partial \beta} \right) dv,$$

and this transforms in the usual way by Green's theorem and becomes

$$+ \frac{1}{4\pi} \int \Phi \frac{\partial}{\partial \beta} (\text{div } \mathbf{E}) \, dv$$

$$= \int \Phi \frac{\partial \rho}{\partial \beta} dv = 0.$$

It follows therefore that
$$\frac{\partial L}{\partial \beta} = c\mathbf{M}_e,$$

or since
$$\beta = \frac{|\mathbf{v}|}{c},$$

$$\frac{\partial L}{\partial |\mathbf{v}|} = \frac{1}{c} \frac{\partial L}{\partial \beta} = \mathbf{M}_e,$$

and this relation again exhibits the analogy between the so-called electromagnetic momentum of the system and an ordinary momentum in dynamical theory: it also provides us with the simplest means of calculating $\mathbf{M}_e$.

It is important to notice either as a deduction from this last equation or as a consequence of the particular type of field determined that the vector of linear momentum of a rigid system of charges moving in a straight line is directed entirely along that line: there is no tendency to motion across the direction of translation.

730. Next let us turn to another fundamental aspect of these matters. In ordinary dynamical theories the existence of momentum implies the presence of a material mass moving with a velocity. Now we have seen that for instance an electrostatic system in uniform motion in a straight line would possess something akin to momentum, viz. what we have called electromagnetic momentum, in the direction of its motion, even if its material mass were negligibly small. Thus if we are prepared to adopt the analogy between electromagnetic and material momentum it appears as, at least convenient to extend the analogy still farther and to say that the existence of electromagnetic momentum implies also an electromagnetic mass moving with a velocity. All that it is intended to imply in such a statement is that an electrical system possesses a certain amount of inertia on account of the field and charges in it, just as if it had an additional mass of ordinary type: the inertia offered by this additional mass of the system to accelerated motion

exactly accounts for the reaction which according to our theories arises from the electromagnetic field surrounding the system.

We can put this point in another way. The equations of linear translation of an electrical system were obtained in the form

$$\mathbf{F} = \frac{d}{dt}(\mathbf{M}_0 + \mathbf{M}_e),$$

$\mathbf{F}$ being the applied force vector: now suppose that the motion is of the quasi-stationary type so that both $\mathbf{M}_0$ and $\mathbf{M}_e$ are functions of geometrical configuration and velocities only, and are both parallel to the direction of the velocity $\mathbf{v}$ of the system. If then we use $\mathbf{R}$ for the radius of curvature of the path of the system, and $\mathbf{R}_1$ a unit vector along $\mathbf{R}$ then

$$\mathbf{F} = \frac{d}{d\,|\mathbf{v}|}(\mathbf{M}_0 + \mathbf{M}_e)\frac{d\,|\mathbf{v}|}{dt} + \left|\frac{\mathbf{M}_0 + \mathbf{M}_e}{\mathbf{v}}\right| . \frac{\mathbf{R}_1\,|\mathbf{v}|^2}{\mathbf{R}},$$

and now we recognise the ordinary definition of inertia mass in dynamics on the basis of Newton's second law of motion. Thus even if $\mathbf{M}_0$, the material mass of the body is zero there will still be an apparent mass of electrodynamic origin which for accelerations in the direction of the line of motion is of amount

$$\frac{d\,|\mathbf{M}_e|}{d\,|\mathbf{v}|} = \frac{1}{c}\frac{d\,|\mathbf{M}_e|}{d\beta} = \frac{1}{c^2}\frac{d^2L}{d\beta^2},$$

while it is
$$\left|\frac{\mathbf{M}_e}{\mathbf{v}}\right| = \frac{1}{c}\frac{|\mathbf{M}_e|}{\beta} = \frac{1}{c^2}\frac{1}{\beta}\frac{dL}{d\beta},$$

for accelerations perpendicular to the direction of motion.

Thus the quasi-stationary motion of an electromagnetic system is effectively modified on account of the charges on it just as if it possessed additional mass of an amount however depending on the relative direction of its main velocity and its acceleration.

731. These results may be further illustrated by application to the two special cases examined in the previous paragraph. When the moving system consists solely of a uniformly charged sphere the convection potential assumes the constant value

$$\Phi = \frac{Q}{2a}\frac{1-\beta^2}{\beta}\log\frac{1+\beta}{1-\beta},$$

on the surface of the sphere: the force function or, if we prefer it, the Lagrangian function of the motion with the sign changed is therefore

$$\tfrac{1}{2}\int \Phi\rho\,dv = -L = \frac{Q^2}{4a}\frac{1-\beta^2}{\beta}\log\frac{1+\beta}{1-\beta}.$$

Thus we have
$$\mathbf{M}_e = \frac{1}{c}\frac{dL}{d\beta}$$
$$= \frac{Q^2}{2ac\beta}\left\{\left(\frac{1+\beta^2}{2\beta}\right)\log\frac{1+\beta}{1-\beta} - 1\right\},$$

and then the longitudinal and transverse masses turn out to be*

$$m_e = \frac{Q^2}{2ac^2} \cdot \frac{1}{\beta^2} \left\{ \frac{2}{1-\beta^2} - \frac{1}{\beta} \log \frac{1+\beta}{1-\beta} \right\},$$

$$m_t = \frac{Q^2}{2ac^2} \cdot \frac{1}{\beta^2} \left\{ -1 + \frac{1+\beta^2}{2\beta} \log \frac{1+\beta}{1-\beta} \right\}.$$

These masses, functions of the velocity, increase indefinitely as $\beta = \frac{|\mathbf{v}|}{c}$ is increased and become infinite for $\beta = 1$, i.e. when the velocity of the electron attains the velocity of light. This means of course that such a velocity would never be attained. They have the common limit

$$m_0 = \frac{2}{3} \frac{Q^2}{ac^2},$$

when β is small.

732. In the second case the moving system is of variable dimensions, having the form of an oblate spheroid with axes $(\kappa a, a, a)$ when the motion is with velocity $\mathbf{v}$, where

$$\kappa^2 = 1 - \frac{|\mathbf{v}|^2}{c^2} = 1 - \beta^2.$$

In this case the convection potential assumes the constant value

$$\Phi = \frac{\kappa Q}{a} = (1 - \beta^2)^{\frac{1}{2}} \frac{Q}{a},$$

on the surface of the body where the charge is confined: and thus now

$$L = \frac{\kappa Q^2}{2a}.$$

It follows therefore, just as above, that

$$\mathbf{M}_e = \frac{1}{c} \frac{dL}{d\beta} = \frac{Q^2}{2ac} \cdot \frac{\beta}{\sqrt{1-\beta^2}},$$

so that †

$$m_e = \frac{Q^2}{2ac^2} \cdot \frac{1}{(1-\beta^2)^{\frac{3}{2}}},$$

$$m_t = \frac{Q^2}{2ac^2} \cdot \frac{1}{(1-\beta^2)^{\frac{1}{2}}},$$

which again become infinite as the velocity approaches that of light, starting from the same limit.

$$m_0' = \frac{Q^2}{2ac^2} = \frac{4}{3} m_0.$$

733. The great interest in these results lies in their application to the explanation of the properties of the electron. It is impossible to charge an ordinary piece of matter with sufficient electricity to produce an electromagnetic inertia at all comparable with its ordinary inertia; but an electron

* Abraham, *Ann. d. Phys.* 10 (1903), p. 105, *Die Theorie der Elektrizität*, II. p. 181.

† Lorentz, *Theory of Electrons*, p. 210.

carries a charge which is so enormous, compared with its mass, that its electromagnetic inertia is far greater than its material inertia if it has any. Again pieces of matter even so small as the individual molecules are much too cumbersome to get up a speed comparable with that of light, whereas the electrons are often found under natural circumstances travelling with such speeds, so that they should exhibit the additional characteristic of the electromagnetic mass in increasing with the speed. Some time after Thomson's determination of the ratio of the charge to the mass, Kaufmann* proceeded by the same method to determine the functional form of this ratio in terms of the velocity, with a view to testing the effectiveness of the theoretical formulae. He worked with the negative electrons which are thrown off as the β-rays from radium with velocities ranging up to $\cdot 95\,c$. Now it was found that while the velocity increases from about $\cdot 5\,c$. to the higher value the corresponding value of e/m considerably diminishes, and exactly in the way that we might expect if the charge remains constant and the mass increases according to the theoretical formulae. The first experiments were unable to distinguish between the two types of formulae given, the first by Abraham and the second by Lorentz, but later and more exact methods tend to the view that the simpler functional forms given by Lorentz's method are the more exact of the two.

The conclusion to be drawn from these facts is that at all events the electromagnetic mass of an electron has an appreciable influence; but the experiments went farther, the type of function obtained pointed to the conclusion that the electromagnetic mass greatly preponderates over any material mass that the electrons may have. Indeed Kaufmann's numbers show no trace of an influence of the material mass at all, his ratio of the effective masses for two different velocities agreeing within the limits of experimental error with the theoretical ratio of the electromagnetic masses themselves.

734. If the material mass of an electron is inappreciable under all circumstances it may be treated as absolutely zero. It is no use talking of something that cannot be traced. On such a view the electron consists merely of an element of negative electricity.

Of course by the negation of a material mass, the electron loses much of its substantiality because our theory has been interpreted entirely in terms of charged matter and the ordinary mechanical relations of such matter. In speaking therefore of the electron as having no material mass, we must nevertheless leave it sufficient substantiality to enable us to speak of forces acting on it. In terms of a pure aether theory we might say that the electron, which is necessarily a centre of an electric field of strain in the aether, is

* *Gött. Nachr.* 1 (1901), 5 (1902), 3 (1903); *Phys. Zeitschr.* 1902, p. 55; *Ann. d. Phys.* 19 (1906), p. 487.

nothing more nor less than the centre or nucleus of the strain thus depicted, a knot, so to speak, which can however move freely from one point of the medium to another. This point of view has been advocated by Larmor and it has many points in its favour on account of the simplicity it introduces into the subject.

It is natural to enquire whether there is similarly no material mass for the elements of positive electricity. Unfortunately no decisive answer is yet possible to this question, but the view is now generally held that not only is there no material mass in the ordinary sense of the word in the positive particles, but that the material mass of any ordinary piece of matter is nothing but the inertia mass of the electric charges involved in it*. This is the 'electromagnetic theory of matter' and although it is not possible for us to enter here into any further details, it must be admitted that the evidence of both the experimental and theoretical researches of the last 10 years is gradually tending to confirm the substantiality of the view.

735. Optical phenomena in moving media. We now leave these interesting questions concerning moving charges to turn to another aspect of electrodynamic theory of moving systems in general. We have so far merely treated of steady electromagnetic fields convected with a system of bodies having a motion of translation, but it is equally important to consider the other extreme case of the propagation of very rapid electromagnetic disturbances across the field between such moving systems. The practical aspect of these considerations lies in their application to the explanation of the optical phenomena in systems having a motion of translation, as for instance all terrestrial bodies have by the annual motion of the earth.

We consider in the first place the propagation of electric waves across a dielectric medium, which is moving with uniform velocity $\mathbf{v}$ parallel to a definite direction which we shall take to be the x-axis of a system of rectangular coordinates. According to the general scheme of equations formulated above we have the electromotive force expressed in terms of the vector and scalar potentials and the magnetic induction by the relation

$$\mathbf{F} = -\frac{1}{c}\frac{d\mathbf{A}}{dt} - \operatorname{grad}\phi + \frac{1}{c}[\mathbf{vB}],$$

where

$$\mathbf{B} = \operatorname{curl}\mathbf{A},$$

and if $\mathbf{C}$ is the total current of Maxwell's theory

$$\frac{4\pi\mathbf{C}}{c} = \operatorname{curl}\mathbf{H},$$

and

$$\mathbf{B} = \mathbf{H} + \frac{4\pi}{c}[\mathbf{Pv}],$$

* Cf. Larmor, *Phil. Trans.* A, 186 (1895), p. 697. Cf. also *Aether and Matter*.

if there are no magnetic substances about; $\mathbf{P}$ is the dielectric polarisation: thus*

$$\nabla^2\mathbf{A} = -\frac{4\pi\mathbf{C}}{c} - 4\pi \operatorname{curl}[\mathbf{Pv}].$$

These equations are satisfied by the propagation of a train of transverse waves along the x-axis, in which $\mathbf{F}_x$, $\mathbf{H}_x$, $\mathbf{C}_x$, $\mathbf{A}_x$ are all null, while ϕ is a function of y, z.

Thus for such a wave train

$$\begin{aligned}[\mathbf{vB}] &= [\mathbf{v}[\nabla\mathbf{A}]] \\ &= \nabla(\mathbf{Av}) - (\mathbf{v}\nabla)\mathbf{A} \\ &= -v\frac{\partial\mathbf{A}}{\partial x}\end{aligned}$$

and thus

$$\mathbf{F}_y = -\frac{1}{c}\left(\frac{d}{dt} + v\frac{\partial}{\partial x}\right)\mathbf{A}_y - \frac{\partial\phi}{\partial y},$$

$$\mathbf{F}_z = -\frac{1}{c}\left(\frac{d}{dt} + v\frac{\partial}{\partial x}\right)\mathbf{A}_z - \frac{\partial\phi}{\partial z}.$$

Now since $\operatorname{div}\mathbf{A} = 0$ any theory that makes

$$\operatorname{div}\mathbf{F} = 0,$$

will also make $\nabla^2\phi = 0$, that is, will make ϕ merely the static potential of the electric charges in the field. We shall then have

$$\nabla^2\mathbf{F} = \frac{4\pi}{c^2}\left(\frac{d}{dt} + v\frac{\partial}{\partial x}\right)\{\mathbf{C} + c\operatorname{curl}[\mathbf{Pv}]\},$$

and the remainder of the analysis depends on the relation between $\mathbf{C}$ and $\mathbf{F}$.

736. Now we have seen that the total current $\mathbf{C}$ is composed of two parts: the first, an aethereal part, of density

$$\frac{1}{4\pi}\frac{d\mathbf{E}}{dt},$$

which is not convected, presumably because the aether does not participate in the motion of the matter; the second part depends on the material polarisation and is measured by its time rate of change

$$\frac{\partial\mathbf{P}}{\partial t}.$$

Again
$$\operatorname{curl}[\mathbf{Pv}] = (\mathbf{v}\nabla)\mathbf{P} - \mathbf{v}(\nabla\mathbf{P})$$
and as far as concerns the two components under consideration this is the same as

$$v\frac{\partial\mathbf{P}}{\partial x}$$

whilst
$$\mathbf{P} = \epsilon'\mathbf{F},$$

* We are using Maxwell's instantaneous vector potential.

where $\epsilon = 1 + 4\pi\epsilon'$ is the dielectric constant of the medium*. Thus

$$\mathbf{C} + c \operatorname{curl} [\mathbf{P}\mathbf{v}] = \frac{1}{4\pi}\frac{d\mathbf{E}}{dt} + \frac{\epsilon - 1}{4\pi}\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\mathbf{F}.$$

Now putting ϕ null for purely transverse waves, which will be found to cause no discrepancy, we have

$$\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{A}}{dt},$$

and thus keeping $\mathbf{A}$ as the more convenient independent variable

$$\nabla^2\mathbf{A} = \frac{1}{c^2}\left[\frac{\partial^2}{\partial t^2} + (\epsilon - 1)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)^2\right]\mathbf{A}.$$

If the period of the waves propagated is $\frac{2\pi}{p}$ all the functions may be taken to depend on the time and position coordinates by the factor

$$e^{ip(c't - x)},$$

where c' is the velocity of propagation; substituting this form we find that

$$c^2 = c'^2 + (\epsilon - 1)(c' - v)^2,$$

whence since v/c' is very small we have approximately

$$c' = \frac{c}{\sqrt{\epsilon}}\left(1 + \left(1 - \frac{1}{\epsilon}\right)\frac{v}{c'}\right)$$
$$= \frac{c}{\sqrt{\epsilon}} + \left(1 - \frac{1}{\epsilon}\right)v.$$

Thus the velocity of propagation of radiation in the moving medium is increased from its value in the same medium at rest by the amount

$$\left(1 - \frac{1}{\epsilon}\right)v.$$

This result was formulated by Fresnel† long before the electromagnetic theory had been formulated, and it has been experimentally verified first by Fizeau but much more accurately by Michelson and Morley‡ in connection with the propagation of light in flowing water. In the experiments water was made to flow in opposite directions through two parallel tubes placed side by side and closed at both ends by glass plates: the two interfering beams of light were passed through the tubes in such a manner that throughout their course one went with the water and the other against it.

These experiments provide the most decisive, as well as the most accurate test of Fresnel's hypothesis of a stagnant aether, on which the above theory has been based, and they definitely exclude the more complicated hypothesis

* It is to be understood that the constant here involved is that appropriate to rapidly alternating fields and is a function of the frequency of alternation.

† *Paris C. R.* 33 (1851), p. 349; *Ann. d. Phys.* (1853), p. 377.

‡ *Amer. Jour. of Sc.* 31 (1885).

advanced by Stokes. This is easily seen because if the motion of a piece of matter through the aether drags that medium with it, all that is contained in the flowing column of water in the experiments will share the translatory motion of flow: the propagation of light in the water will then go on in its interior in exactly the same manner as if it were at rest; so that its actual velocity in space will be $\left(\frac{c}{\sqrt{\epsilon}} + v\right)$.

737. A further important point is also emphasised by the result obtained above. It essentially involves not only the distinction between aethereal and material polarisation currents but also the distinction between the forces producing these currents, which is brought out in the general dynamical theory of the previous chapter. It was there shown that the force of electrodynamic origin which acts on the electrons and produces the true current of material displacement is the electromotive force $\mathbf{F}$, whereas the force which strains the aether and produces the aethereal displacement current is the electric force, differing from the electromotive force $\mathbf{F}$ by the dynamical part $\frac{1}{c}[\mathbf{vB}]$.

The fundamental hypotheses on which the theory is based are therefore the only ones which are consistent with the results of experiments made especially with a view to testing them: we must now follow the theory through into some of its more important consequences in order to see whether it is entirely sufficient or whether it still requires modification and amplification. Before proceeding at once to the optical theory it will however be convenient to elaborate at the present stage a few further details of a question already dealt with at some length on a previous occasion.

738. The ponderomotive force which an electromagnetic field exerts on any interface separating a perfectly conducting or absorbing medium from the surrounding free space is determined mainly in the form of a traction per unit area

$$\mathbf{T} = \frac{1}{8\pi}\{2\mathbf{E}\mathbf{E}_n + 2\mathbf{H}\mathbf{H}_n - \mathbf{n}_1(\mathbf{E}^2 + \mathbf{H}^2)\},$$

$\mathbf{n}_1$ being the unit vector along the normal to the surface.

The resultant force exerted by the field on any such body is obtained by integration of this surface force over the outer surface of the body.

Now let us find how this surface force is altered when the body moves in a vacuum: the above expression is then increased for momentum is taken up as a result of the motion. If $\mathbf{v}$ is the velocity of the typical element df of the surface, the momentum picked up by the body on account of the motion of df is then

$$\frac{\mathbf{v}_n}{4\pi c}[\mathbf{EH}]\,df.$$

It follows that the pressure per unit area on the moving surface is of the form

$$\mathbf{T}' = \frac{1}{4\pi}\left\{\mathbf{E}\mathbf{E}_n + \mathbf{H}\mathbf{H}_n - \frac{1}{2}\,\mathbf{n}\,(\mathbf{E}^2 + \mathbf{H}^2) + \frac{\mathbf{v}_n}{c}\,[\mathbf{E}\mathbf{H}]\right\}.$$

We can transform this into a more convenient form: to do this we start from the identity

$$\mathbf{v}_n\,[\mathbf{E}\mathbf{H}] + \mathbf{E}_n\,[\mathbf{H}\,.\,\mathbf{v}] + \mathbf{H}_n\,[\mathbf{v}\mathbf{E}] = \mathbf{n}_1\,(\mathbf{v}\,.\,[\mathbf{E}\mathbf{H}]),$$

so that we may write

$$8\pi\mathbf{T}' = 2\mathbf{E}'\mathbf{E}_n + 2\mathbf{H}'\mathbf{H}_n - \mathbf{n}\left\{\mathbf{E}^2 + \mathbf{H}^2 - \frac{2}{c}\,(\mathbf{v}\,.\,[\mathbf{E}\mathbf{H}])\right\},$$

where we have written $\mathbf{E}'$ and $\mathbf{H}'$ for the vectors

$$\mathbf{E}' \equiv \mathbf{E} + \frac{1}{c}\,[\mathbf{v}\mathbf{H}], \qquad \mathbf{H}' = \mathbf{H} + \frac{1}{c}\,[\mathbf{E}\,.\,\mathbf{v}].$$

We may notice also that

$$(\mathbf{E}\mathbf{E}') = \mathbf{E}^2 - \frac{1}{c}\,(\mathbf{v}\,.\,[\mathbf{E}\mathbf{H}]), \qquad (\mathbf{H}\mathbf{H}') = \mathbf{H}^2 - \frac{1}{c}\,(\mathbf{v}\,.\,[\mathbf{E}\mathbf{H}]),$$

so that

$$\mathbf{T}' = \frac{1}{4\pi}\left\{\mathbf{E}'\mathbf{E}_n + \mathbf{H}'\mathbf{H}_n - \frac{\mathbf{n}}{2}\,((\mathbf{E}\mathbf{E}') + (\mathbf{H}\mathbf{H}'))\right\}$$

is the general expression for the electromagnetic surface force on a moving surface.

739. If the problem concerns the pressure of radiation on a perfectly absorbing surface in motion, then $\mathbf{E}$, $\mathbf{H}$ are simply the vectors of the field of the incident radiation, there being no reflexion. But with a moving mirror things are different; here the reflexion takes place so that at all points of the surface the tangential components of the electromotive force ($\mathbf{E}'$) and the normal components of magnetic induction ($\mathbf{H}$ and therefore also of the vector $\mathbf{H}'$) must vanish*. Thus at the surface

$$\mathbf{H}_n = 0, \qquad [\mathbf{n}\mathbf{E}'] = 0,$$

so that

$$0 = [\mathbf{E}\,.\,[\mathbf{E}'\mathbf{n}]] = \mathbf{E}'\mathbf{E}_n - \mathbf{n}\,(\mathbf{E}\mathbf{E}'),$$

so that we have now

$$\mathbf{T}' = \frac{1}{8\pi}\,\mathbf{n}_1\,(\mathbf{E}\mathbf{E}' - \mathbf{H}\mathbf{H}')$$

$$= \frac{\mathbf{n}_1}{8\pi}\,(\mathbf{E}^2 - \mathbf{H}^2).$$

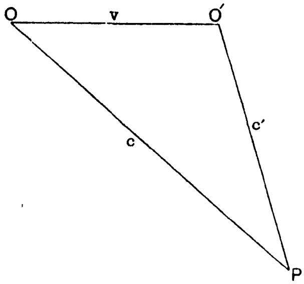
Each of these two last formulae determine the surface forces exerted by the electromagnetic field on the moving mirror. Since $\mathbf{n}_1$ is a unit vector parallel to the normal to the surface $\mathbf{T}'$ is always perpendicular to this surface. The radiation pressure gives no tangential forces on the perfectly reflecting surface.

* Provided of course the substance of the mirror is perfectly reflecting, so that there is no penetration.

740. Now let us return to our purely optical considerations in moving media.

The fundamental conception of optical science is the relation between the direction of the ray and that in which the radiant waves are travelling. When the material medium is at rest in the aether, the ray, or the path of the radiant energy, is the same relative to the matter as to the aether. In the circumstances of moving matter there are however two kinds of rays to be distinguished, one of them being the paths of the radiant energy with respect to the particles of the moving matter, the other the absolute path of the radiant energy in the stagnant aether, or as we may say in space. As radiation is revealed to us wholly by its action on matter it is the former type of rays that is of objective importance.

Let us suppose that a ray from a fixed source passes through a small opening in a screen at O* and after traversing the distance OP in the free aether arrives at P. An observer situated at P and the screen at O may be supposed to have a common velocity $\mathbf{v}$. The opening O has then arrived



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at the new position O' by the time that the radiation which passed through it at O has reached the observer at P and this observer, supposed ignorant of the motion, would regard $O'P$ as the direction of the ray: this is therefore the direction of the *relative ray* and it is parallel to the direction of the relative velocity of the actual light and the observer

$$\mathbf{c}' = \mathbf{c} - \mathbf{v}.$$

Bradley explained by this simple construction the aberration of the starlight which arises from the yearly motion of the earth: the periodic motion of the earth round the sun gives rise to a periodic change in the direction of the relative ray of light from a star and therefore also to yearly periodic change in the apparent position of the star.

741. This simple definition of the relative ray can also be obtained by purely electromagnetic reasoning without introducing the conceptions of geometrical optics.

* For example the object glass of a telescope.

The absolute ray of light is determined by the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}[\mathbf{EH}],$$

and it gives the flux of energy per unit area through a small surface placed perpendicular to its direction. If this surface be regarded as perfectly black all the energy which is thus transferred is absorbed by the surface.

The relative flux of energy per unit area through a small surface perpendicular to the direction denoted by $\mathbf{n}$ and moving with the velocity $\mathbf{v}$ is

$$\mathbf{S}_n - \frac{\mathbf{v}_n}{8\pi}(\mathbf{E}^2 + \mathbf{H}^2),$$

the second term arising from the fact that the energy provided in the flux $\mathbf{S}$ is partly used up in establishing the energy in the ray which gets longer and longer on account of the motion of the surface. Suppose now that the moving surface is perfectly absorbing; then we may regard the component of the relative ray in the direction perpendicular to this surface as defined by the amount of energy absorbed per unit time by the surface and transformed into heat. Now only part of the energy which reaches the surface is thus transformed, the remainder being used up in doing mechanical work as a radiation pressure on the surface: moreover the rate of working of this latter force per unit area is

$$-\frac{1}{8\pi}\left\{2\mathbf{E}_n(\mathbf{vE}) + 2\mathbf{H}_n(\mathbf{v}\,.\,\mathbf{H}) - \mathbf{v}_n(\mathbf{E}^2 + \mathbf{H}^2) + \frac{2\mathbf{v}_n}{c}(\mathbf{v}\,.\,[\mathbf{EH}])\right\},$$

so that the component of the relative ray perpendicular to the direction of the surface is determined by

$$\mathbf{S}_n + \frac{1}{4\pi}\left\{\mathbf{E}_n(\mathbf{vE}) + \mathbf{H}_n(\mathbf{vH}) - \mathbf{v}_n(\mathbf{E}^2 + \mathbf{H}^2) + \frac{\mathbf{v}_n}{c}(\mathbf{v}\,.\,[\mathbf{EH}])\right\},$$

which may be regarded as the component in the same direction of the vector

$$\mathbf{S}' = \mathbf{S} + \frac{1}{4\pi}\left\{\mathbf{E}(\mathbf{vE}) + \mathbf{H}(\mathbf{vH}) - \mathbf{v}(\mathbf{E}^2 + \mathbf{H}^2) + \frac{\mathbf{v}}{c}(\mathbf{v}\,.\,[\mathbf{EH}])\right\},$$

which determines the relative ray vector. If we write

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}[\mathbf{v}\,.\,\mathbf{H}], \qquad \mathbf{H}' = \mathbf{H} - \frac{1}{c}[\mathbf{vE}],$$

then we see that

$$[\mathbf{E}'\mathbf{H}'] = [\mathbf{EH}] - \frac{1}{c}[\mathbf{E}\,.\,[\mathbf{vE}]] - \frac{1}{c}[\mathbf{H}\,.\,[\mathbf{v}\,.\,\mathbf{H}]] + \frac{1}{c^2}[[\mathbf{vE}]\,.\,[\mathbf{vH}]],$$

and

$$\begin{aligned}[\mathbf{E}\,.\,[\mathbf{v}\,.\,\mathbf{E}]] &= \mathbf{vE}^2 - \mathbf{E}(\mathbf{vE}),\\ [\mathbf{H}[\mathbf{vH}]] &= \mathbf{vH}^2 - \mathbf{H}(\mathbf{vH}),\\ [[\mathbf{vE}]\,.\,[\mathbf{vH}]] &= \mathbf{v}([\mathbf{vE}],\ \mathbf{H}) - \mathbf{H}([\mathbf{vE}]\,.\,\mathbf{v})\\ &= \mathbf{v}(\mathbf{v}\,.\,[\mathbf{EH}]),\end{aligned}$$

so that finally

$$\mathbf{S}' = \frac{c}{4\pi}[\mathbf{E}',\ \mathbf{H}'],$$

which is the simplest form of the general expression of the relative ray vector: its component in any direction determines the amount of heat developed per unit time per unit area in a perfectly absorbing surface moving with the appropriate velocity and placed perpendicular to this direction.

742. It remains to show that for plane waves this definition of the relative ray direction agrees with the elementary rule given in the earlier part of the paragraph.

For a plane polarised wave train the absolute vector velocity $\mathbf{c}$ of propagation, and the vectors of electric and magnetic force intensities are mutually perpendicular to one another: and since the intensities of these forces are equal to one another we have

$$\mathbf{H} = \frac{1}{c}[\mathbf{c}\,.\,\mathbf{E}], \qquad \mathbf{E} = \frac{1}{c}[\mathbf{H}\,.\,\mathbf{c}].$$

Thus
$$\mathbf{E}' = \frac{1}{c}[\mathbf{H}\,.\,\mathbf{c}'], \qquad \mathbf{H}' = \frac{1}{c}[\mathbf{c}'\mathbf{E}],$$

so that
$$\begin{aligned}\mathbf{S}' &= \frac{\mathbf{c}}{4\pi}\,.\,\frac{1}{c}[[\mathbf{H}\mathbf{c}']\,.\,[\mathbf{c}'\mathbf{E}]] \\ &= \frac{c}{4\pi}\,.\,\frac{\mathbf{c}'}{c^2}(\mathbf{c}'\,.\,[\mathbf{E}\mathbf{H}]) \\ &= \frac{\mathbf{c}'}{c^2}(\mathbf{c}'\,.\,\mathbf{S}),\end{aligned}$$

so that $\mathbf{S}'$ is parallel to the relative velocity of the light to the receiving surface: the elementary rule for the construction of the path is therefore completely in accordance with the electromagnetic theory of these things.

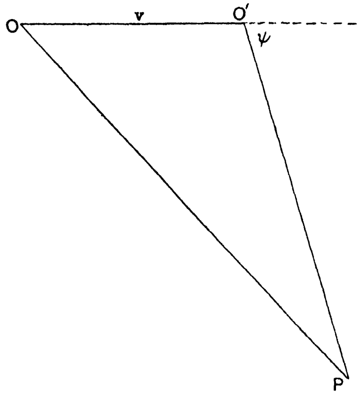
743. These considerations are confined to cases where the source of light is at rest in the aether and they therefore only apply to such problems as the aberration of light from fixed stars. We have next to enquire whether the motion of a system has any influence on optical phenomena taking place inside it. Such a question is of practical importance in considerations respecting the effect of the motion of the earth on optical phenomena taking place on its surface. Can we determine the motion of the earth by optical measurements in a laboratory? Let us imagine that at the time $t = 0$ a light signal is emitted from the point O. At the instant t it will have arrived at P, the absolute direction of its path being OP, which we can denote by the vector $\mathbf{r}$. In the interval thus occupied by the actual radiation in travelling from O to P the source of light will have moved to O' with the velocity $\mathbf{v}$ of the system to which it and the observer at P belong. The radius $O'P$ in the vector sense is given by

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t,$$

and of course
$$\mathbf{r} = \mathbf{c}t,$$

so that
$$\mathbf{r}' = (\mathbf{c} - \mathbf{v})\frac{r}{c} = \frac{r}{c}\mathbf{c}', \qquad r = |\mathbf{r}|.$$

Thus the observed direction of the ray is identical with the actual direction of the line joining the instantaneous positions of source and observer. Thus in a uniformly moving system the source of light is seen just where it happens to be at the instant. The common motion of source and observer cannot therefore be detected by an observation of the direction of the relative ray. We might however still hope to detect this motion by a measurement of the



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length of the path of the ray, for the surface through P of constant absolute light distances is a sphere of centre O and the point O' from which the relative ray starts lies eccentric to this sphere so that given lengths r of the absolute light path would correspond to different lengths r' of the relative light path $O'P$ depending on its direction. Now we see from the figure as given

$$r^2 = \beta^2 r^2 + r'^2 + 2\beta r r' \cos\psi,$$

where β is used for $\frac{|\mathbf{v}|}{c}$ just as before so that again using $\kappa^2 = 1 - \beta^2$

$$\kappa r = \frac{\beta r' \cos\psi}{\kappa} + \sqrt{r'^2 + \frac{\beta^2 r'^2 \cos^2\psi}{\kappa^2}}.$$

If now we choose moving axes with origin at P and the x-axis in the direction of motion of the observer and source we can write

$$\kappa r = \frac{\beta x'}{\kappa} + \sqrt{\frac{x'^2}{\kappa^2} + y'^2 + z'^2},$$

where (x', y', z') are the coordinates of O': this relation determines the absolute length of the path in terms of the relative coordinates of source and observer.

744. Now let us compare with this moving system a stationary system obtained from it by a uniform expansion in the ratio $1 : \kappa$ parallel to the

direction of motion, i.e. the x-axis. The analytical transformation is expressed by the relations

$$x_0' = \frac{x'}{\kappa}, \quad y_0' = y', \quad z_0' = z',$$

giving the coordinates of the point P_0' corresponding to P'. If we use

$$r_0'^2 = x_0'^2 + y_0'^2 + z_0'^2,$$

then

$$\kappa r = r_0' + \beta x_0'.$$

The coordinates of the absolute effective position of the source in the moving system relative to P are therefore connected with the coordinates of the relative position in the stationary system by the relations

$$\kappa x_0' = x - \beta r, \quad y_0' = y, \quad z_0' = z,$$

and we have also

$$\kappa r_0' = r - \beta x,$$

$$\kappa x = x_0' + \beta r_0', \quad y = y_0^1, \quad z = z_0'.$$

745. Now let us examine the question as to whether a measurement of the length of the relative light path by an observer moving with the system will enable a determination of the motion of the system to be made. The method is to send two parts of the same beam along different paths and then unite them and determine by an interference method whether the one has had to traverse a longer path. Suppose that light is sent along the relative path $O'P'$ and then reflected back by a mirror to O'. The length of the absolute path of the incident beam is

$$r_1 = \kappa^{-1} r_0' + \beta \kappa^{-1} x_0',$$

and that of the reflected beam is

$$r_2 = \kappa^{-1} r_0' - \beta \kappa^{-1} x_0',$$

the sum of the two being

$$r_1 + r_2 = 2\kappa^{-1} r_0'.$$

Now draw a sphere round O' as centre and of radius r'. For any point P on this sphere the path $O'PO'$ would be the same if the system were at rest, but when it is in motion this is by no means the case. In this case the absolute path is determined not by the relative radius in $\mathbf{S}$ but by the corresponding length r_0' in the expanded system $\mathbf{S}_0$,

$$r_0'^2 = \frac{x'^2}{\kappa^2} + y'^2 + z'^2,$$

whilst

$$r'^2 = x'^2 + y'^2 + z'^2.$$

Thus a given total relative path corresponds to different lengths of the absolute path. If the relative path is parallel to the direction of motion

$$r_0'^2 = \frac{r'^2}{\kappa^2},$$

so that

$$r_1 + r_2 = \frac{2r'}{\kappa^2},$$

whilst if the relative path is perpendicular to the direction of motion

$$r_0' = r',$$

and
$$r_1 + r_2 = \frac{2r'}{\kappa}.$$

Thus with equal relative paths the ratio of the absolute light paths in the two cases is $1 : \kappa$ greater in the first than in the second. The difference of the light paths is then Δl when

$$\frac{\Delta l}{l} = (1 - \beta^2)^{-\frac{1}{2}} - 1 = \tfrac{1}{2}\beta^2,$$

if we neglect higher powers of β.

746. This difference in length of the absolute light paths for a beam propagated over a given length in a moving system parallel and perpendicular to the motion was first noticed by Maxwell. It formed the object of investigation in a now famous experiment conducted by Michelson and Morley* with a view to determining the motion of the earth. Two pencils of light coming from the same source are brought to inference after traversing perpendicular paths of equal length, the one perpendicular to the direction of motion of the earth and the other parallel to it. Although the precision attained in the measurements was such that a difference equal to $\frac{1}{20}$ of that expected would have been observed and measured no difference whatever was observed at all which was not within the very close limits of experimental error. The conclusion must therefore be drawn that no effect of the kind mentioned can be experimentally detected.

Thus if we retain the view tacitly assumed throughout that the dimensions of a body are unaltered by imparting a uniform motion to it, the theory as we have expounded it is inconsistent with experience. In order however to surmount this difficulty it is suggested by Fitzgerald† and Lorentz‡ that we make the further not unnatural assumption that the dimensions of a body in motion are altered in such a way that its diameter parallel to the direction of motion are contracted in the corresponding ratio

$$\kappa : 1 \quad \text{or} \quad \left(1 - \frac{|\mathbf{v}|^2}{c^2}\right)^{\frac{1}{2}} : 1.$$

This of course disposes of the difficulty of the Michelson-Morley experiment for it throws the points in the stationary system S_0 which correspond to loci of constant relative paths round O' on to a sphere and constant relative paths then correspond to constant absolute paths.

* *Phil. Mag.* Dec. 1887. Cf. also Morley and Miller, *Phil. Mag.* 9 (1905).

† Cf. Lodge, *Phil. Trans.* 184 A (1893), p. 727.

‡ 'Versuch einer Theorie der elektrischen u. optischen Erscheinungen in bewegters Körpen,' Leiden (1895).

747. However extraordinary this hypothesis may appear at first sight, it must be admitted that it is by no means gratuitous, if we assume that the intermolecular forces act through the mediation of the aether in a manner similar to that which we know to be the case in regard to electric and magnetic forces. If that is so, the translation of the matter will most likely alter the action between two molecules or atoms in a manner similar to that in which it alters the attraction or repulsion between electrically charged particles. As then the form and dimensions of a solid body are determined in the last resort solely by the intensity of the molecular forces, an alteration of the dimensions cannot well be left out of consideration.

In its theoretical aspect there is thus nothing to be urged against the hypothesis. As regards its experimental aspect we at once notice that the elongation or contraction which it implies is extraordinarily minute. It would involve a shortening in the diameter of the earth of about $6\frac{1}{2}$ centimetres. The only experimental arrangements in which it could come into evidence would be just of the type of this one of Michelson's which first suggested it.

748. Relativity. Before finally closing this work reference must be made to a fundamental property of the electromagnetic equations which is of great importance in the discussions relating to the conditions in moving electrical systems and which co-ordinates in one principle many of the properties of these systems treated separately above. We shall simplify our discussion by assuming that the whole electrodynamic properties of matter can be explained on the basis of a stationary aether and electrons, and we shall so far substantiate these latter elements as to assume that they are extremely small but finite entities consisting essentially and solely of electricity distributed with a finite volume density throughout their small extent, where the fundamental equations are also presumed to hold. In this case the only constituents of the complete current of the theory are its purely aethereal part

$$\frac{1}{4\pi}\frac{d\mathbf{E}}{dt}$$

and a part due to the convection of the electrons which, at any place, is of density

$$\rho\mathbf{u},$$

ρ being the density of the charge in the point of the electron passing over it.

The equations of the theory referred to a frame of reference fixed in the aether thus assume the form

$$\text{curl } \mathbf{H} = \frac{1}{c}\frac{d\mathbf{E}}{dt} + 4\pi\rho\mathbf{u}$$

$$\text{curl } \mathbf{E} = -\frac{1}{c}\frac{d\mathbf{H}}{dt}$$

with

$$\text{div } \mathbf{E} = 4\pi\rho, \ \text{div } \mathbf{H} = 0.$$

It is now at least of philosophical interest to determine how the electrical phenomena in such a system would behave to an observer moving uniformly relative to the fixed frame of reference with the velocity v, which we may, for simplicity, take to be parallel to the x-axis of the fixed co-ordinate system. The first step naturally will be to take a set of axes moving with the observer, but parallel to the old ones. When this is done the equations lose their original simplicity; but it is found that the original form can be recovered by a slight modification of the variables*.

749. We write $\kappa^2 = 1 - \frac{v^2}{c^2}$ and then put

$$x' = \frac{1}{\kappa}(x - vt), \quad y' = y, \quad z' = z$$

$$t' = \frac{1}{\kappa}\left(t - \frac{vx}{c^2}\right)$$

and then if ϕ is any function of (x, y, z, t)

$$\frac{\partial\phi}{\partial x} = \kappa^{-1}\left(\frac{\partial\phi}{\partial x'} - \frac{v}{c^2}\frac{\partial\phi}{\partial t'}\right), \quad \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial y'}, \quad \frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial z'}.$$

and

$$\frac{\partial\phi}{\partial t} = \kappa^{-1}\left(\frac{\partial\phi}{\partial t'} - v\frac{\partial\phi}{\partial x'}\right).$$

Thus in terms of the new variables the fundamental equations assume the form

$$\frac{\kappa^{-1}}{c}\left(\frac{\partial\mathbf{E}_x}{\partial t'} - v\frac{\partial\mathbf{E}_x}{\partial x'}\right) + \frac{4\pi\rho\mathbf{u}_x}{c} = \frac{\partial\mathbf{H}_z}{\partial y'} - \frac{\partial\mathbf{H}_y}{\partial z'}$$

$$\frac{\kappa^{-1}}{c}\left(\frac{\partial\mathbf{E}_y}{\partial t'} - v\frac{\partial\mathbf{E}_y}{\partial x'}\right) + \frac{4\pi\rho\mathbf{u}_x}{c} = \frac{\partial\mathbf{H}_x}{\partial z'} - \kappa\left(\frac{\partial\mathbf{H}_z}{\partial x'} - \frac{v}{c^2}\frac{\partial\mathbf{H}_z}{\partial t'}\right)$$

$$\frac{\kappa^{-1}}{c}\left(\frac{\partial\mathbf{E}_z}{\partial t'} - v\frac{\partial\mathbf{E}_z}{\partial x'}\right) + \frac{4\pi\rho\mathbf{u}_z}{c} = \kappa\left(\frac{\partial\mathbf{H}_y}{\partial x'} - \frac{v}{c^2}\frac{\partial\mathbf{H}_y}{\partial t'}\right) - \frac{\partial\mathbf{H}_x}{\partial y'}.$$

The second and third of these can be written in the form

$$\frac{1}{c}\frac{\partial\mathbf{E}_y'}{\partial t'} + \frac{4\pi\rho\mathbf{u}_y}{c} = \frac{\partial\mathbf{H}_x'}{\partial z'} - \frac{\partial\mathbf{H}_z'}{\partial x'}$$

$$\frac{1}{c}\frac{\partial\mathbf{E}_z'}{\partial t} + \frac{4\pi\rho\mathbf{u}_z}{c} = \frac{\partial\mathbf{H}_y'}{\partial x'} - \frac{\partial\mathbf{H}_x'}{\partial y'}$$

where

$$\mathbf{E}_y' = \kappa^{-1}\left(\mathbf{E}_y - \frac{v}{c}\mathbf{H}_z\right), \quad \mathbf{E}_z' = \kappa^{-1}\left(\mathbf{E}_z + \frac{v}{c}\mathbf{H}_y\right)$$

$$\mathbf{H}_y' = \kappa^{-1}\left(\mathbf{H}_y + \frac{v}{c}\mathbf{E}_z\right), \quad \mathbf{H}_z' = \kappa^{-1}\left(\mathbf{H}_z - \frac{v}{c}\mathbf{E}_y\right).$$

* Cf. Lorentz, 'Versuch einer Theorie der elektrischen u. optischen Erscheinungen,' etc.; Larmor, *Aether and Matter*, chaps. x and xi; Lorentz, *Amsterdam Proc.* (1904), p. 809; Einstein, *Annalen d. Phys.* 17 (1905).

These last equations are equivalent to

$$\mathbf{E}_y = \kappa^{-1}\left(\mathbf{E}_y' + \frac{v}{c}\mathbf{H}_z'\right), \quad \mathbf{E}_z = \kappa^{-1}\left(\mathbf{E}_z' - \frac{v}{c}\mathbf{H}_y'\right)$$

$$\mathbf{H}_y = \kappa^{-1}\left(\mathbf{H}_y' - \frac{v}{c}\mathbf{E}_z'\right), \quad \mathbf{H}_z = \kappa^{-1}\left(\mathbf{H}_z' + \frac{v}{c}\mathbf{E}_y'\right).$$

If now we substitute these values in the first equation and use also

$$\mathbf{E}_x' = \mathbf{E}_x, \quad \mathbf{H}_x' = \mathbf{H}_x$$

it becomes

$$\frac{\kappa^{-1}}{c}\left\{\frac{\partial \mathbf{E}_x'}{\partial t'} - v\left(\frac{\partial \mathbf{E}_x'}{\partial x'} + \frac{\partial \mathbf{E}_y'}{\partial y'} + \frac{\partial \mathbf{E}_z'}{\partial z'}\right)\right\} + \frac{4\pi\rho \mathbf{u}_x}{c} = \kappa^{-1}\left(\frac{\partial \mathbf{H}_z'}{\partial y'} - \frac{\partial \mathbf{H}_y'}{\partial z'}\right)$$

while the equation

$$\operatorname{div} \mathbf{E} = 4\pi\rho$$

becomes under similar circumstances

$$\kappa^{-1}\left(\frac{\partial \mathbf{E}_x'}{\partial x'} - \frac{v}{c^2}\frac{\partial \mathbf{E}_x'}{\partial t'}\right) + \kappa^{-1}\left(\frac{\partial \mathbf{E}_y'}{\partial y'} + \frac{v}{c}\frac{\partial \mathbf{H}_z'}{\partial y'}\right) + \kappa^{-1}\left(\frac{\partial \mathbf{E}_z'}{\partial z'} - \frac{v}{c}\frac{\partial \mathbf{H}_y'}{\partial z'}\right) = 4\pi\rho.$$

Multiplying this by $\frac{v}{c}$ and adding it to the first equation and remembering that

$$\kappa^2 = 1 - \frac{v^2}{c^2},$$

we get

$$\frac{1}{c}\left\{\frac{\partial \mathbf{E}_x'}{\partial t'} + \frac{4\pi\rho}{\kappa}(u_x - v)\right\} = \frac{\partial \mathbf{H}_z'}{\partial y'} - \frac{\partial \mathbf{H}_y'}{\partial z'}$$

and also

$$\frac{\partial \mathbf{E}_x'}{\partial x'} + \frac{\partial \mathbf{E}_y'}{\partial y'} + \frac{\partial \mathbf{E}_z'}{\partial z'} = \frac{4\pi\rho}{\kappa}\left(1 - \frac{vu_x}{c^2}\right).$$

750. On treating Faraday's equations in the same way we find that they are equivalent to

$$-\frac{1}{c}\frac{\partial \mathbf{H}_x'}{\partial t'} = \frac{\partial \mathbf{E}_z'}{\partial y'} - \frac{\partial \mathbf{E}_y'}{\partial z'}$$

$$-\frac{1}{c}\frac{\partial \mathbf{H}_y'}{\partial t'} = \frac{\partial \mathbf{E}_x'}{\partial z'} - \frac{\partial \mathbf{E}_z'}{\partial x'}$$

$$-\frac{1}{c}\frac{\partial \mathbf{H}_z'}{\partial t'} = \frac{\partial \mathbf{E}_y'}{\partial x'} \quad \frac{\partial \mathbf{E}_x'}{\partial y'}$$

together with the conditional equation

$$\frac{\partial \mathbf{H}_x'}{\partial x'} + \frac{\partial \mathbf{H}_y'}{\partial y'} + \frac{\partial \mathbf{H}_z'}{\partial z'} = 0.$$

Thus finally if we write

$$\mathbf{u}_x' = \frac{\mathbf{u}_x - v}{1 - \dfrac{v\mathbf{u}_x}{c}} = \frac{dx'}{dt'}$$

$$\mathbf{u}_y' = \frac{\kappa \mathbf{u}_y}{1 - \dfrac{v\mathbf{u}_x}{c}} = \frac{dy'}{dt'}$$

$$\mathbf{u}_z' = \frac{\kappa \mathbf{u}_z}{- \dfrac{v\mathbf{u}_x}{c}} = \frac{dz'}{dt'}$$

and $$\rho' = \rho\kappa^{-1}\left(1 - \frac{v\mathbf{u}_x}{c}\right) = \rho\frac{dt'}{dt}$$

the original form of the equations is completely restored.

Moreover since the determinant of the transformation is unity and the transformation itself is an orthogonal one we must have

$$dx'dy'dz'dt' = dx\,dy\,dz\,dt$$

so that also $$\rho'\,dx'\,dy'\,dz' = \rho\,dx\,dy\,dz$$

or the charges in corresponding elements of volume in the two systems are the same.

Thus if we transform any electrical system S in the manner specified above into another system S' in such a way that the charges in corresponding volume elements in the two systems are the same, the electromagnetic equations defining the sequence of events in the system remain unaltered, so long as the vectors $\mathbf{E}'$, $\mathbf{H}'$ in S' correspond respectively to the vectors $\mathbf{E}$, $\mathbf{H}$ in S. This mathematical law is called by Minkowski the 'Theorem of Relativity.'*

751. According to the transformation as a mere geometrical correspondence, the length of a line in the direction of the axis of x, measured in the co-ordinates (x', y', z'), is greater than its length measured in the co-ordinates (x, y, z) in the ratio $1 : \kappa$. Thus the Fitzgerald-Lorentz hypothesis of the reduction in the dimensions of a body when it moves relatively to an observer is reduced by this geometrical transformation to the assumption that in the variables associated with it, the shape of the body is unaltered.

A similar result holds as regards the time. Suppose we have a clock moving with the origin of the (x', y', z') system of co-ordinates: then $x' = 0$ and so $x = vt$. Thus if t_1', t_2' are the times of two consecutive events as recorded on the moving clock and t_1, t_2 the times for the same events as registered in the fixed system, then

$$\kappa^{-1}(t_2' - t_1') = t_2 - t_1.$$

We may for instance take t_2', t_1' to represent the two consecutive hour strokes of the clock. It is then clear from the equation above that the moving clock as observed from the fixed system will appear to have its periodic time increased in the ratio $1 : \kappa$. The frequency will be decreased in the inverse ratio.

752. Let us† now imagine an observer whom we shall call A and to whom we shall assign a fixed position in the aether, to be engaged in the study of the phenomena in the stationary electromagnetic system S. We shall suppose him to be provided with a measuring rod and a clock. By these

* Cf. H. Minkowski, 'Zwei Abhandlungen über die Grundgleichungen der Elektrodynamik' (Leipzig, 1910); the first appeared in *Göttinger Nachr.* (*math.-phys. Kl.*), 1907.

† Cf. Lorentz, *Theory of Electrons*, ch. v.

means he will be able to determine the co-ordinates (x, y, z) of any point and the time t for any instant, and by studying the electromagnetic field as it manifests itself to him at different places he will be led to the introduction of the vectors **E** and **H** and the fundamental equations connecting them.

Let A' be a second observer, whose task it is to examine the phenomena in the system S', and who himself also moves through the aether with the velocity v, without being aware of his motion or that of the system S'. Let this observer also be provided with a similar measuring rod, which will however be contracted on account of its possessing the motion v, and a clock working properly to his time. If this observer study the electromagnetic phenomena in the system S' he will introduce the vectors $\mathbf{E}'$, $\mathbf{H}'$ and find that they are connected by exactly the same equations as were the vectors used by the observer A in the stationary system.

Thus if both A and A' were to keep a record of their observations and the conclusions they draw from them, these records would, on comparison, be found to be identical. In other words neither observer can detect, by an examination of circumstances in his own system, whether he is in motion or at rest.

753. Thus far it has been the task of each observer to examine the conditions in his own system. Let us now suppose that each observer is able to see the system to which the other belongs and to study the phenomena in it. The observer A, in studying the electromagnetic field in S' will be led to introduce the new variables x', y', z', t'; $\mathbf{E}', \mathbf{H}'$, etc., and so will establish our relative equations which are exactly those deduced directly by A'. On the other hand A', in studying the field of S, would introduce new variables (x, y, z, t) defined by the reversed relations (the v to him is now negative) and eventually he would deduce the original equations found by A.

Thus the impressions received by the two observers in examining their own or any other electromagnetic system would be exactly identical, so that in reality neither could be sure of his own or the other's state of motion: all that they know is that their relativity velocity is v. We conclude that if the transformation mathematically defined above has any truth in fact then no measurements of electromagnetic conditions will ever enable anyone to determine whether he is in motion or not, except relatively to other bodies. The idea of absolute motion, frequently employed in our discussions, thus proves to be immaterial to our physical theories.

754. The conclusion just stated can hardly be avoided on purely philosophical grounds, and if it is granted as intuitive then a sufficient scientific reason is provided for the general validity of the transformation on which

it was based. In fact Einstein*, starting from the philosophical principle that absolute motion is physically indeterminate, derives the transformation by purely mathematical considerations. As a consequence of the principle he assumes that the velocity of propagation of light in any direction referred to any axes must be the same and then proceeds to determine what amount of arbitrariness in the space and time variables is consistent with this assumption. If the changes between the variables from the fixed to a moving system is linear and such that a point at rest in the fixed system corresponds to a point moving with a uniform velocity v parallel to the direction of the x-axis in the other, then we are limited to transformations of the type

$$x' = k(x - vt), \quad y' = ly, \quad z' = lz$$
$$t' = \alpha x + \beta y + \gamma z + \delta t.$$

The analytical criterion is that the equation

$$dx^2 + dy^2 + dz^2 - c^2dt^2 = 0$$

must lead to the equation

$$dx'^2 + dy'^2 + dz'^2 - c^2dt'^2 = 0$$

and it is easy to verify that the only forms of the above transformation are those in which

$$k = l\kappa^{-1} = l\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\alpha = -\frac{lv}{c^2\kappa}, \quad \beta = \gamma = 0, \quad \delta = l\kappa^{-1}.$$

The transformation thus only differs from that adopted above by the presence of the arbitrary constant factor l; this can however easily be reduced to unity by a proper choice of units.

A full discussion of the developments of this theory is quite beyond the scope of the present work. There are however several treatises† specially devoted to the subject and the reader who desires to pursue the subject is referred to these for further information. He will find in them much elegant mathematical analysis and several important applications of the fundamental principle to the dynamics of moving electrical systems; to radiation problems, and finally also to the fundamental problem of gravitation.

* *Ann. d. Phys.* 17 (1905).

† Cf. M. Lane, *Das Relativitätsprinzip* (Brunswick, 1913); E. Cunningham, *The Principle of Relativity* (Cambridge, 1914); L. Silberstein. *The Theory of Relativity* (London, 1914).

APPENDIX

EXAMPLES

I. ELECTROSTATICS

1. Charges $4q$, $-q$ are placed at the points A_1, A_2 and B is the point of equilibrium. Prove that the line of force which passes through B meets A_1A_2 at an angle of 60° at A_1 and at right angles at B. Find the angle at A_1 between A_1A_2 and the line of force which leaves A_2 at right angles to A_1A_2.

2. Two positive charges q_1 and q_2 are placed at the points A_1 and A_2 respectively. Show that the tangent at infinity to the line of force which starts from q_1 making an angle α with A_2A_1 produced, makes an angle

$$2 \sin^{-1}\left(\sqrt{\frac{q_1}{q_1+q_2}} \sin\frac{\alpha}{2}\right)$$

with A_2A_1, and passes through B in A_1A_2 such that

$$A_1B : BA_2 = q_2 : q_1.$$

3. Point charges $+q$, $-q$ are placed at the points A_1, A_2. The line of force which leaves A_1 making an angle α with A_1A_2 meets the plane which bisects A_1A_2 at right angles in P. Show that

$$\sin\frac{\alpha}{2} = \sqrt{2} \sin\frac{PA_1A_2}{2}.$$

4. The potential is given at four points near each other and not all in one plane. Obtain an approximate construction for the direction of the field in their neighbourhood.

5. The potentials at the four corners of a small tetrahedron $A_1A_2A_3A_4$ are ϕ_1, ϕ_2, ϕ_3, ϕ_4 respectively. G is the centre of gravity of masses m_1, m_2 m_3, m_4 at A_1, A_2, A_3, A_4 respectively. Show that the potential at G is

$$\frac{m_1\phi_1 + m_2\phi_2 + m_3\phi_3 + m_4\phi_4}{m_1 + m_2 + m_3 + m_4}.$$

6. Charges $3q$, $-q$, $-q$ are placed at A_1, A_2, A_3 respectively, where A_2 is the middle point of A_1A_3. Draw a rough diagram of the lines of force: shew that a line of force which starts from A_1 making an angle α with $A_1A_2 > \cos^{-1}(-\frac{1}{3})$ will not reach A_2 or A_3, and show that the asymptote of the line of force for which $\alpha = \cos^{-1}(-\frac{2}{3})$ is at right angles to A_1A_3.

7. If there are three electrified points A_1, A_2, A_3 in a straight line, such that $A_1A_3 = f$, $A_2A_3 = \frac{a^2}{f}$, and the charges are q, $-\frac{aq}{f}$ and ϕa respectively, show that there is always a spherical equipotential surface, and discuss the position of the points of equilibrium on the line A_1A_3 when $\phi = q\frac{f+a}{(f-a)^2}$ and when $\phi = q\frac{f-a}{(f+a)^2}$.

8. A negative point charge $-q_2$ lies between two positive point charges q_1 and q_3 on the line joining them and at distances a_1, a_2 from them respectively. Show that, if the magnitudes of the charges are given by

$$\frac{q_1}{a_2} = \frac{q_3}{a_1} = \frac{\lambda^3 q_2}{a_1 + a_2}, \text{ and if } 1 < \lambda^2 < \left(\frac{a_1 + a_2}{a_1 - a_2}\right)^2,$$

there is a circle at every point of which the force vanishes. Determine the general form of the equi-potential surface on which this circle lies.

9. Charges of electricity q_1, $-q_2$, q_3 $(q_3 > q_1)$ are placed in a straight line, the negative charge being midway between the other two. Show that, if $4q_2$ lie between $(q_3^{\frac{1}{3}} - q_1^{\frac{1}{3}})^3$ and $(q_3^{\frac{1}{3}} + q_1^{\frac{1}{3}})^3$, the number of unit tubes of force that pass from q_1 to q_2 is

$$\tfrac{1}{2}(q_1 + q_2 - q_3) + \frac{3}{4\sqrt{2}}(q_3^{\frac{2}{3}} - q_1^{\frac{2}{3}})(q_1^{\frac{2}{3}} - 2^{\frac{1}{3}}q_2^{\frac{2}{3}} + q_3^{\frac{2}{3}})^{\frac{1}{2}}.$$

10. Three infinite parallel wires cut a plane perpendicular to them in the angular points A_1, A_2, A_3 of an equilateral triangle, and have charges q, q, $-q'$ per unit length respectively. Prove that the extreme lines of force which pass from A_1 to A_3 make at starting angles $\frac{2q - 5q'}{6q}\pi$ and $\frac{2q + q'}{6q}\pi$ with A_1A_3, provided that $q' \not> 2q$.

11. Draw the equi-potential surfaces when the field is that due to two small conductors with charges q and $4q$ placed at a distance $3a$ apart, and find the capacity of a conductor in the form of the equi-potential surface, whose potential is $\lambda q/a$, where $\lambda < 3$.

12. If the algebraic sum of the charges on a system of conductors be positive, then on one at least the surface density is everywhere positive.

13. Two conductors carrying respectively a positive charge q_1 and a negative charge q_2 are introduced into the interior of a larger conductor maintained at potential ϕ; show that if $q_1 > q_2$ the potential of the first conductor is greater than ϕ.

14. There are a number of insulated conductors in given fixed positions. The capacities of any two of them in their given positions are C_1 and C_2, and their mutual coefficient of induction is B. Prove that if these conductors be joined by a thin wire, the capacity of the combined conductor is

$$C_1 + C_2 + 2B.$$

15. A system of insulated conductors having been charged in any manner, charges are transferred from one conductor to another till they are all brought to the same potential ϕ. Show that

$$\phi = Q/(s_1 + 2s_2),$$

where s_1, s_2 are the algebraic sums of the coefficients of capacity and induction respectively and Q is the sum of the charges.

16. Two Leyden jars of capacities C, C' are placed at such a distance apart that their mutual induction is negligible. The outer coating of the former is connected to a distant charged conductor of very large capacity which maintains its potential at a constant value ϕ, whilst that of the latter is earthed. A charge Q is given to the inner coating of the former and the knobs are then connected by a wire. Show that the total dissipation of energy due to the discharge is

$$\frac{C'(Q + C\phi)^2}{2C(C + C')}.$$

17. Prove that the effect of the operation described in the last question is a decrease of the electrostatic energy equal to what would be the energy of the system if each of the original potentials were diminished by ϕ.

18. Two equal similar condensers, each consisting of two spherical shells, radii a, b, are insulated and placed at a great distance r apart. Charges q_1 and q_2 are given to the inner shells. If the outer surfaces are now joined by a wire, show that the loss of energy is approximately

$$\tfrac{1}{4}(q_1 - q_2)^2\left(\frac{1}{b} - \frac{1}{r}\right).$$

19. Four equal uncharged conductors are placed symmetrically at the corners of a regular tetrahedron, and are touched in turn by a moving spherical conductor at the points nearest to the centre of the tetrahedron, receiving charges q_1, q_2, q_3, q_4. Show that the charges are in geometrical progression.

Replace the word 'tetrahedron' by 'square' and prove that

$$(q_1 - q_2)(q_1q_3 - q_2^2) = q_1(q_2q_3 - q_1q_4).$$

20. Show that if the distance x between two conductors is so great compared with the linear dimensions of either, that the square of the ratio of these linear dimensions to x may be neglected, then the coefficient of induction between them is $-\dfrac{CC'}{x}$, where C, C' are the capacities of the conductors when isolated.

21. Prove (i) that if a conductor, insulated in free space and raised to unit potential, produce at any external point P a potential denoted by (P), then a unit charge placed at P in the presence of this conductor uninsulated will induce on it a charge $-(P)$;

(ii) that if the potential at a point Q due to the induced charge be denoted by (P, Q), then (P, Q) is a symmetrical function of the positions of P and Q.

22. Two small uninsulated spheres are placed near together between two large parallel planes, one of which is charged, and the other connected to earth. Show by figures the nature of the disturbance so produced in the uniform field, when the line of centres is (i) perpendicular, (ii) parallel to the planes.

23. A portion P of a conductor, the capacity of which is C, can be separated from the conductor. The capacity of this portion, when at a long distance from other bodies, is c. The conductor is insulated, and the part P when at a considerable distance from the remainder is charged with a quantity q and allowed to move under the mutual attraction upon it; describe and explain the changes which take place in the electrical energy of the system.

24. A hollow conductor A is at zero potential, and contains in its cavity two other insulated conductors, B and C, which are mutually external: B has a positive charge, and C is uncharged. Analyse the different types of lines of force within the cavity which are possible, classifying with respect to the conductor from which the line starts, and the conductor at which it ends, and proving the impossibility of the geometrically possible types which are rejected.

Hence prove that B and C are at positive potentials, the potential of C being less than that of B.

25. A conductor having a charge Q_1 is surrounded by a second conductor with charge Q_2. The inner is connected by a wire to a very distant uncharged conductor. It is then disconnected, and the outer conductor connected. Show that the charges Q_1', Q_2' are now

$$Q_1' = \frac{mQ_1 - nQ_2}{m + n + mn}, \qquad Q_2' = \frac{(m + n)Q_2 + mnQ_1'}{m + n},$$

where C, $C(1 + m)$ are the coefficients of capacity of the near conductors, and Cn is the capacity of the distant one.

26. The inner sphere of a spherical condenser (radii a, b) has a constant charge Q, and the outer conductor is at zero potential. Under the internal forces the outer conductor contracts from radius b to radius b_1. Prove that the work done by the electric forces is

$$\tfrac{1}{2}Q^2 \frac{b - b_1}{bb_1}.$$

27. Two equal and similar conductors A and B are charged and placed symmetrically with regard to each other; a third moveable conductor C is carried so as to occupy successively two positions, one practically wholly within A, the other within B, the positions being similar and such that the coefficients of potential of C in either position are p, q, r in ascending order of magnitude. In each position C is in turn connected with the conductor surrounding it, put to earth, and then insulated. Determine the charges on the conductors after any number of cycles of such operations, and show that they ultimately lead to the ratios

$$1 : -\beta : \beta^2 - 1,$$

where β is the positive root of

$$rx^2 - qx + p - r = 0.$$

28. Three conductors A_1, A_2, A_3 are such that A_3 is practically inside A_2. A_1 is alternately connected with A_2 and A_3 by means of a fine wire, the first contact being with A_3. A_1 has a charge Q initially, A_2, A_3 being uncharged. Prove that the charge on A_1 after it has been connected n times with A_2 is

$$\frac{Q\beta}{\alpha + \beta}\left\{1 + \frac{\alpha(\gamma - \beta)}{\beta(\alpha + \gamma)}\left(\frac{\alpha + \beta}{\alpha + \gamma}\right)^{n-1}\right\},$$

where α, β, γ stand for $p_{11} - p_{12}$, $p_{22} - p_{12}$ and $p_{33} - p_{12}$ respectively.

29. Two equal and similar conductors are placed symmetrically with regard to each other, one of them being uncharged. Another insulated conductor is made to touch them alternately in a symmetrical manner, beginning with the one which has a charge. If q_1, q_2 be their charges when it has touched each once, show that their charges, when it has touched each r times, are respectively

$$\frac{q_1^2}{2q_1 - q_2}\left\{1 + \left(\frac{q_1 - q_2}{q_1}\right)^{2r-1}\right\},$$

and

$$\frac{q_1^2}{2q_1 - q_2}\left\{1 - \left(\frac{q_1 - q_2}{q_1}\right)^{2r}\right\}.$$

30. Two closed equi-potentials ϕ_1, ϕ_0 are such that ϕ_1 contains ϕ_0, and ϕ_P is the potential at any point P between them. If now a charge Q be put at P, and both equipotentials be replaced by conducting shells and earth connected, then the charges Q_1, Q_0 induced on the two surfaces are given by

$$\frac{Q_1}{\phi_0 - \phi_P} = \frac{Q_0}{\phi \ \ - \phi_1} = \frac{Q}{\phi_1 - \phi_0}.$$

31. A condenser is formed of two thin spherical shells of radii a, b. A small hole exists in the outer sheet through which an insulated wire passes connecting the inner sheet with a third conductor of capacity c, at a great distance r from the condenser. The outer sheet of the condenser is put to earth, and the charge on the two connected conductors is Q. Prove that approximately the force on the third conductor is

$$ac^2Q\bigg/\left(\frac{ab}{a - b} + c\right)^2 r^3.$$

32. Two insulated fixed condensers are at given potentials when alone in the electric field and charged with quantities Q_1, Q_2 of electricity. Their coefficients of potential are

p_{11}, p_{12}, p_{22}. But if they are surrounded by a spherical conductor of very large radius R at potential zero with its centre near them, the two conductors require charges Q_1', Q_2' to produce the given potentials. Prove, neglecting $\frac{1}{R^2}$, that

$$\frac{Q_1' - Q_1}{Q_2' - Q_2} = \frac{p_{22} - p_{12}}{p_{11} - p_{12}}.$$

33. Show that the locus of the positions, in which a unit charge will induce a given charge on a given uninsulated conductor is an equi-potential surface of that conductor supposed freely charged.

34. A condenser consists of a spherical shell of internal radius b, surrounding a concentric sphere of radius a. If the thickness t of the shell is such that a, b, $b + t$ are in harmonic progression, and if it is kept at a constant potential while the sphere is connected to earth, show that the presence of a small distant sphere of radius r, also connected to earth, will increase the charge of the shell by $\frac{rt}{R^2}$ of itself, R being the distance between the centres.

35. With the usual notation, prove that

$$p_{11} + p_{23} > p_{12} + p_{13},$$
$$p_{11}p_{23} > p_{12}p_{13}.$$

36. A, B, C are three condensers at large distances from each other, which can be connected by wires. Their charges are originally q_1, q_2, q_3. A and B are then connected, and after they are disconnected, B is connected with C. If after this the charges on A and C are q_1' and q_3', prove that the capacity of B : to the capacity of C

$$= q_1 + q_2 + q_3 - q_1' - q_3' : q_3'.$$

37. Show that if p_{rr}, p_{rs}, p_{ss} be three coefficients before the introduction of a new conductor, and p'_{rr}, p'_{rs}, p'_{ss} the same coefficients afterwards, then

$$(p_{rr}p_{ss} - p'_{rr}p'_{ss}) \not< (p_{rs} - p'_{rs})^2.$$

38. A, B, C, D are four conductors, of which B surrounds A and D surrounds C. Given the coefficients of capacity and induction

(i) of A and B when C and D are removed,

(ii) of C and D when A and B are removed,

(iii) of B and D when A and C are removed,

determine those for the complete system of four conductors.

39. A system of conductors consists of three, C_1, C_2, C_3; and C_1 completely surrounds C_2. If C_2 were annihilated the coefficients of potential of C_1 and C_3 would be P_{11}, P_{13}, P_{33}; and if C_3 were annihilated the coefficients of potential of C_1 and C_2 would be ϖ_{11}, ϖ_{12}, ϖ_{22}. Show that the actual coefficient of potential of C_2 on itself is

$$P_{11} - \varpi_{11} + \varpi_{22},$$

and write down the remaining coefficients of potential of the system.

40. If one conductor contains all the others, and there are $(n + 1)$ in all, shew that there are $(n + 1)$ relations between either the coefficients of potential or the coefficients of induction, and if the potential of the largest be ϕ_0, and that of the others $\phi_1, \phi_2, \dots \phi_n$, then the most general expression for the energy is $\frac{1}{2}C\phi_0^2$ increased by a quadratic function of $\phi_1 - \phi_0$, $\phi_2 - \phi_0$, $\dots \phi_n - \phi_0$; where C is a definite constant for all positions of the inner conductors.

41. Two conductors are of capacities C_1 and C_2 when each is alone in the field. They are both in the field at potentials ϕ_1 and ϕ_2 respectively, at a great distance r apart. Prove that the repulsion between the conductors is

$$\frac{C_1C_2(r\phi_1 - C_2\phi_2)(r\phi_2 - C_1\phi_1)}{(r^2 - C_1C_2)^2}.$$

As far as what power of $\frac{1}{r}$ is this result accurate?

42. A system of conductors consists of one large conductor and n conductors whose dimensions and distances from the large conductor and from each other are small compared with the dimensions of the large conductor. Prove that if ϕ_0 is the potential of the large conductor, $\phi_1, \phi_2, \dots \phi_n$ the potentials of the other conductors, the potential energy of the system is approximately

$$\tfrac{1}{2}C\phi_0^2 + \tfrac{1}{2}\Sigma' a_{rr}(\phi_r - \phi_0)^2 + \Sigma' a_{rs}(\phi_r - \phi_0)(\phi_s - \phi_0).$$

Discuss the significance of this result relatively to the theory of electrostatic measurements.

43. An electrometer consists of two fixed similar conductors A and B, together with a conductor C which can turn round an axis placed symmetrically with respect to A and B. The position of C is specified by the angle θ between its axis of symmetry and a line which is symmetrically situated with respect to A and B; in any position, C is acted on by a restoring couple $\kappa\theta$ towards the symmetrical position ($\theta = 0$). From the following data, valid for positive values of θ, find the position of equilibrium of C when A, B and C are insulated and at potentials ϕ_1, ϕ_2 and ϕ_3 respectively:

A and B earthed, C at unit potential, charge on

$$C = a - a\theta^2,$$

A and B earthed, C at unit potential, charge on

$$A = -b - \beta\theta,$$

B and C earthed, A at unit potential, charge on

$$A = C + \beta\theta \text{ and on } B = -d - \gamma\theta^2.$$

Show further that for given values of ϕ_1 and ϕ_2, if these are small compared with ϕ_3, the maximum displacement of C occurs when

$$\phi_3 = \sqrt{\kappa/a}.$$

44. Two small pith balls, each of mass m, are connected by a light insulating rod. The rod is supported by parallel threads, and hangs in a horizontal position in front of an infinite vertical plane at potential zero. If the balls when charged with q units of electricity are at a distance a from the plate equal to half the length of the rod, show that the inclination θ of the strings to the vertical is given by

$$\tan\theta = \frac{q^2}{4mga^2}\left\{1 + \frac{1}{2\sqrt{2}}\right\}.$$

45. An uninsulated conducting sphere is under the influence of an external electric charge; find the ratio in which the induced charge is divided between the parts of its surface in direct view of the external charge and the remaining part.

46. A conducting surface consists of two infinite planes which meet at right angles, and a quarter of a sphere of radius a fitted into the right angle. If the conductor is at zero potential, and a point charge q is symmetrically placed with regard to the planes and the spherical surface at a great distance f from the centre, show that the charge induced on the spherical portion is approximately $-5qa^3/\pi f^3$.

47. If two infinite plane uninsulated conductors meet at an angle of 60°, and there is a charge q at a point equi-distant from each, and distant r from the line of intersection, find the electrification at any point of the planes. Show that at a point in a principal plane through the charged point at a distance $r\sqrt{3}$ from the line of intersection, the surface density is

$$-\frac{q}{4\pi r^2}\left(\frac{3}{4}+\frac{1}{7\sqrt{7}}\right).$$

49. An infinite conducting plane at zero potential is under the influence of a charge of electricity at a point O. Show that the charge on any area of the plane is proportional to the angle it subtends at O.

49. A charged particle is placed in the space between two uninsulated planes which intersect at right angles. Sketch the sections of the equi-potentials made by an imaginary plane through the charged particle, at right angles to the planes.

Supposing the particle to have a charge q and be equi-distant from the planes, show that the total charge on a strip, of which one edge is the line of intersection of the planes, and of which the width is equal to the distance of the particle from this line of intersection, is $-\frac{1}{4}q$.

50. Two equal parallel plates of area A are connected by a wire, and an equal thin plate with a charge Q is placed between them at distances a, b from them. Prove that it is repelled from the nearer one with a force

$$\frac{2\pi Q^2}{A}\left(\frac{a-b}{a+b}\right).$$

51. Within a spherical hollow in a conductor connected to earth, equal point charges q are placed at equal distances f from the centre, on the same diameter. Shew that each is acted on by a force equal to

$$q^2\left[\frac{4a^3f^3}{(a^4-f^4)^2}+\frac{1}{4f^2}\right].$$

52. An uncharged insulated conductor formed of two equal spheres of radius a cutting one another at right angles, is placed in a uniform field of force of intensity E, with the line joining the centres parallel to the lines of force. Prove that the charges induced on the two spheres are $\frac{11}{8}Ea^2$ and $-\frac{11}{8}Ea^2$.

53. A point charge q is brought near to a spherical conductor of radius a having a charge Q. Show that the particle will be repelled by the sphere, unless its distance from the nearest point of its surface is less than $\frac{1}{2}a\sqrt{\frac{q}{Q}}$, approximately.

54. A hollow conductor has the form of a quarter of a sphere bounded by two perpendicular diametral planes. Find the image of a charge placed at any point inside.

55. Show that the image of a doublet of moment m in a sphere is a doublet of strength $\frac{ma^3}{c^3}$ at the inverse point together with a distinct charge $\frac{m_1 a}{c^2}$ at the same point, where m_1 denotes the component of m_1 along the direction of the diameter through the point.

Interpret these results when the initial doublet is at a great distance from the sphere.

56. An infinite plate with a hemispherical boss of radius a is at zero potential under the influence of a point charge q on the axis of the boss distant f from the plate. Find the surface density at any point of the plate, and show that the charge is attracted towards the plate with a force

$$\frac{q^2}{4f^2}+\frac{4q^2a^3f^3}{(f^4-a^4)^2}.$$

57. A conductor is formed by the outer surfaces of two equal spheres, the angle between their radii at a point of intersection being $2\pi/3$. Show that the capacity of the conductor so formed is

$$\frac{5\sqrt{3}-4}{2\sqrt{3}}\,a,$$

where a is the radius of either sphere.

58. A conducting plane has a hemispherical boss of radius a, and at a distance f from the centre of the boss and along its axis there is a point charge q. If the plane and the boss be kept at zero potential, prove that the charge induced on the boss is

$$-q\left\{1-\frac{f^2-a^2}{f\sqrt{f^2+a^2}}\right\}.$$

59. A conductor is bounded by the larger portions of two equal spheres of radius a cutting at an angle $\frac{1}{3}\pi$, and of a third sphere of radius c cutting the two former orthogonally. Show that the capacity of the conductor is

$$c+a\left(\tfrac{5}{2}-\tfrac{2}{3}\sqrt{3}\right)-ac\left\{2(a^2+c^2)^{-\frac{1}{2}}-2(a^2+3c^2)^{-\frac{1}{2}}+(a^2+4c^2)^{-\frac{1}{2}}\right\}.$$

60. A conductor formed by the outer portions of two equal spheres cutting at right angles is placed in a uniform field of force of strength E parallel to the line of centres. Show that the external field of the charge induced on the spheres is the same as that of the field due to doublets of strength Ea^3 at the centres and $\dfrac{Ea^6}{c^3}$ midway between them together with charges $\dfrac{Ea^3}{c}$ at one centre and $-\dfrac{Ea^3}{c}$ at the other.

Work out the corresponding results when the spheres are unequal.

61. A conducting spherical shell of radius a is placed, insulated and without charge, in a uniform field of force of intensity E. Show that if the sphere be cut into two hemispheres by a plane perpendicular to the field, these hemispheres tend to separate and require forces equal to $\frac{9}{16}a^2E^2$ to keep them together.

62. A conducting sphere of radius a is electrified to potential ϕ. If the sphere consists of two separate hemispheres, shew that the force between them is $\frac{1}{8}\phi^2$; and if the whole be surrounded by an uninsulated concentric spherical conductor of internal radius b and the potential of the solid sphere is still ϕ, prove that the force between the hemispheres is

$$\frac{1}{8}\frac{b^2\phi^2}{(b-a)^2}.$$

63. If a non-conducting portion of an electrostatic field suddenly becomes conducting, discuss the effect upon the electrostatic energy. If there was previously only one conductor how is its capacity affected by the change?

64. Two spherical insulated conducting shells of radius a are placed with their centres at a distance h apart, and carry equal charges q; within one of them there is situated a concentric spherical conductor of radius b carrying a charge q'. Show that, if a/h is small, the joining of the equal shells by a wire of negligible capacity will result in a loss of electrostatic energy whose measure is approximately

$$\frac{1}{4}\left(\frac{1}{a}-\frac{1}{h}\right)q'^2.$$

65. If a particle charged with a quantity q of electricity be placed at the middle point of the line joining the centres of two equal spherical conductors kept at zero potential, show that the charge induced on each sphere is

$$-2qm(1-m+m^2-3m^3+4m^4),$$

neglecting higher powers of m, which is the ratio of the radius to the distance between the centres of the spheres.

66. Two insulating conducting spheres of radii a, b, the distance c of whose centres is large compared with a and b, have charges Q_1, Q_2 respectively. Show that the potential energy is approximately

$$\frac{1}{2}\left\{\left(\frac{1}{a}-\frac{b^3}{c^4}\right)Q_1^2+\frac{2}{c}Q_1Q_2+\left(\frac{1}{b}-\frac{a^3}{c^4}\right)Q_2^2\right\}.$$

67. Show that the force between two insulated spherical conductors of radius a placed in an electric field of uniform intensity E perpendicular to their line of centres is

$$3E^2\frac{a^6}{c^4}\left(1-\frac{2a^3}{c^3}-\frac{8a^5}{c^5}+\dots\right),$$

c being the distance between their centres.

68. Two uncharged insulated spheres, radii a, b, are placed in a uniform field of force so that their line of centres is parallel to the lines of force, the distance c between their centres being great compared with a and b. Prove that the surface density at the point at which the line of centres cuts the first sphere (a) is approximately

$$\frac{E}{4\pi}\left\{3+\frac{6b^3}{c^3}+\frac{15ab^3}{c^4}+\frac{28a^2b^3}{c^5}+\frac{57a^3b^3}{c^6}+\dots\right\}.$$

69. Assuming that the force on an element df of a conductor is $2\pi\sigma^2 df$, show that if a conductor with a charge Q is slightly deformed so that the outward normal displacement of the element df is Δn, the alteration in the capacity C is given by

$$\frac{1}{2}\frac{Q^2}{C^2}\delta C=\int 2\pi\sigma^2\Delta n\,.\,df.$$

A conductor has the form of a circular cylinder with hemispherical ends; show that if the length l is small the capacity is $a+\frac{1}{4}l$, where a is the radius.

70. A prolate spheroid, semi-axes a, b, has a charge Q of electricity. Show that the repulsion between the two halves into which it is divided by its diametral plane is

$$\frac{Q^2}{4(a^2-b^2)}\log\frac{a}{b}.$$

71. One face of a condenser is a circular plate of radius a: the other is a segment of a sphere of radius R, R being so large that the plate is almost flat. Show that the capacity is $\frac{1}{2}R\log(t_1/t_0)$, where t_1, t_0 are the thickness of the dielectric at the middle and edge of the condenser. Determine also the distribution of the charge.

72. Prove that if v is a solution of Laplace's equation the capacity of the condenser formed by the conducting surfaces $v=\alpha$, $v=\beta$ separated by a heterogeneous dielectric whose specific inductive capacity is $f(v)$ is

$$\frac{1}{4\pi}\int\frac{\partial v}{\partial n}df\bigg/\int_\alpha^\beta\frac{dv}{f(v)},$$

where the integral in the numerator is an area integral over the surface $v=\alpha$ and ∂n is an element of the normal measured into the dielectric.

Prove that the function v which is determined in terms of Cartesian coordinates by the relation

$$\tanh v = 2ax/(x^2 + y^2 + a^2)$$

satisfies Laplace's equation, and ascertain what surfaces are represented by the equation $v = \text{const.}$

Complete the integrations in this case when $f(v)$ is e^v.

73. A condenser is formed of the two prolate spheroids

$$\frac{z^2}{a^2} + \frac{r^2}{b^2} = 1,$$

and

$$\frac{z^2}{a^2} + \frac{r^2}{b^2} = 1 - \mu,$$

where μ is very small. Show that its capacity is

$$a\left\{\frac{2}{\mu} - \frac{2e^2}{3\mu} - \frac{1-e^2}{4e}\log\frac{1+e}{1-e} + \frac{e^2}{6} - 1\right\}$$

where

$$e^2 = 1 - \frac{b^2}{a^2}.$$

74. The surface of a conductor being one of revolution whose equation is

$$\frac{4}{r} + \frac{1}{r'} = \frac{7}{12},$$

where r, r' are the distances of any two fixed points at distance 8 apart, find the electric density at either vertex when the conductor has a given charge.

75. The curve

$$\frac{1}{(x^2+y^2)^{\frac{1}{2}}} - \frac{9a}{16}\left\{\frac{a+x}{(x+a^2+y^2)^{\frac{3}{2}}} + \frac{a-x}{(a-x^2+y^2)^{\frac{3}{2}}}\right\} = \frac{1}{a},$$

when rotated round the axis of a, generates a single closed surface, which is made the bounding surface of a conductor. Show that its capacity will be a, and that the surface density at the end of the axis will be $q/3\pi a^2$, where q is the total charge.

76. An uninsulated conductor consisting of two equal spheres in contact is under the influence of a charge q at a point in the tangent plane at the point of contact. Prove that the charge is attracted with a force $\frac{Cq^2}{f^2}$, where C is the capacity of the conductor and f is the distance of the charge from the point of contact of the spheres.

77. A conducting sphere of radius a is in contact with an infinite conducting plane. Show that if a unit charge be placed beyond the sphere and on the diameter through the point of contact at distance c from that point, the charges induced on the plane and sphere are

$$-\frac{\pi a}{c}\cot\frac{\pi a}{c} \text{ and } \frac{\pi a}{c}\cot\frac{\pi a}{c} - 1.$$

78. Show that the capacity of a spherical conductor of radius a with its centre at a distance c from an infinite conducting plane is

$$a \sinh \alpha \sum_{1}^{\infty} \operatorname{cosech} n\alpha,$$

where $c = a \cosh \alpha$.

79. An insulated conducting sphere of radius a is placed midway between two parallel infinite uninsulated planes at a great distance $2c$ apart. Neglecting $\left(\frac{a}{c}\right)^2$, show that the capacity of the sphere is approximately

$$a\left\{1 + \frac{a}{c}\log 2\right\}.$$

80. Two spheres of radii a, b touch each other, and their capacities in this position are c, d. Show that

$$c = b\left\{f^2 \sum_1^\infty \frac{1}{n^2} + f^3 \sum_1^\infty \frac{1}{n^3} + \ldots\right\},$$

where

$$f = \frac{a}{a+b}.$$

81. If the centres of the two shells of a spherical condenser be separated by a small distance d, prove that the capacity is approximately

$$\frac{ab}{b-a}\left\{1 - \frac{abd^2}{(b-a)(b^3-a^3)}\right\}.$$

82. A condenser is formed of two spherical conducting sheets, one of radius b surrounding the other of radius a. The distance between the centres is c, this being so small that $(c/a)^2$ may be neglected. The surface densities on the inner conductor at the extremities of the axis of symmetry of the instrument are σ_1, σ_2 and the mean surface density over the inner conductor is $\bar{\sigma}$. Prove that

$$\frac{\sigma_2 \sim \sigma_1}{\bar{\sigma}} = \frac{6ca^2}{b^3 - a^3}.$$

83. The equation of the surface of a conductor is $r = a(1 + \alpha P_n)$, where α is very small, and the conductor is placed in a uniform field of force E parallel to the axis of harmonics. Show that the surface density of the induced charge at any point is greater than it would be if the surface were perfectly spherical, by the amount

$$\frac{3n\alpha E}{4\pi(2n+1)}\{(n+1)P_{n+1} + (n-2)P_{n-1}\}.$$

84. The conductor of the last question is uninsulated. Show that the charge induced on it by a unit charge at a distance f from the origin and of angular coordinate θ, is approximately

$$-\frac{a}{f}\left\{1 + \left(\frac{a}{f}\right)^n \alpha P_n(\theta)\right\}.$$

85. An uninsulated conducting sphere is placed in a field whose potential before the introduction of the sphere was

$$\phi = \sum_0^\infty A_n r^n P_n.$$

Determine the charge distribution induced on the sphere and prove that there is a force urging it in the positive direction of the polar axis of amount

$$\sum^\infty (n+1) A_n A_{n+1} a^{2n+1}.$$

86. A circular disc of radius a is under the influence of a charge q at a point in its plane at distance b from the centre of the disc. Show that the density of the induced distribution at a point on the disc is

$$\frac{q}{2\pi^2 R^2}\sqrt{\frac{b^2 - a^2}{a^2 - r^2}},$$

where r, R are the distances of the point from the centre of the disc and the charge.

87. An ellipsoidal conductor differs but little from a sphere. Its volume is equal to that of a sphere of radius a, its axes are $2a(1+\alpha_1)$, $2a(1+\alpha_2)$, $2a(1+\alpha_3)$. Show that, neglecting cubes of $\alpha_1, \alpha_2, \alpha_3$, its capacity is

$$a\{1+\tfrac{2}{15}(\alpha_1^2+\alpha_2^2+\alpha_3^2)\}.$$

88. Electricity is induced on an uninsulated spherical conductor of radius a, by a uniform distribution, density σ, over an external concentric non-conducting spherical segment of radius c. Prove that the surface density at the point A of the conductor at the nearer end of the axis of the segment is

$$-\tfrac{1}{2}\sigma\frac{c(c+a)}{a^2}\left(1-\frac{AB}{AD}\right),$$

where B is the point of the segment on its axis, and D is any point on its edge.

89. Two conducting discs of radii a, a' are fixed at right angles to the line which joins their centres, the length of this line being r, large compared with a. If the first has potential ϕ and the second is uninsulated, prove that the charge on the first is

$$\frac{2a\pi r^2\phi}{\pi^2r^2-4aa'}.$$

90. Two equal spheres of radius a are in contact. Show that the capacity of the conductor so formed is $2a\log_e 2$.

91. Two spheres of radii a, b are in contact, a being large compared with b. Show that if the conductor so formed is raised to potential ϕ, the charges on the two spheres are

$$\phi a\left(1-\frac{\pi^2b^2}{\sigma(a+b)^2}\right) \text{ and } \phi a\frac{\pi^2b^2}{\sigma(a+b)^2}.$$

92. Prove that if the centres of two equal uninsulated spherical conductors of radius a be at a distance $2c$ apart, the charge induced on each by a unit charge at a point midway between them is

$$\sum_1^\infty(-1)^n \operatorname{sech} n\alpha,$$

where $$c = a\cosh\alpha.$$

93. A spherical shell of radius a with a little hole in it is freely electrified to potential ϕ. Prove that the charge on its inner surface is less than $\dfrac{\phi s}{8\pi a}$, where s is the area of the whole.

94. A thin spherical conducting shell from which any portions have been removed is freely electrified. Prove that the difference of densities inside and outside at any point is constant.

95. Prove that the capacity of a hemispherical shell of radius a is

$$a\left(\frac{1}{2}+\frac{1}{\pi}\right).$$

Prove that the capacity of an elliptic plate of small eccentricity e and area A is approximately

$$\sqrt{\frac{A}{\pi}}\,\frac{2}{\pi}\left(1+\frac{e^4}{64}+\frac{e^6}{64}\right).$$

96. A thin circular disc of radius a is electrified with charge Q and surrounded by a spheroidal conductor with a charge Q_1 placed so that the edge of the disc is the locus of the focus S of the generating ellipse. Show that the energy of the system is

$$\frac{1}{2}\frac{Q^2}{a}\widehat{BSC} + \frac{1}{2}\frac{(Q+Q_1)^2}{a}\widehat{SBC},$$

B being an extremity of the polar axis of the spheroid, and C the centre.

97. If the two surfaces of a condenser are concentric and coaxial oblate spheroids of small ellipticities ϵ and ϵ' and polar axes $2c$ and $2c'$, prove that the capacity is

$$cc'(c'-c)^{-2}\{c'-c+\tfrac{2}{5}(\epsilon c'-\epsilon' c)\},$$

neglecting squares of the ellipticities; and find the distribution of electricity on each surface to the same order of approximation.

98. A thin spherical bowl is formed by the portion of the sphere $x^2+y^2+z^2=cz$ bounded by and lying within the cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$, and is put in connection with the earth by a fine wire. O is the origin, and C, diametrically opposite to O, is the vertex of the bowl; Q is any point on the rim, and P is any point on the great circle arc CQ. Show that the surface density induced at P by a charge Q placed at O is

$$-\frac{Qc}{4\pi abI}\cdot\frac{CQ}{OP^2(OP^2-OQ^2)^{\frac{1}{2}}},$$

where
$$I=\int_0^{\frac{\pi}{2}}\frac{d\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}}}.$$

99. A flat piece of corrugated metal ($y=a\sin mx$) is charged with electricity. Find the surface density at any point, and show that it exceeds the average density approximately in the ratio $my:1$.

100. A long hollow cylindrical conductor is divided into two parts by a plane through the axis and the parts are separated by a small interval. If the two parts are kept at potentials ϕ_1 and ϕ_2, the potential at any point within the cylinder is

$$\frac{\phi_1+\phi_2}{2}+\frac{\phi_1-\phi_2}{\pi}\tan^{-1}\frac{2ar\cos\theta}{a^2-r^2},$$

where r is the distance from the axis, and θ is the angle between the plane joining the point to the axis and the plane through the axis normal to the plane of separation.

101. Show that the capacity per unit length of a telegraph wire of radius a at height h above the surface of the earth is

$$\left[4\tanh^{-1}\sqrt{\frac{h-a}{h+a}}\right]^{-1}.$$

102. A cylindrical conductor of infinite length, whose cross section is the outer boundary of three orthogonal circles of radius a, has a charge q per unit length. Prove that the electric density at distance r from the axis is

$$\frac{q}{6\pi a}\frac{(3r^2+a^2)(3r^2-a^2-\sqrt{6}ar)(3r^2-a^2+\sqrt{6}ar)}{r^2(9r^4-3a^2r^2+a^4)}.$$

103. If the cylinder $r=a+b\cos\theta$ be freely charged, show that in free space the resultant force varies as

$$r^{-1}(r^2+2rc\cos\theta+c^2)^{-\frac{1}{4}},$$

and makes with the line $\theta = 0$ an angle

$$\theta + \tfrac{1}{2}\tan^{-1}\left(\frac{r\sin\theta}{c + r\cos\theta}\right),$$

where $$a^2 - b^2 = 2bc.$$

104. If $\phi + i\psi = f(x + iy)$, and the curves for which $\phi =$ const. be closed, show that the capacity C of a condenser with boundary surfaces $\phi = \phi_1$, $\phi = \phi_0$ is

$$\frac{[\psi]}{4\pi(\phi_1 - \phi_0)}$$

per unit length, where $[\psi]$ is the increment of ψ on passing once round a ϕ-curve.

105. Using the transformation

$$x + iy = c\cot\tfrac{1}{2}(\phi + i\psi),$$

show that the capacity per unit length of a condenser formed by two right circular cylinders (radii a, b) one inside the other, with parallel axes at a distance d apart, is

$$2\cosh^{-1}\frac{a^2 + b^2 - d^2}{2ab}.$$

106. A plane infinite grating is made up of equal and equi-distant parallel thin metal plates, the distance between their successive central lines being π, and the breadth of each plate $2\sin^{-1}\left(\frac{1}{K}\right)$. Show that when the grating is electrified to constant potential, the potential and charge functions ϕ, ψ on the surrounding space are given by the equation

$$\sin(\phi + i\psi) = K\sin(x + iy).$$

Deduce that when the grating is to earth and is placed in a uniform field of force of unit intensity at right angles to its plane, the charge and potential functions of the portion of the field which penetrates through the grating are expressed by

$$\phi + i\psi - (x + iy),$$

and expand the potential in the latter problem in a Fourier series.

107. Two insulated uncharged circular cylinders outside each other, given by $\eta = a$ and $\eta = -\beta$, where $x + iy = c\tan\frac{1}{2}(\xi + i\eta)$, are placed in a uniform field of force of potential $\phi_0 - Ex$. Show that the potential due to the distribution on the cylinders is

$$2Ec\sum_1^\infty(-1)^n\frac{e^{n(\eta-a)}\sinh n\beta + e^{-n(\eta+\beta)}\sinh na}{\sinh n(a + \beta)}\sin n\xi,$$

the summation being taken for all odd positive integral values of n.

108. Two circular cylinders outside each other, given by $\eta = a$ and $\eta = -\beta$, where

$$x + iy = c\tan\tfrac{1}{2}(\xi + i\eta),$$

are put to earth under the influence of a line-charge Q on the line $x = 0$, $y = 0$. Show that the potential of the induced charge outside the cylinders is

$$-4Q\sum\frac{1}{n}\frac{e^{-na}\sinh n(\eta + \beta) + e^{-n\beta}\sinh n(a - \eta)}{\sinh n(a + \beta)}\cos n\xi + \text{const.},$$

the summation being taken for all odd positive integral values of n.

109. What problems are solved by the transformation

$$\frac{d}{dt}(x + iy) = \frac{c(t^2 - 1)^{\frac{1}{2}}}{a^2 - t^2}, \quad \pi(\psi + i\phi) = \log\frac{a + t}{a - t},$$

where $a > 1$?

110. What problem in electrostatics is solved by the transformation

$$x + iy = cn\,(\phi + i\psi),$$

where ψ is taken as the potential function, ϕ being the function conjugate to it?

111. Verify that, if r, s be real positive constants, $z = x + iy$, $a = \rho e^{i\beta}$, $\frac{1}{c} = \frac{1}{r} + \frac{1}{s}$, the field of force outside the conductors $x^2 + y^2 + 2sx = 0$, $x^2 + y^2 - 2rx = 0$ due to a doublet at the point $z = a$, outside both the circles, of strength μ and inclination α to the axis, is given by putting

$$\phi + i\psi = \frac{c\mu\pi}{\rho^2}\left\{e^{i(\alpha-2\beta)}\cot c\pi\left(\frac{1}{z} - \frac{1}{a}\right) - e^{-i(\alpha-2\beta)}\cot c\pi\left(\frac{1}{z} - \frac{1}{a_0}\right)\right\},$$

where $z = a_0$ is the inverse point to $z = a$ with regard to either of the circles.

112. A semi-infinite conducting plane is at zero potential under the influence of an electric charge q at a point Q outside it. Show that the potential at any point P is given by

$$\frac{q}{\pi\sqrt{2rr_1}}\left[\{\cosh\eta - \cos(\theta - \theta_1)\}^{-\frac{1}{2}}\tan^{-1}\sqrt{\frac{\cosh\frac{1}{2}\eta + \cos\frac{1}{2}(\theta - \theta_1)}{\cosh\frac{1}{2}\eta - \cos\frac{1}{2}(\theta - \theta_1)}}\right.$$
$$\left. - \{\cosh\eta - \cos(\theta + \theta_1)\}^{-\frac{1}{2}}\tan^{-1}\sqrt{\frac{\cosh\frac{1}{2}\eta + \cos\frac{1}{2}(\theta + \theta_1)}{\cosh\frac{1}{2}\eta - \cos\frac{1}{2}(\theta + \theta_1)}}\right],$$

where r, θ, z are the cylindrical coordinates of the point P, $(r_1, \theta_1, 0)$ of the point Q, $\theta = 0$ is the equation of conducting plane, and

$$2rr_1\cosh\eta = r^2 + r_1^2 + z^2.$$

Hence obtain the potential at any point due to a spherical bowl at constant potential and shew that the capacity of the bowl is

$$\frac{a}{\pi}\left(1 + \frac{\pi - \alpha}{\sin\alpha}\right),$$

where a is the radius of the aperture, and α is the angle subtended by this radius at the centre of the sphere of which the bowl is a part.

113. Show that the potential at any point P of a circular bowl, electrified to potential ϕ, is

$$\frac{\phi}{\pi}\left\{\sin^{-1}\frac{AB}{AP + PB} + \frac{OA}{OB}\sin^{-1}\left(\frac{OP}{OA}\cdot\frac{AB}{AP + PB}\right)\right\},$$

where O is the centre of the bowl, and A, B are the points in which a plane through P and the axis of the bowl cuts the circular rim.

Find the density of electricity at a point on either side of the bowl and shew that the capacity is

$$\frac{a}{\pi}(\alpha + \sin\alpha),$$

where a is the radius of the sphere, and 2α is the angle subtended at the centre.

114. Two spheres are charged to potentials ϕ_1 and ϕ_2. The ratio of the distances of any point from the two limiting points of the spheres being denoted by e^η and the angle between them by ξ, prove that the potential at the point ξ, η is

$$\phi_1\sqrt{\{2(\cosh\eta - \cos\xi)\}}\sum_0^\infty\frac{\sinh(n + \frac{1}{2})(\beta + \eta)}{\sinh(n + \frac{1}{2})(\beta + \alpha)}P_n(\cos\xi)\,e^{-(n+\frac{1}{2})\alpha}$$
$$+ \phi_2\sqrt{\{2(\cosh\eta - \cos\xi)\}}\sum_0^\infty\frac{\sinh(n + \frac{1}{2})(\alpha - \eta)}{\sinh(n + \frac{1}{2})(\beta + \alpha)}P_n(\cos\xi)\,e^{-(n+\frac{1}{2})\beta},$$

where $\eta = \alpha$, $\eta = -\beta$ are the equations of the spheres. Hence find the charge on either sphere.

115. Two large parallel conducting plates are maintained at potentials ϕ_1 and ϕ_2 and the space between them is filled up by slabs of dielectric whose inductive capacities are ϵ_1 and ϵ_2, whose thicknesses are d_1 and d_2 and whose common face is parallel to the plates. Find the potential at any point between the plates and shew that it is everywhere the same as if the dielectrics were replaced by an insulated conducting sheet along their common face, and the charge on this plate per unit of area were

$$\frac{(\epsilon_1 - \epsilon_2)(\phi_1 - \phi_2)}{4\pi (d_1\epsilon_2 + d_2\epsilon_1)}.$$

116. Three thin conducting sheets are in the form of concentric spheres of radii $a + d$, a, $a - c$ respectively. The dielectric between the outer and middle sheet is of specific inductive capacity ϵ, and that between the middle and inner sheet is air. At first the outer sphere is uninsulated, the inner sheet is uncharged and insulated and the middle coating is charged to potential ϕ and insulated. The inner sheet is now uninsulated without connection with the middle sheet. Prove that the potential of the middle sheet falls to

$$\frac{\epsilon\phi c (a + d)}{\{\epsilon c (a + d) + d (a - c)\}}.$$

117. A conductor has a charge Q and ϕ_1, ϕ_2 are the potentials of two equi-potential surfaces completely surrounding it ($\phi_1 > \phi_2$). The space between these two surfaces is now filled with a dielectric of inductive capacity ϵ. Show that the change in the energy of the system is

$$\tfrac{1}{2}Q (\phi_1 - \phi_2)\frac{\epsilon - 1}{\epsilon}.$$

118. The surfaces of an air-condenser are concentric spheres. If half the space between the spheres be filled with solid dielectric of specific inductive capacity ϵ, the dividing surface between the solid and the air being a plane through the centre of the spheres, show that the capacity will be the same as though the whole dielectric were of uniform specific inductive capacity $\frac{1}{2}(1 + \epsilon)$.

119. The radii of the inner and outer shells of two equal spherical condensers, remote from each other and immersed in an infinite dielectric of inductive capacity ϵ, are respectively a and b, and the inductive capacities of the dielectric inside the condensers are ϵ_1, ϵ_2. Both surfaces of the first condenser are insulated and charged, the second being uncharged. The inner surface of the second condenser is now connected to earth, and the outer surface is connected to the outer surface of the first condenser by a wire of negligible capacity. Show that the loss of energy is

$$\frac{Q^2 \{2 (b - a) \epsilon + a\epsilon_2\}}{2\epsilon b \{(b - a) \epsilon + a\epsilon_2\}},$$

where Q is the quantity of electricity which flows along the wire.

120. The outer coating of a long cylindrical condenser is a thin shell of radius a, and the dielectric between the cylinders has inductive capacity ϵ on one side of a plane, through the axis, and ϵ' on the other side. Show that when the inner cylinder is connected to earth, and the outer has a charge q per unit length, the resultant force on the outer cylinder is

$$\frac{4q^2 (\epsilon - \epsilon')}{\pi a (\epsilon + \epsilon')}$$

per unit length.

121. A heterogeneous dielectric is formed of n concentric spherical layers of specific inductive capacities $\epsilon_1, \epsilon_2, \epsilon_3, \ldots \epsilon_n$, starting from the innermost dielectric, which forms a solid sphere; also the outermost dielectric extends to infinity. The radii of the spherical boundary surfaces are $a_1, a_2, \ldots a_{n-1}$ respectively. Prove that the potential due to a quantity Q of electricity at the centre of the spheres at a point distant r from the centre in the dielectric ϵ_s is

$$\frac{Q}{\epsilon_s}\left(\frac{1}{r} - \frac{1}{a_s}\right) + \frac{Q}{\epsilon_{s+1}}\left(\frac{1}{a_s} - \frac{1}{a_{s+1}}\right) + \ldots + \frac{Q}{\epsilon_n} \cdot \frac{1}{a_n}.$$

122. A condenser is formed by two rectangular parallel conducting plates of breadth b and area A at distance d from each other. Also a parallel slab of a dielectric of thickness t and of the same area is between the plates. This slab is pulled along its length from between the plates, so that only a length x is between the plates. Prove that the electric force sucking the slab back again to its original position is

$$\frac{2\pi Q^2 dbt'(d - t')}{\{A(d - t') + xbt'\}^2},$$

where $t' = \frac{\epsilon - 1}{\epsilon} t$, ϵ is the specific inductive capacity of the slab, Q is the charge, and the disturbances produced by the edges are neglected.

123. Three closed surfaces 1, 2, 3 are equi-potentials in an electric field. If the space between 1 and 2 is filled with a dielectric ϵ, and that between 2 and 3 is filled with a dielectric ϵ', show that the capacity of a condenser having 1 and 3 for faces is C, given by

$$\frac{1}{C} = \frac{1}{A\epsilon} + \frac{1}{B\epsilon'},$$

where A, B are the capacities of air-condensers having as faces the surfaces 1, 2 and 2, 3 respectively.

124. The surface separating two dielectrics (ϵ_1, ϵ_2) has an actual charge σ per unit area. The electric forces on the two sides of the boundary are F_1, F_2 at angles c_1, c_2 with the common normal. Show how to determine F_2, and prove that

$$\epsilon_2 \cot c_2 = \epsilon_1 \cot c_1 \left(1 - \frac{4\pi\sigma}{F_1 \cos c_1}\right).$$

125. The space between two concentric spheres radii a, b which are kept at potentials A, B is filled with a heterogeneous dielectric of which the inductive capacity varies as the nth power of the distance from their common centre. Show that the potential at any point between the surfaces is

$$\frac{Aa^{n+1} - Bb^{n+1}}{a^{n+1} - b^{n+1}} - \frac{a^{n+1}b^{n+1}}{r^{n+1}} \cdot \frac{A - B}{a^{n+1} - b^{n+1}}.$$

126. An infinitely long elliptic cylinder of inductive capacity ϵ, given by $\xi = a$, where $x + iy = c\cosh(\xi + i\eta)$, is in a uniform field E parallel to the major axis of any section. Show that the potential at any point inside the cylinder is

$$-Ex\frac{1 + \coth a}{\epsilon + \coth a}.$$

127. An infinite slab of a dielectric of constant ϵ and thickness t has air on either side of it. A point charge q is situated at a point A in the air on one side of the slab. Prove that the potential at any point on the other side is the same as if the slab were removed,

the charge at A altered to $\dfrac{4q\epsilon}{(\epsilon+1)^2}$ and point charges placed at points situated at distances $2t$, $4t$, $6t$, ... from A, the charge at the nth point being

$$4q\epsilon \cdot \frac{(\epsilon-1)^{2n}}{(\epsilon+1)^{2n+2}}.$$

128. A dielectric hemisphere of radius a and inductive capacity ϵ is laid flat against an unlimited plane conductor charged to surface density σ; prove that the disturbance which its presence produces in the field of force is derived from a potential

$$4\pi\sigma \frac{\epsilon-1}{\epsilon+2} \frac{a^3 x}{r^3},$$

where x is the distance from the plane and r the distance from the centre of the hemisphere.

129. A condenser is formed of two very long circular and coaxal cylindrical conductors. If a portion of the intermediate space, bounded by planes through the axes inclined to one another at an angle a, be filled with a dielectric of specific inductive capacity ϵ, show that the capacity per unit length is increased in the ratio

$$1 + \frac{a}{2\pi}(\epsilon - 1) : 1.$$

130. The plates of a condenser are vertical and at a difference of potential equal to ϕ; show that if a slab of dielectric (constant ϵ) of thickness t is suspended partly inside the condenser and partly outside, it will be sucked in by a force

$$\frac{b\phi^2}{8\pi}\left(\frac{1}{a_1} - \frac{1}{a}\right)$$

where b is the horizontal breadth of the slab, a the distance between the plates and

$$a_1 = a - t + t/\epsilon.$$

131. A point charge is placed in front of an infinite slab of dielectric, bounded by a plane face. The angle between a line of force in the dielectric and the normal to the face of the slab is a; the angle between the same two lines in the immediate neighbourhood of the charge is β. Prove that

$$\sin\frac{\beta}{2} = \sqrt{\frac{2\epsilon}{1+\epsilon}} \sin\frac{a}{2}.$$

132. An electrified particle is placed in front of an infinitely thick plate of dielectric. Show that the particle is urged towards the plate by a force

$$\frac{\epsilon-1}{\epsilon+1} \cdot \frac{q^2}{4d^2},$$

where d is the distance of the point from the plate.

133. Two dielectrics of inductive capacities ϵ_1 and ϵ_2 are separated by an infinite plane face. Charges q_1, q_2 are placed at points on a line at right angles to the plane, each at a distance a from the plane. Find the forces on the two charges, and explain why they are unequal.

134. A conducting sphere of radius a is placed in air, with its centre at a distance c from the plane face of an infinite dielectric. Show that its capacity is

$$a \sinh a \sum_1^\infty \left(\frac{\epsilon-1}{\epsilon+1}\right)^{n-1} \operatorname{cosech} na,$$

where $a = c/a$.

135. Two conductors of capacities q_1 and q_2 in air are on the same normal to the plane boundary between the dielectrics ϵ_1, ϵ_2 at great distances a, b from the boundary. They are connected by a thin wire and charged. Prove that the charge is distributed between them approximately in the ratio

$$\epsilon_1 \left\{\frac{1}{c_2} - \frac{\epsilon_1 - \epsilon_2}{2b(\epsilon_1 + \epsilon_2)} - \frac{2\epsilon_2}{(\epsilon_1 + \epsilon_2)(a + b)}\right\} : \epsilon_2 \left\{\frac{1}{c_1} + \frac{\epsilon_1 - \epsilon_2}{2a(\epsilon_1 + \epsilon_2)} - \frac{2\epsilon_1}{(\epsilon_1 + \epsilon_2)(a + b)}\right\}.$$

136. A conducting sphere of radius a is embedded in a dielectric (ϵ) whose outer boundary is a concentric sphere of radius $2a$. Show that if the system be placed in a uniform field of force E equal quantities of positive and negative electricity are separated of amount

$$\frac{9Ea^2\epsilon}{5\epsilon + 7}.$$

137. A sphere of glass of radius a is held in air with its centre at a distance c from a point at which there is a charge q. Prove that the resultant attraction is

$$\tfrac{1}{2}\beta q^2 \frac{a^3}{c^3}\left\{\frac{1 + \beta}{c^2 - a^2} + \frac{2c^2}{(c^2 - a^2)^2} - \frac{c}{a^3}(1 - \beta^2)\left(\frac{a}{c}\right)\int_0^{\frac{a}{c}} \frac{x^{2-\beta}\,dx}{1 - x^2}\right\},$$

where $$\beta = \frac{\epsilon - 1}{\epsilon + 1}.$$

138. A spherical conductor of radius a is surrounded by a uniform dielectric ϵ, which is bounded by a sphere of radius b having its centre at a small distance γ from the centre of the conductor. Prove that, if the potential of the conductor is ϕ, and there are no other conductors in the field, the surface density at a point where the radius makes an angle θ with the line of centres is

$$\frac{\epsilon\phi b}{4\pi a\{(\epsilon - 1)a + b\}}\left[1 + \frac{6(\epsilon - 1)\gamma a^2 \cos\theta}{2(\epsilon - 1)a^3 + (\epsilon + 2)b^3}\right].$$

139. A sphere of S.I.C. ϵ is placed in air in a field of force due to a potential X_n (before the introduction of the sphere) referred to rectangular axes through the centre of the sphere, where X_n is a solid harmonic of order n. Prove that the potential inside the sphere is

$$\frac{2n + 1}{n + 1 + \epsilon n} X_n.$$

140. A charge q is placed at a distance c from the centre of a sphere of S.I.C. ϵ and outside the sphere. Prove that the potential at any point inside the sphere at a distance r from the centre is

$$\frac{q}{c}\sum_0^\infty \frac{2n + 1}{\epsilon n + n + 1}\left(\frac{r}{c}\right)^n P_n.$$

141. Find the potential at any point when a sphere of specific inductive capacity ϵ is placed in air in a field of uniform force.

A circle has its centre on the line of force which passes through the centre of the sphere and its plane perpendicular to this line of force. Prove that if the plane of the circle does not cut the sphere, the presence of the sphere increases the induction through the circle in the ratio

$$1 + 2\frac{\epsilon - 1}{\epsilon + 2}\sin^2 a : 1,$$

where $2a$ is the angle of the enveloping cone from any point on the circumference of the circle to the sphere.

142. A shell of glass of inductive capacity ϵ, which is bounded by concentric spherical surfaces of radii a, b $(a < b)$, surrounds an electrified particle with charge Q which is at a point P' at a small distance c from O, the centre of the spheres. Show that the potential at a point P outside the shell at a distance r from P' is approximately

$$\frac{Q}{r} + \frac{2Qc\,(b^3 - a^3)\,(\epsilon - 1)^2}{2a^3(\epsilon - 1)^2 - b^3(\epsilon + 2)(2\epsilon + 1)}\,\frac{\cos\theta}{r^2},$$

where θ is the angle PP' makes with OP' produced.

143. A conductor at potential ϕ whose surface is of the form $r = a(1 + \epsilon P_n)$ is surrounded by a dielectric (ϵ) whose boundary is the surface $r = b(1 + \epsilon' P_n)$, and outside this the dielectric is air. Show that the potential in the air at a distance r from the origin is

$$\frac{\epsilon ab\phi}{(\epsilon - 1)a + b}\left[\frac{1}{r} + \frac{(2n+1)\,\epsilon a^n b^{2n+1} + (\epsilon - 1)\,\epsilon' b^n\,\{nb^{2n+1} + (n+1)\,a^{2n+1}\}}{(1 + n + n\epsilon)\,b^{2n+1} + (\epsilon - 1)(n+1)\,a^{2n+1}}\,\frac{P_n}{r^{n+1}}\right],$$

where squares and higher powers of ϵ and ϵ' are neglected.

144. A dielectric sphere is surrounded by a thin circular wire of large radius b carrying a charge Q. Prove that the potential within the sphere is

$$\frac{Q}{b}\left\{1 + \sum_1^\infty (-1)^n\,\frac{1 + 4n}{1 + 2n(1 + \epsilon)}\cdot\frac{1\,.\,3\,.\,5\,\ldots\,2n - 1}{2\,.\,4\,.\,6\,\ldots\,2n}\left(\frac{r}{b}\right)^{2n} P_{2n}\right\}.$$

145. An electrified line with charge Q per unit length is parallel to a circular cylinder, of radius a and inductive capacity ϵ, the distance of the wire from the centre of the cylinder being c. Show that the force on the wire per unit length is

$$\frac{\epsilon - 1}{\epsilon + 1}\cdot\frac{4a^2q^2}{c\,(c^2 - a^2)}.$$

146. The space between two concentric conducting spheres is filled on one side of a diametral plane with dielectric of specific inductive capacity ϵ, and on the other side with dielectric of specific capacity ϵ'. The inner sphere is of radius a and has a charge Q. Prove that the force on it perpendicular to this diametral plane is

$$\frac{\epsilon - \epsilon'}{(\epsilon + \epsilon')^2}\cdot\frac{Q^2}{2a^2}.$$

147. A dielectric hemisphere of radius a and inductive capacity ϵ is placed with its base in contact with the plane boundary of an otherwise unlimited conductor. Prove that the potential at any point of the field outside both the conductor and the dielectric is

$$\phi = -\,4\pi\sigma\cos\theta\left(r - \frac{\epsilon - 1}{\epsilon + 2}\cdot\frac{a^3}{r^2}\right),$$

where the origin is at the centre of the hemisphere and σ is the surface density of the charge on the plane conductor at a great distance from the hemisphere.

148. A point charge q is within a sphere of homogeneous dielectric (ϵ) of radius a and is a short distance c from the centre. Show that the force on the point is approximately

$$\frac{2\,(\epsilon - 1)}{2\,(\epsilon + 2)}\,\frac{q^2c}{a^3}.$$

149. A condenser is formed of two parallel plates, distant h apart, one of which is at zero potential. The space between the plates is filled with a dielectric whose inductive capacity increases uniformly from one plate to the other. Show that the capacity per unit area is

$$\frac{\epsilon_2 - \epsilon_1}{4\pi h\log\epsilon_2/\epsilon_1},$$

where ϵ_1 and ϵ_2 are the values of the inductive capacity at the surfaces of the plate. The inequalities of distribution at the edges of the plates are neglected.

150. A spherical conductor of radius a is surrounded by a concentric spherical conducting shell whose internal radius is b, and the intervening space is occupied by a dielectric whose specific inductive capacity at a distance r from the centre is $\frac{c+r}{r}$. If the inner sphere is insulated and has a charge Q, the shell being connected with the earth, prove that the potential in the dielectric at a distance r from the centre is $\frac{Q}{c}\log\frac{b(c+r)}{r(c+b)}$.

151. A spherical conductor of radius a is surrounded by a concentric spherical shell of radius b, and the space between them is filled with a dielectric of which the inductive capacity at distance r from the centre is $\mu e^{-p^2} p^{-3}$, where $p = \frac{r}{a}$. Prove that the capacity of the condenser so formed is

$$2\mu a\,(e^{\frac{b^2}{a^2}} - e)^{-1}.$$

152. If the specific inductive capacity varies as $e^{-\frac{r}{a}}$, where r is the distance from a fixed point in the medium, verify that a solution of the differential equation satisfied by the potential is

$$\left(\frac{a}{r}\right)^2\left[e^{\frac{r}{a}} - 1 - \frac{r}{a} - \frac{r^3}{2a^2}\right]\cos\theta,$$

and hence determine the potential at any point of a sphere, whose inductive capacity is the above function of the distance from the centre, when placed in a uniform field of force.

153. Show that the capacity of a condenser consisting of the conducting spheres $r = a$, $r = b$ and a heterogeneous dielectric of inductive capacity $\epsilon = f(\theta, \phi)$ is

$$\frac{ab}{4\pi(b-a)}\iint f(\theta, \phi)\sin\theta\, d\theta\, d\phi.$$

154. If the electricity in the field is confined to a given system of conductors at given potentials, and the inductive capacity of the dielectric is slightly altered according to any law such that at no point is it diminished, and such that the differential coefficients of the increment are also small at all points, prove that the energy of the field is increased.

155. A slab of dielectric of inductive capacity ϵ and of thickness x is placed inside a parallel plate condenser so as to be parallel to the plates. Show that the surface of the slab experiences a tension

$$2\pi\sigma^2\left\{1 - \frac{1}{\epsilon} - x\frac{d}{dx}\left(\frac{1}{\epsilon}\right)\right\}.$$

156. For a gas $\epsilon = 1 + \theta\rho$, where ρ is the density and θ is small. A conductor is immersed in the gas: show that if θ^2 is neglected the mechanical force on the conductor is $2\pi\sigma^2$ per unit area.

157. A metallic shell is surrounded by a thin concentric conducting shell formed by two hemispheres with their rims in contact, the space between the sphere and shell being filled with a dielectric of specific inductive capacity ϵ. If charges Q, Q' be given to the shell and the sphere, show that if the halves of the shell remain in contact the charges must be of opposite signs and the ratio of their magnitudes must lie between the limits $1 \pm \frac{1}{\sqrt{\epsilon}}$.

158. If a spherical conductor, of radius a, with no other conductor in the neighbourhood is coated with a uniform thickness d of shellac of which ϵ is the specific inductive capacity, shew that the capacity of the conductor is increased in the ratio $\epsilon(a+d) : \epsilon a + d$. If the spherical conductor consist of two separate hemispherical portions, what would be the force tending to separate one hemisphere from the other?

II. MAGNETOSTATICS

159. A small magnet ACB, free to turn about its centre C, is acted on by a small fixed magnet PQ. Prove that in equilibrium the axis ACB lies in the plane PQC, and that $\tan\theta = -\frac{1}{2}\tan\theta'$, where θ, θ' are the angles which the two magnets make with the line joining them.

160. The axis of a small magnet makes an angle ϕ with the normal to a plane. Prove that the line from the magnet to the point in the plane where the number of lines crossing it per unit area is a maximum makes an angle θ with the axis of the magnet, such that

$$2\tan\theta = 3\tan 2(\phi - \theta).$$

161. Two small magnets lie in the same plane, and make angles θ, θ' with the line joining their centres. Show that the line of action of the resultant force between them divides the line of centres in the ratio

$$\tan\theta' + 2\tan\theta : \tan\theta + 2\tan\theta'.$$

162. Two small magnets having their centres at distance r apart, make angles θ, θ' with the line joining them and an angle ϵ with each other. Show that the force on the first magnet in its own direction is

$$\frac{3mm'}{r^4}(5\cos^2\theta\cos\theta' - \cos\theta' - 2\cos\epsilon\cos\theta).$$

Show also that the couple about the line joining them which the magnets exert on one another is

$$\frac{mm'}{r^4}d\sin\epsilon,$$

where d is the shortest distance between their axes.

163. Two magnetic needles of moments M, M' are soldered together so that their directions include an angle a. Show that when they are suspended so as to swing freely in a uniform horizontal field, their directions will make angles θ, θ' with the lines of force, given by

$$\frac{\sin\theta}{M'} = \frac{\sin\theta'}{M} = \frac{\sin a}{(M'^2 + M^2 + 2MM'\cos a)^{\frac{1}{2}}}.$$

164. Two small magnets at distance r are in one plane and are inclined at angles $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$ to the line joining their centres. Prove that the action between them reduces to a single force parallel to this line and at a distance $1/9r$ from it.

165. Show that the action between two small magnets with centres A, A' reduces to a single force if the image of each magnet in the plane perpendicularly bisecting AA' is perpendicular to the other magnet. Show also that if the axis of the magnets meet the plane in the points B, B' and if $AO = 2OB$ and $A'O' = 2O'B'$, then OO' is the line of action of the force.

166. A sphere of hard steel is magnetised uniformly in a constant direction and a magnetic particle is held at an external point with the axis of the particle parallel to the direction of magnetisation of the sphere. Find the couples acting on the sphere and on the particle.

167. Two magnetic particles of equal moment are fixed with their axes parallel to the axis of z, and in the same direction, and with their centres at the points $(\pm a, 0, 0)$.

Shew that if another magnetic molecule is free to turn about its centre, which is fixed at the point $(0, y, z)$, its axis will rest in the plane $x = 0$ and will make with the axis of z the angle

$$\tan^{-1}\frac{3yz}{2z^2 - a^2 - y^2}.$$

Examine which of the two positions of equilibrium is stable.

168. Two small equal magnets have their centres fixed and can turn about them in a magnetic field of uniform intensity H, whose direction is perpendicular to the line r joining the centres. Show that the position in which the magnets both point in the direction of the lines of force of the uniform field is stable only if

$$H > \frac{3M}{r^3}.$$

169. Obtain the equation

$$T = 2\pi\sqrt{I/mH}$$

for the time of a small oscillation of a magnet of moment m swung about its centre of inertia in a horizontal plane in a uniform field of horizontal intensity H, the moment of inertia of the magnet being I.

170. Assuming the earth to be a sphere uniformly magnetised parallel to the axis of rotation with intensity M, show that the time of a small horizontal oscillation in latitude ϕ is

$$\sqrt{3I\pi/mM\cos\phi}.$$

171. A small magnet of moment m is held fixed at the origin of coordinates, with its axis in the direction (l, m, n); another small magnet of moment m' has its centre fixed at the point $(x, 0, z)$, and is free to turn so that its axis moves in a plane parallel to the plane $z = 0$. Find the position of stable equilibrium of m' and show that the period of its free oscillations about this position is

$$2\pi I^{\frac{1}{2}} m^{-\frac{1}{2}} m'^{-\frac{1}{2}} (x^2 + z^2)^{\frac{5}{4}} [\{l(x^2 + z^2) - 3x(lx + nz)\}^2 + m^2(x^2 + z^2)^2]^{-\frac{1}{4}},$$

where I is the moment of inertia of m'.

172. Four small equal magnets are placed at the corners of a square and oscillate under the actions they exert on each other. Prove that the times of vibration of the principal oscillations are

$$2\pi\left\{\frac{Id^2}{m^2 3(2 + \frac{1}{2}\sqrt{2})}\right\}^{\frac{1}{2}}, \quad 2\pi\left\{\frac{Id^2}{m^2(3 - \frac{1}{2}\sqrt{2})}\right\}^{\frac{1}{2}}, \quad 2\pi\left(\frac{Id^2\, 2\sqrt{2}}{3m^2}\right)^{\frac{1}{2}},$$

where m is the magnetic moment, I the moment of inertia of a magnet and d is a side of the square.

173. If a small cylindrical cavity be made within a magnetised body with its axis parallel to the direction of magnetisation at the point, prove that the magnetic force within the cavity is simply H if the length of the cavity is large compared with its radius but is B if the radius is large compared with the length.

174. A uniformly magnetised substance has an ellipsoidal cavity in it with a principal axis in the direction of magnetisation of the substance; prove that the ratio of the magnetic force in the cavity to the magnetic force at a great distance from the cavity is

$$\left\{1 + \alpha\left(\frac{1}{\mu} - 1\right)\right\}^{-1},$$

where
$$\alpha = \tfrac{1}{2}a^3bc\int_0^\infty \frac{du}{(a^2 + u)^{\frac{3}{2}}(b^2 + u)^{\frac{1}{2}}(c^2 + u)^{\frac{1}{2}}},$$

the direction of the principal axis a being that of magnetisation of the substance, and show that when the cavity is an oblate spheroid with its least axis in the direction of magnetisation the ratio tends to μ as the least axis diminishes.

175. A steel shell, bounded internally and externally by non-concentric spherical surfaces, is magnetised uniformly. Prove that there is no magnetic field in the hollow, and that the external field is the same as would be due to two magnetic particles at the centres of the spheres, whose moments are proportional to the respective volumes of the spheres.

176. A small magnet of moment m is held in the presence of a very large fixed mass of soft iron of permeability μ with a very large plane face: the magnet is at a distance a from the plane face and makes an angle θ with the shortest distance from it to the plane. Show that a certain force, and a couple

$$\frac{(\mu - 1)\, m^2 \sin\theta \cos\theta}{8a^3 (\mu + 1)},$$

are required to keep the magnet in position.

177. A sphere of soft iron of radius a is placed in a field of uniform magnetic force parallel to the axis of z. Show that the lines of force external to the sphere lie on surfaces of revolution, the equation of which is of the form

$$\left\{1 + \frac{2(\mu - 1)}{\mu + 2}\left(\frac{a}{r}\right)^2\right\}(x^2 + y^2) = \text{const.},$$

r being the distance from the centre of the sphere.

178. A sphere of soft iron of permeability μ is introduced into a field of force in which the potential is a homogeneous polynomial of degree n in (x, y, z). Show that the potential inside the sphere is reduced to

$$\frac{2n + 1}{n\mu + n + 1}$$

of its original value.

179. If a shell of radii a, b is introduced in place of the sphere in the last question, show that the force inside the cavity is altered in the ratio

$$(2n + 1)^2 \mu : (n\mu + n + 1)(n\mu + n + \mu) - n(n + 1)(\mu - 1)^2 \left(\frac{a}{b}\right)^{2n+1}.$$

180. If the magnetic field within a body of permeability μ be uniform, shew that any spherical portion can be removed and the cavity filled up with a concentric spherical nucleus of permeability μ_1 and a concentric shell of permeability μ_2 without affecting the external field, provided μ lies between μ_1 and μ_2 and the ratio of the volume of the nucleus to that of the shell is properly chosen. Prove also that the field inside the nucleus is uniform and that its intensity is greater or less than that outside according as μ is greater or less than μ_1.

181. A sphere of radius a has at any point (x, y, z) components of permanent magnetisation $\left(I_x \frac{x}{a},\ I_y \frac{y}{a},\ 0\right)$, the origin of coordinates being at its centre. It is surrounded by a spherical shell of uniform permeability μ, the bounding radii being a and b. Determine the complete circumstances of the field.

182. An infinitely long hollow iron cylinder of permeability μ, the cross section being concentric circles of radii a, b, is placed in a uniform field of magnetic force the direction

of which is perpendicular to the generators of the cylinder. Show that the number of lines of induction through the space occupied by the cylinder is changed by inserting the cylinder in the field, in the ratio

$$b^2(\mu+1)^2 - a^2(\mu-1)^2 : 2\mu\{b^2(\mu+1) - a^2(\mu-1)\}.$$

183. An infinite cylinder of soft iron is placed in a uniform field of potential

$$\psi_0 - H_x x - H_y y,$$

the equation of the cylinder being $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the potential of the induced magnetism at any internal point is

$$-(\mu-1)\left(\frac{b}{a+\mu b}H_x x + \frac{a}{\mu a + b}H_y y\right).$$

184. A circular wire of radius a is concentric with a spherical shell of soft iron of radii b and c. Show that, if a steady unit current flow round the wire, the presence of the iron increases the number of lines of induction through the wire by

$$\frac{2\pi^2 a^4 (c^3 - b^3)(\mu-1)(\mu+2)}{b^3\{(2\mu+1)(\mu+2)c^3 - 2(\mu-1)^2 b^3\}}.$$

185. A solid elliptic cylinder of iron whose equation is $\xi = a$ given by

$$x + iy = c\cosh(\xi + i\eta)$$

is placed in a field of magnetic force whose potential is $A(x^2 - y^2)$. Show that in the space external to the cylinder the potential of the induced magnetism is

$$-\tfrac{1}{4}Ac^2 \operatorname{cosech} 2(a+\beta)\sin 4a e^{2(a-\beta-\xi)}\cos 2\eta,$$

where $\coth 2\beta$ is the permeability.

186. An infinite elliptic cylinder of permeability μ is placed in a uniform magnetic field of strength H so that the direction of the force is perpendicular to the axis of the cylinder. Show that there is a couple on the cylinder tending to set the major axis of a principal section in the direction of the force; the moment of the couple per unit length being

$$\frac{ab(a^2-b^2)(\mu-1)^2 H^2 \sin 2\theta}{8(a+\mu b)(b+\mu a)},$$

where θ is the angle between the major axis of a section and the direction of the force.

187. A unit magnetic pole is placed on the axis of z at a distance f from the centre of a sphere of soft iron of radius a. Show that the potential of the induced magnetism at any external point is

$$-\frac{1}{\pi}\frac{\mu-1}{\mu+1}\frac{a^3}{f^2}\int_0^\pi\int_0^1 \frac{t^{\frac{1}{\mu+1}}\,dt\,d\theta}{\left(z + i\varpi\cos\theta - \frac{a^2 t}{f}\right)^2},$$

where z, ϖ are the cylindrical coordinates of the point. Find also the potential at an external point.

188. A magnetic pole of strength m is placed in front of an iron plate of permeability μ and thickness c. If this pole be the origin of rectangular coordinates x, y and if x be perpendicular and y parallel to the plate, show that the potential behind the plate is given by

$$\psi = m(1-\rho^2)\int_0^\infty \frac{e^{-xt}J_0(yt)\,dt}{1-\rho^2 e^{-2ct}},$$

where

$$\rho = \frac{\mu-1}{\mu+1}.$$

III. ELECTROKINETICS

189. A network is formed of uniform wire in the shape of a rectangle of sides $2a$, $3a$, with parallel wires arranged so as to divide the internal space into six squares of sides a, the contact at the points of intersection being perfect. Show that if a current enter the framework by one corner and leave it by the opposite, the resistance is equivalent to that of a length $121a/69$ of the wire.

190. A fault of given earth resistance developes in a telegraph line. Prove that the current at the receiving end, generated by an assigned battery at the signalling end, is least when the fault is at the middle of the line.

191. The resistances of three wires BC, CA, AB, of the same uniform section and material, are a, b, c respectively. Another wire from A of constant resistance d can make a sliding contact with BC. If a current enter at A and leave at the point of contact with BC, show that the maximum resistance of the network is

$$\frac{(a+b+c)\,d}{a+b+c+4d},$$

and determine the least resistance.

192. The resistances in the arms AC, CB, BD, DA of a Wheatstone's bridge are P, R, S, Q respectively. The points A and C are also connected through a condenser of capacity K in parallel with the resistance P; and the arm BD includes a coil of inductance L. A and B are connected to the battery and C and D to the galvanometer. If there is no flow, instantaneous or permanent, through the galvanometer on making the battery circuit, show that $L = PSK$.

193. A quadrilateral is formed of wire and A, B, C, D are its corners taken in order. The resistances of the wires AB, BC, CD, DA are p, q, r, s respectively. The excess of the potential of A over that of B when unit current enters the quadrilateral at C and leaves it at D by wires applied at these points is denoted by $[AB, CD]$. The resistance of the quadrilateral when A and B are the electrodes is denoted by $[AB]$. The other symbols have corresponding meanings. Calculate $[AB, CD]$ in terms of p, q, r, s and show that

$$[AD] + [BC] - [AC] - [BD] = 2\,[AB\,.\,CD],$$
$$[AB\,.\,CD] + [BC\,.\,AD] + [CA\,.\,BD] = 0.$$

194. Two planes A, B are connected by a telegraph of which the end at A is connected to one terminal of a battery, and the end at B to one terminal of a receiver, the other terminals of the battery and receiver being connected to earth. At a point C of the line a fault is developed, of which the resistance is r. If the resistances of AC, CB be p, q respectively, show that the current in the receiver is diminished in the ratio

$$r\,(p+q) : qr + rp + pq,$$

the resistance of the battery, receiver and earth circuit being neglected.

195. Two cells of electromotive forces e_1, e_2 and resistances r_1, r_2 are connected in parallel to the ends of a wire of resistance R. Show that the current in the wire is

$$\frac{e_1 r_2 + e_2 r_1}{r_1 R + r_2 R + r_1 r_2},$$

and find the ratio at which the cells are working.

196. A network of conductors is in the form of a tetrahedron $PQRS$; there is a battery of electromotive force E in PQ, and the resistance of PQ, including the battery, is R. If the resistances in QR, RP are each equal to r, and the resistances in PS, RS are each equal to $\frac{1}{2}r$, and that in $QS = \frac{3}{2}r$, find the currents in each branch.

197. A cell of resistance r is connected to the ends of a wire AB. The cell is then replaced by two different cells, of resistances R, R' arranged in parallel, producing the same current in AB and having the combined resistance r when in parallel. Show that the total heat production is greater in the second case than in the first, by the amount which would be produced in the circuit of the two cells if the wire AB were broken.

198. An electric circuit contains a galvanometer and a battery of constant electromotive force ϕ. The resistance of the galvanometer is G and that of the rest of the circuit including the battery R. Show that on shunting the galvanometer with resistance S the current through the galvanometer is decreased by

$$\frac{\phi RG}{(R+G)(RS+RG+SG)}.$$

199. A circuit contains two lamps, each of resistance R, in parallel on leads each of resistance S. The resistance of the rest of the circuit, including the battery of constant voltage ϕ, is r. Show that if one lamp is broken, the heat emitted in unit time by the other is increased by

$$\frac{\phi^2 R\{3r^2 + 2r(R+S)\}}{(r+R+S)^2(2r+R+S)^2}.$$

200. A, B, C, D are the four junctions of a Wheatstone's bridge, and the resistances c, β, b, γ on AB, BD, AC, CD respectively are such that the battery sends no current through the galvanometer in BC. If now a new battery of electromotive force E be introduced into the galvanometer circuit, and so raise the total resistance in that circuit to a, find the current that will flow through the galvanometer.

201. A wire is interpolated in a circuit of given resistance and electromotive force. Find the resistance of the interpolated wire in order that the rate of generation of heat may be a maximum.

202. The resistances of the opposite sides of a Wheatstone's bridge are a, a' and b, b' respectively. Show that when the two diagonals which contain the battery and galvanometer are interchanged,

$$\frac{E}{C} - \frac{E}{C'} = \frac{(a-a')(b-b')(G-R)}{aa'-bb'},$$

where C and C' are the currents through the galvanometer in the two cases, G and R are the resistances of the galvanometer and battery conductors, and E the electromotive force of the battery.

203. A current C is introduced into a network of linear conductors at A, and taken out at B, the heat generated being H_1. If the network be closed by joining A, B by a resistance r in which an electromotive force E is inserted, the heat generated is H_2. Prove that

$$\frac{H_1}{C^2 r} + \frac{rH_2}{E^2} = 1.$$

204. A number N of incandescent lamps, each of resistance r, are fed by a machine of resistance R (including the leads). If the light emitted by any lamp is proportional to the square of the heat produced, prove that the most economical way of arranging the lamps is to place them in parallel arc, each arc containing n lamps, where n is the integer nearest to $\sqrt{NR/r}$.

205. A system of 30 conductors of equal resistances are connected in the same way as the edges of a dodecahedron. Show that the resistance of the network between a pair of opposite corners is $\frac{7}{6}$ of the resistance of a single conductor.

206. Four points A, B, C, D are connected, in order, by a battery of electromotive force E and resistance b, and three resistances c, d, a. A and C are connected by a resistance λ, and B, D by a galvanometer of resistance G. Find the current through the galvanometer, and prove that it is independent of λ if $ac = bd$.

207. Six wires of equal length and resistance are arranged so as to form the sides and the lines joining the middle points of the opposite sides of a square; prove that the resistance of the network between diagonally opposite corners is $\frac{3}{4}$ that of each wire.

208. An octahedron is formed of twelve bars of equal length and thickness and of the same material; a current enters the system at one end of a bar and leaves at the other end of the same bar; show that the resistance of the octahedron is $\frac{5}{12}$ of that of a single bar.

209. Two conducting circuits OPQ, $OP'Q'$ are connected from P to P' and Q to Q' by wires of resistances r and r'. A current enters the circuit at O and leaves it at O'. Show that, if the resistances of OP, PQ, QO are A, B, C and of $O'P'$, $P'Q'$, $Q'O'$ are a, b, c respectively, the currents in PP' and QQ' are in the rat o

$$\frac{BC}{A+B+C}+\frac{bc}{a+b+c}+r' : \frac{AB}{A+B+C}+\frac{ab}{a+b+c}+r.$$

210. A wire forms a regular hexagon and the angular points are joined to the centre by wires of equal resistance. Show that it is possible to adjust this resistance so that the resistance to a current entering at one angular point of the hexagon and leaving by the opposite is equal to that of a side of the hexagon.

211. A battery of mn equal cells is such that when it is arranged in m parallel sets of n cells in series the maximum current C is produced for a given external circuit. Show that when the cells are arranged in n parallel sets of m cells the current is

$$2mnC/(m^2+n^2).$$

212. If there be n points A and n points B such that the resistance between two A points and two B points is r and that between an A and a B point R, then the resistance to a current entering at an A point and leaving at a B point is

$$\frac{R(R+\overline{2n-1}r)}{R+r}.$$

213. A telegraph wire joining two places A, B drops from one of its supports at a place C and rests on another wire which is earthed at both ends. If λ is the ratio of the current strength at A to that at B when the current in AB is sent from A, and μ is the ratio when the current is sent from B, show that C divides AB in the ratio

$$(\mu^{-1}-1):(\lambda-1).$$

214. An assemblage of n points, of which A, B, C, D, P are any five, has each point connected to every other point by wires of resistance r. A current enters at any point X in the wire AB and leaves from any point Y in the wire CD. Prove (i) that the sum of the currents in AP, BP is $1/n$ of the whole current; (ii) that the mean of the currents in AC, BD or in AD, BC is also $1/n$ of the whole current; and (iii) that the currents in AP, BP are inversely proportional to the resistances of AX, BX.

Show also that the whole resistance of the network lies between $r(n+2)/2n$ and $2r/n$.

215. A, B, C, D are four points in succession at equal distances along a wire; and A, C and B, D are also joined by two other wires of the same length as the distances between those pairs of points measured along the original wire. A current enters the network thus formed at A and leaves at D: show that $\frac{1}{5}$ of it passes along BC.

216. A battery of electromotive force E and of resistance B is connected with the tw terminals of two wires arranged in parallel. The first wire includes a voltameter whic contains discontinuities of potential such that a unit current passing through it for a uni time does p units of work. The resistance of the first wire, including the voltameter, i R: that of the second is r. Show that if E is greater than $p\,(B + r)/r$, the current throug the battery is

$$\frac{E\,(R + r) - pr}{Rr + B\,(R + r)}.$$

217. In a network $PA, PB, PC, PD, AB, BC, CD, DA$, the resistances are $\alpha, \beta, \gamma,$ δ $\gamma + \delta, \delta + \alpha, \alpha + \beta, \beta + \gamma$ respectively. Show that if AD contains a battery of electrc motive force E, the current in BC is

$$\frac{P\,(\alpha\beta + \gamma\delta)\,E}{2P^2Q + (\beta\delta - \alpha\gamma)^2},$$

where $$P = \alpha + \beta + \gamma + \delta, \qquad Q = \beta\gamma + \gamma\alpha + \alpha\beta + \alpha\delta + \beta\delta + \gamma\delta.$$

218. A wire forms a regular hexagon and the angular points are joined to the centr by wires each of which has a resistance $1/n$ of the resistance of a side of the hexagon. Sho that the resistance to a current entering at one angular point of the hexagon and leavin it by the opposite point is

$$\frac{2\,(n + 3)}{(n + 1)\,(n + 4)}$$

times the resistance of a side of the hexagon.

219. Two long equal parallel wires AB, $A'B'$ of length l have their ends B, B' joine by a wire of negligible resistance, while A, A' are joined to the poles of a cell whos resistance is equal to that of a length r of the wire. A similar cell is placed as a bridg across the wire at a distance x from A, A'. Show that the effect of the second cell is t increase the current in BB' in the ratio

$$\frac{2\,(2l + r)\,(x + r)}{r\,(4l + r) + 2x\,(2l - r) - 4x^2}.$$

220. There are n points $1, 2 \ldots n$ joined in pairs by linear conductors. On introducin a current C at electrode 1 and taking it out at 2, the potentials of these are $\phi_1, \phi_2, \ldots \phi_n$ If x_{12} is the actual current in the direction 12, and x'_{12} any other that merely satisfies th conditions of introduction at 1 and abstraction at 2, show that

$$\Sigma\, r_{12}x_{12}x_{12}' = (\phi_1 - \phi_2)\,C = \Sigma\,(r_{12}x_{12}^2),$$

and interpret the result physically. If x typify the actual current when the current enter at 1 and leaves at 2, and y typify the actual current when the current enters at 3 and leave at 4, shew that

$$\Sigma\,(r_{12}x_{12}y_{12}) = (X_3 - X_4)\,C = (Y_1 - Y_2)\,C,$$

where the X's are potentials corresponding to the currents x, and the Y's are potential corresponding to the currents y.

221. In a simple network of conductors joining n points, the currents $C_1, C_2, \ldots C_{n-}$ are supplied at $(n - 1)$ of them and taken out at the nth point. There are also electro motive forces in the conductors. Show that the heat function $\Sigma R_{pq}C_{pq}^2$ can be expresse as the sum of two quadratic functions H_c and H_e of the entering amounts and the electro motive forces respectively and that the current in any conductor pq is given by

$$C_{pq} = \frac{1}{R_{pq}}\left(\frac{\partial H_c}{\partial C_p} - \frac{\partial H_c}{\partial C_q}\right) + \frac{\partial H_e}{\partial E_{pq}}.$$

Develope from this point of view the theorems of minimum heat dissipation and th reciprocal relations between the currents and potentials in the circuits.

222. A, B, C are three stations on the same telegraph wire. An operator at A knows that there is a fault between A and B, and observes that the current at A when he uses a given battery is i, i' or i'', according as B is insulated and C to earth, B to earth, or B and C both insulated. Show that the distance of the fault from A is

$$\{ka - k'b + (b-a)^{\frac{1}{2}}(ka - k'b)^{\frac{1}{2}}\}/k - k',$$

where $\qquad AB = a, \qquad BC = b - a, \qquad k = \dfrac{i'}{i - i''}, \qquad k' = \dfrac{i'}{i' - i''}.$

223. Six conductors join four points A, B, C, D in pairs, and have resistances a, α, b, β, c, γ, where a, α refer to BC, AD respectively, and so on. If this network be used as a resistance coil, with A, B as electrodes, show that the resistance cannot lie outside the limits

$$\left[\frac{1}{c} + \frac{1}{a+b} + \frac{1}{\alpha+\beta}\right]^{-1}$$

and $$\left[\frac{1}{c} + \left\{\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^{-1}\right\}^{-1}\right]^{-1}.$$

224. Two equal straight pieces of wire A_0A_n, B_0B_n are each divided into n equal parts at the points $A_1, A_2, \ldots A_{n-1}$ and $B_1, B_2, \ldots B_{n-1}$ respectively, the resistance of each part and that of A_nB_n being R. The corresponding points of each wire from 1 to n inclusive are joined by cross wires, and a battery is placed in A_1B_0. Show that, if the current through each cross wire is the same, the resistance of the cross wire A_sB_s is

$$\{(n-s)^2 + (n-s) + 1\} R.$$

225. A network is formed of a system of conductors joining every pair of a set of n points, the resistances of the conductors being all equal, and there is an electromotive force in the conductor joining the points A_1, A_2. Show that there is no current in any conductor except those which pass through A_1 or A_2 and find the current in these conductors.

226. Each member of the series of n points $A_1, A_2, A_3, \ldots A_n$ is united to its successor by a wire of resistance ρ, and similarly for the series of n points $B_1, B_2, \ldots B_n$. Each pair of points corresponding in the two series, such as A_r and B_r, is united by a wire of resistance R. A steady current i enters the network at A_1 and leaves it at B_n. Show that the current at A_1 divides itself between A_1A_2 and A_1B_1 in the ratio

$$\sinh \alpha + \sinh(n-1)\alpha + \sinh(n-2)\alpha : \sinh \alpha + \sinh(n-1)\alpha - \sinh(n-2)\alpha,$$

where $$\cosh \alpha = \frac{R+\rho}{R}.$$

227. An underground cable of length a is badly insulated so that it has faults throughout its length indefinitely near to one another and uniformly distributed. The conductivity of the faults is $1/\rho'$ per unit length of cable, and the resistance of the cable is ρ per unit length. One pole of a battery is connected to one end of the cable and the other pole is earthed. Prove that the current at the farther end is the same as if the cable were free from faults and of total resistance

$$\sqrt{\rho\rho'} \tanh\left(a\sqrt{\frac{\rho}{\rho'}}\right).$$

228. Two parallel conducting wires at unit distance are connected by $(n+1)$ cross pieces of the same wire, so as to form n squares. A current enters by an outer corner

of the first square, and leaves by the diagonally opposite corner of the last. Show that, if the resistance is that of a length $\frac{1}{2}n + a_n$ of the wire,

$$a_{n+1} = \frac{a_n + \frac{1}{2}}{a_n + 2}.$$

229. A, B are the ends of a long telegraph wire with a number of faults, and C is an intermediate point on the wire. The resistance to a current sent from A is R when C is earth-connected, but if C is not earth-connected the resistance is S or T according as the end B is to earth or insulated. If R', S', T' denote the resistances under similar circumstances when a current is sent from B towards A, show that

$$T'(R - S) = R'(R - T).$$

230. The inner plates of two condensers of capacities C, C' are joined by wires of resistances R, R' to a point P, and their outer plates by wires of negligible resistance to a point Q. If the inner plates be also connected through a galvanometer, show that the needle will suffer no sudden deflection on joining P, Q to the poles of a battery, if $CR = C'R'$.

231. An infinite cable of capacity and resistance K and R per unit length is at zero potential. At the instant $t = 0$ one end is suddenly connected to a battery for an infinitesimal interval of time and then insulated. Show that, except for very small values of t, the potential at any instant at a distance x from this end of the cable will be proportional to

$$\frac{1}{\sqrt{t}} e^{-\frac{KRx^2}{4t}}.$$

232. A condenser is formed of two large parallel plates of area A, separated by a thickness d of a medium with dielectric constant ϵ and resistance k. If it is charged, show that the time in which the charge will sink to $\frac{1}{n}$ of its original amount is $\frac{A\epsilon k}{4\pi d}\log n$.

233. The sectional area of a wire of uniform material of specific resistance σ is equal to $A + Bx^2$, where x is the distance from one end. Determine the resistance of a length of the wire; and find the position of a point at which the electric potential is the mean of those at the ends, when a steady current is flowing through it.

234. The ends of a rectangular conducting lamina of breadth c, length a, and uniform thickness τ, are maintained at different potentials. If $\rho = f(x, y)$ be the specific resistance of a point whose distances from an end and a side are x, y, prove that the resistance of the lamina cannot be less than

$$\int_0^a \frac{dx}{\tau \int_0^c \frac{dy}{\rho}},$$

or greater than

$$\frac{1}{\tau \int_0^c \frac{dy}{\int_0^a \rho\, dx}}.$$

235. Two large vessels filled with mercury are connected by a capillary tube of uniform bore. Find superior and inferior limits to the conductivity.

236. A cylindrical cable consists of a conducting core of copper surrounded by a thin insulating sheath of material of given specific resistance. Show that if the sectional areas of the core and sheath are given, the resistance to lateral leakage is greatest when the surfaces of the two materials are coaxal right circular cylinders.

237. Prove that the product of the resistance to leakage per unit length between two practically infinitely long parallel wires insulated by a uniform dielectric and at different potentials, and the capacity per unit length, is $\frac{\epsilon\rho}{4\pi}$, where ϵ is the inductive capacity and ρ the specific resistance of the dielectric. Prove also that the time that elapses before the potential difference sinks to a given fraction of its original value is independent of the sectional dimensions and relative positions of the wires.

238. If the right sections of the wires in the last question are semi-circles described on opposite sides of a square as diameters, and outside the square, while the cylindrical space whose section is the semi-circles similarly described on the other two sides of the square is filled up with a dielectric of infinite specific resistance, and all the neighbouring space is filled up with a dielectric of resistance ρ, prove that the leakage per unit length in unit time is $2V/\rho$, where V is the potential difference.

239. If $\phi + i\psi = f(x + iy)$, and the curves $\phi =$ const. be closed curves, show that the insulation resistance between lengths l of the surfaces $\phi = \phi_0$, $\phi = \phi_1$ is

$$\frac{\rho\,(\phi_1 \sim \phi_0)}{l\,[\psi]},$$

where $[\psi]$ is the increment of ψ on passing once round a ϕ-curve, and ρ is the specific resistance of the dielectric.

240. Current enters and leaves a uniform circular disc through two circular wires of small radius e whose central lines pass through the edge of the disc at the extremities of a chord of length d. Show that the total resistance of the sheet is

$$(2\sigma/\pi)\log(d/e).$$

241. Using the transformation

$$\log x + iy = \xi + iy,$$

prove that the resistance of an infinite strip of uniform breadth π between two electrodes distant $2a$ apart, situated on the middle line of the strip and having equal radii δ, is

$$\frac{\sigma}{\pi}\log\left(\frac{2}{\delta}\tanh a\right).$$

242. Show that the transformation

$$x' + iy' = \cosh\frac{\pi(x + iy)}{a}$$

enables us to obtain the potential due to any distribution of electrodes upon a thin conductor in the form of the semi-infinite strip bounded by $y = 0$, $y = a$, and $x = 0$.

If the margin be uninsulated, find the potential and flow due to a source at the point $x = c$, $y = \frac{a}{2}$. Show that if the flows across the three edges are equal, then

$$\pi c = a\cosh^{-1} 2.$$

243. Equal and opposite electrodes are placed at the extremities of the base of an isosceles triangular lamina, the length of one of the equal sides being a, and the vertical angle $\frac{2\pi}{3}$. Show that the lines of flow and equal potentials are given by

$$\sinh^{\frac{2}{3}}\frac{w}{2} + 1 = \sqrt{3}\,\frac{1 + \operatorname{cn} u}{1 - \operatorname{cn} u},$$

where $$3^{\frac{3}{4}}\,\Gamma\left(\frac{1}{2}\right) ua = \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{6}\right)\left(ze^{-\frac{\pi i}{6}} - a\right),$$

and the modulus of $\operatorname{cn} u$ is $\sin 75°$, the origin being at the vertex.

244. A circular sheet of copper, of specific resistance σ_1 per unit area, is inserted in a very large sheet of tinfoil (σ_0) and currents flow in the composite sheet, entering and leaving at the electrodes. Prove that the current function in the tinfoil corresponding to an electrode at which a current e enters the tinfoil is the coefficient of i in the imaginary part of

$$-\frac{\sigma_0 e}{2\pi}\left[\log(z-c)+\frac{\sigma_0-\sigma_1}{\sigma_0+\sigma_1}\log\frac{cz}{cz-a^2}\right],$$

where a is the radius of the copper sheet, z is a complex variable with its origin at the centre of the sheet, and c is the distance of the electrode from the origin, the real axis passing through the electrode.

Investigate the corresponding expressions determining the lines of flow in the copper.

245. A uniform conducting sheet has the form of the catenary of revolution

$$y^2+z^2=c^2\cosh^2\frac{x}{c}.$$

Prove that the potential at any point due to an electrode at (x_0, y_0, z_0) introducing a current C is

$$\text{const.}-\frac{C\sigma}{4\pi}\log\left\{\cosh\frac{x-x_0}{c}-\frac{yy_0+zz_0}{\sqrt{(y^2+z^2)(y_0{}^2+z_0{}^2)}}\right\}.$$

246. Electric currents are introduced into an infinite plane sheet at a series of electrodes. Show that the asymptotes of the lines of flow all meet in the mean centre (centre of gravity) of the electrodes.

247. The molecules of a metal are assumed to be fixed centres of force repelling the electrons in collision with a force μ/r^n. If it is still assumed that the average duration of a collision is small compared with the time in an average path between two collisions, show that the velocity distribution function is given by an equation of the type

$$X\frac{\partial f}{\partial \xi}+Y\frac{\partial f}{\partial \eta}+Z\frac{\partial f}{\partial \zeta}+\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}+\frac{\partial f}{\partial t}=\frac{f_0-f}{\tau_m},$$

where the notation is as in the text except that now

$$\tau_m=l_m u^{\frac{4}{n+1}-1},$$

and l_m is a constant depending on μ.

Hence develope a general theory of conductivity on this basis and shew that the fundamental laws are still satisfied.

248. Prove that the velocity distribution function determined in the last question may be determined directly from the fact that the electrons at any instant started the free paths which they are then pursuing with the velocities assigned to them by Maxwell's law.

249. Prove by direct calculation that the energy dissipated per unit volume of a metal subject to an applied field of strength E, but in which the conditions are otherwise uniform, is

$$\tfrac{1}{2}\sigma E^2,$$

where σ is the ordinary expression for the conductivity.

250. Prove that a theoretically sufficient explanation of the Volta, Peltier and Thomson effects in metals can be obtained on the assumption that a part of the force exerted by an atom on an electron on collision varies with the conditions and type of the metal molecules.

How far does this explanation differ from that given by Lorentz which is reproduced in the text above?

IV. ELECTRODYNAMICS.

251. A current J flows in a very long straight wire. Find the forces and couples it exerts upon a small magnet.

Show that if the centre of the small magnet be fixed at a distance c from the wire, it has two free small oscillations about its position of equilibrium, of equal period

$$2\pi\sqrt{\frac{2mJ}{Ic}},$$

where I is the moment of inertia, and m the magnetic moment of the magnet.

252. Two parallel infinite wires convey equal currents of strength J in opposite directions, their distance apart being $2a$. A magnetic particle of strength μ and moment of inertia I is free to turn about a pivot at its centre, distant c from each of the wires. Show that the time of a small oscillation is that of a pendulum of length l given by

$$4Jaml = gc^2I.$$

253. Regarding the earth as a uniformly and rigidly magnetised sphere of radius a, and denoting the intensity of the magnetic field on the equator by H, show that a wire surrounding the earth along the parallel of south latitude λ, and carrying a current J from west to east, would experience a resultant force towards the south pole of the heavens of amount

$$6\pi aJH \sin\lambda \cos^2\lambda.$$

254. A current J flows in a circuit in the shape of an ellipse of area A and length l. Show that the force at the centre is $\pi Jl/A$.

255. Show that at any point along a line of force the vector potential due to a current in a circle is inversely proportional to the distance from the centre of the circle to the foot of the perpendicular from the point on to the plane of the circle. Hence trace the lines of constant vector potential.

256. A current J flows round a circle of radius a, and a current J' flows in a very long straight wire in the same plane. Show that the mutual attraction is $4\pi JJ'(\sec\alpha - 1)$, where α is the angle subtended by the circle at the nearest point of the wire.

If the circle is placed perpendicular to the straight wire with its centre at a distance c from it, shew that there is a couple tending to set the two wires in the same plane, of moment $2\pi JJ'a^2/c$ or $2\pi JJ'c$, according as $c >$ or $< a$.

257. A current J flows round a circular wire which can turn about a fixed diameter; and a current J' passes through a long straight wire parallel to this diameter and so placed that the plane through the wire and the diameter is perpendicular to the plane of the circle. Show that there is a couple on this circular wire tending to set the two wires in the same plane and that its magnitude is $4\pi JJ'c\left(1 - \frac{c}{\sqrt{a^2+c^2}}\right)$, where c is the distance of the wire from the plane of the circle and a the radius of the circle.

258. A long straight current intersects at right angles a diameter of a circular current, and the plane of the circle makes an acute angle α with the plane through this diameter and the straight current. Show that the coefficient of mutual induction is

$$4\pi\{c\sec\alpha - (c^2\sec^2\alpha - a^2)^{\frac{1}{2}}\} \text{ or } 4\pi c\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right),$$

according as the straight current passes within or without the circle, a being the radius of the circle, and c the distance of the straight current from its centre.

259. Prove that the coefficient of mutual induction between a pair of infinitely long straight wires and a circular one of radius a in the same plane, and with its centre at a distance ($b > a$) from each of the straight wires, is

$$8\pi (b - \sqrt{b^2 - a^2}).$$

260. A circuit contains a straight wire of length $2a$ conveying a current. A second straight wire, infinite in both directions, makes an angle α with the first, and their common perpendicular is of length c and meets the first wire in its middle point. Prove that the additional electromagnetic forces on the first straight wire, due to the presence of a current in the second wire, are equivalent to a wrench of pitch

$$2\left(a \sin\alpha - c \tan^{-1}\frac{a \sin\alpha}{c}\right) \Big/ \sin 2\alpha \tan^{-1}\frac{a \sin\alpha}{c}.$$

261. Two circular wires of radii a, b have a common centre, and are free to turn on an insulating axis which is a diameter of both. Show that when the wires carry currents J, J' a couple of magnitude

$$\frac{2\pi^2 b^2}{a}\left(1 - \frac{9}{16}\frac{b^2}{a^2}\right) JJ'$$

is required to hold them with their planes at right angles, it being assumed that b/a is so small that its fifth power may be neglected.

262. Two circular circuits are in planes at right angles to the line joining their centres. Show that the coefficient of induction

$$= 2\pi (a^2 - c^2) \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta \, d\theta}{\sqrt{a^2 \sin^2\theta + c^2 \cos^2\theta}},$$

where a, c are the longest and shortest lines which can be drawn from one circuit to the other. Find the force between the circuits when currents are flowing in them.

263. Two currents J, J' flow round two squares each of side a, placed with their edges parallel to one another and at right angles to the distance c between their centres. Show that they attract with a force

$$8JJ'\left\{\frac{c\sqrt{2a^2 + c^2}}{a^2 + c^2} + 1 - \frac{a^2 + 2c^2}{c\sqrt{a^2 + c^2}}\right\}.$$

264. A current J flows in a rectangular circuit whose sides are of lengths $2a$, $2b$ and the circuit is free to rotate about an axis through its centre parallel to the sides of length $2a$. Another current J' flows in a long straight wire parallel to the axis and at a distance d from it. Prove that the couple required to keep the plane of the rectangle inclined at an angle ϕ to the plane through its centre and the straight current is

$$\frac{8JJ' abd (b^2 + d^2) \sin\phi}{b^4 + d^4 - 2b^2 d^2 \cos 2\phi}.$$

265. Two circular wires lie with their planes parallel on the same sphere, and carry opposite currents inversely proportional to the areas of the circuits. A small magnet has its centre fixed at the centre of the sphere and moves freely about it. Show that it will be in equilibrium when its axis is either at right angles to the planes of the circuits, or makes an angle $\tan^{-1}\frac{1}{2}$ with them.

266. An infinite long straight wire conveys a current and lies in front of and parallel to an infinite block of soft iron bounded by a plane face. Find the magnetic potential at all points, and the force which tends to displace the wire.

267. A small sphere of radius b is placed in the neighbourhood of a circuit, which when carrying a current of unit strength would produce a magnetic force H at the point where the centre of the sphere is placed. Show that, if κ is the coefficient of induced magnetisation for the sphere, the presence of the sphere increases the coefficient of self-induction of the wire by an amount approximately equal to

$$\frac{8\pi b^3 \kappa (3 + 2\pi\kappa) H^2}{(3 + 4\pi\kappa)^2}.$$

268. A circular wire of radius a is concentric with a spherical shell of soft iron of radii b and c. If a steady unit current flow round the wire, show that the presence of the iron increases the number of lines of induction through the wire by

$$\frac{2\pi^2 a^4 (c^3 - b^3)(\mu - 1)(\mu + 2)}{b^3 \{(2\mu + 1)(\mu + 2) c^3 - 2(\mu - 1)^2 b^3\}},$$

approximately, where a is small compared with b and c.

269. A right circular cylindrical cavity is made in an infinite mass of iron of permeability μ. In this cavity a wire runs parallel to the axis of the cylinder carrying a steady current of strength I. Prove that the wire is attracted towards the nearest part of the surface of the cavity with a force per unit length equal to

$$\frac{2(\mu - 1) I^2}{(\mu + 1) d},$$

where d is the distance of the wire from its electrostatic image in the cylinder.

270. A wire is wound in a spiral of angle α on the surface of an insulating cylinder of radius a so that it makes n complete turns on the cylinder. A current J flows through the wire. Prove that the resultant magnetic force at the centre of the cylinder is

$$\frac{2\pi J n}{a(1 + \pi^2 n^2 \tan^2 \alpha)}$$

along the axis.

271. Coils of wire in the form of circles of latitude are wound upon a sphere and produce a magnetic potential $Ar^n P_n$ at internal points when a current is sent through them. Find the mode of winding and the potential at external points.

272. A current of strength J flows along an infinitely long straight wire, and returns in a parallel wire. These wires are insulated and touch along generators the surface of an infinite uniform circular cylinder of material whose coefficient of induction is k. Prove that the cylinder becomes magnetised as a lamellar magnet whose strength is

$$2\pi k J/(1 + 2\pi k).$$

273. A fine wire covered with insulating material is wound in the form of a circular disc, the ends being at the centre and circumference. A current is sent through the wire such that I is the quantity of electricity that flows per unit time across unit length of any radius in the disc. Show that the magnetic force at any point on the axis of the disc is

$$2\pi I \{\cosh^{-1}(\sec \alpha) - \sin \alpha\},$$

where α is the angle subtended at the point by any radius of the disc.

274. A given current sent through a tangent galvanometer deflects the magnet through an angle θ. The plane of the coil is slowly rotated round the vertical axis through the centre of the magnet. Prove that if $\theta > \frac{1}{4}\pi$ the magnet will describe complete revolutions, but if $\theta < \frac{1}{4}\pi$, the magnet will oscillate through an angle $\sin^{-1} \tan \theta$ on each side of the meridian.

275. Prove that if a slight error is made in reading the angle of deflection of a tangent galvanometer, the percentage error in the deduced value of the current is a minimum if the angle of deflection is $\frac{1}{4}\pi$.

276. A tangent galvanometer is incorrectly fixed, so that equal and opposite currents give angular readings α and β measured in the same sense. Show that the plane of the coil, supposed vertical, makes an angle ϵ with its proper position such that

$$2 \tan \epsilon = \tan \alpha + \tan \beta.$$

Hence show that the real value of the current is the harmonic mean of its apparent magnitudes when sent in opposite directions round the galvanometer circuit.

277. In a tangent galvanometer, the sensibility is measured by the ratio of the increment of deflection to the increment of current, estimated per unit current. Show that the galvanometer will be most sensitive when the deflection is $\pi/4$, and that in measuring the current given by a generator whose electromotive force is E, and internal resistance R, the galvanometer will be most sensitive if there be placed across the terminals a shunt of resistance

$$\frac{HRr}{E - H(R + r)},$$

where r is the resistance of the galvanometer and H is the constant of the galvanometer.

What is the meaning of this result if the denominator vanishes or is negative?

278. A galvanometer coil of n turns is in the form of an anchor ring described by the revolution of a circle of radius b about an axis in its plane distant a from its centre. Show that the constant of the galvanometer

$$= \frac{8n}{a} \int_0^K \mathrm{cn}^2 u \, \mathrm{dn}^2 u \, du \qquad (k = b/a)$$

$$= (8n/3k^2 a) [(1 + k^2) E - (1 - k^2) K].$$

279. A coil is rotated with constant angular velocity ω about an axis in its plane in a uniform field of force perpendicular to the axis of rotation. Find the current in the coil at any time, and show that it is greatest when the plane of the coil makes an angle

$$\tan^{-1}\left(\frac{L\omega}{R}\right)$$

with the lines of magnetic force.

280. The ends B, D of a wire (R, L) are connected with the plates of a condenser of capacity C. The wire rotates about BD which is vertical with angular velocity ω, the area between the wire and BD being A. If H is the horizontal component of the earth's magnetism, show that the average rate at which work must be done to maintain the rotation is

$$\tfrac{1}{2} H^2 A^2 C^2 R\omega^4 / [R^2 C^2 \omega^2 + (1 - CL\omega)^2].$$

281. The resistance and self-induction of a coil are R and L, and its ends A and B are connected with the electrodes of a condenser of capacity C by wires of negligible resistance. There is a current $J \cos pt$ in a circuit connecting A and B, and the charge of the condenser is in the same phase as this current. Show that the charge at any time is

$$\frac{LJ \cos pt}{R},$$

and that
$$C(R^2 + p^2 L^2) = L.$$

Obtain also the current in the coil.

282. A closed solenoid consists of a large number N of circular coils of wire, each of radius a, wound uniformly upon a circular cylinder of height $2h$. At the centre of the cylinder is a small magnet whose axis coincides with that of the cylinder, and whose moment is a periodic quantity $\mu \sin pt$. Show that a current flows in the solenoid whose intensity is approximately

$$\frac{2\pi\mu N p}{\{(a^2+h^2)(R^2+L^2p^2)\}^{\frac{1}{2}}} \sin(pt+\alpha),$$

where R, L are the resistance and self-induction of the solenoid, and $\tan \alpha = \dfrac{R}{L}$.

283. A circular coil of n turns, of radius a and resistance R, spins with angular velocity ω round a vertical diameter in the earth's horizontal magnetic field H: show that the average electromagnetic damping couple which resists its motion is $\frac{1}{2}H^2n^2\pi^2a^2\omega R$.

284. A condenser, capacity C, is discharged through a circuit, resistance R, induction L, containing a periodic electromotive force $E \sin nt$. Show that the 'forced' current in the circuit is

$$E \sin(nt-\theta)\left[R^2+\left(nl-\frac{1}{Cn}\right)^2\right]^{-\frac{1}{2}},$$

where $$\tan\theta = \frac{n^2CL-1}{nCR}.$$

285. Two electric circuits of negligible resistances have self-inductances L_1, L_2 and are tuned separately to the same natural free period by varying the capacities C_1, C_2 of the condensers in the two circuits. Find the periods of free oscillation in the system consisting of the two circuits in the presence of one another, showing how they are affected by changes in the value of the mutual inductance, M.

The condenser of the first circuit is initially charged to a difference of potential ϕ_1, while the second condenser is uncharged and there is no initial current in either circuit; show that if $M < \sqrt{L_1L_2}$ the potential difference of the second condenser cannot exceed

$$\frac{\phi_1\sqrt{C_1}}{\sqrt{C_2}}.$$

286. Two equal long straight coils are placed end to end so that the direction of the winding is the same in both and they are acted on in series by an electromotive force $E \cos pt$. Show that if, without change in position, they are placed in parallel (the current being led in at the middle and out at the two ends) the maximum current in either coil is increased and the lag diminished.

Discuss also the case where the coils are wound in opposite directions.

287. Two circuits, resistances R_1 and R_2, coefficients of induction L, M, N, lie near each other, and an electromotive force E is switched into one of them. Show that the total quantity of electricity that traverses the other is EM/R_1R_2.

288. A current is induced in a coil B by a current $J \sin pt$ in a coil A. Show that the mean force tending to increase any coordinate of position θ is

$$-\frac{1}{2}\frac{J^2p^2LM}{R^2+L^2p^2}\frac{\partial M}{\partial \theta},$$

where L, M, N are the coefficients of induction of the coils, and R is the resistance of B.

289. A plane circuit, area S, rotates with uniform velocity ω about the axis of z, which lies in its plane at a distance h from the centre of gravity of the area. A magnetic

molecule of strength m is fixed in the axis of x at a great distance a from the origin pointing in the direction Ox. Prove that the current at time t is approximately

$$\frac{2S\omega m}{a^3(R^2+L^2\omega^2)^{\frac{1}{2}}}\cos(\omega t-\epsilon)+\frac{9S\omega mh}{a^4(R^2+4L^2\omega^2)^{\frac{1}{2}}}\cos(2\omega t-\eta),$$

where ϵ, η are determinate constants.

290. Two points A, B are joined by a wire of resistance R without self-induction; B is joined to a third point C by two wires each of resistance R, of which one is without self-induction, and the other has a coefficient of induction L. If the ends A, C are kept at a potential difference $E\cos pt$, prove that the difference of potentials at B and C will be

$$E'\cos(pt-\gamma),$$

where

$$E'=E\left\{\frac{E^2+p^2L^2}{9R^2+4p^2L^2}\right\}^{\frac{1}{2}},$$

$$\tan\gamma=\frac{pLR}{2p^2L^2+3R^2}.$$

291. A condenser, capacity C, charge Q, is discharged through a circuit of resistance R, there being another circuit of resistance S in the field. If $LN=M^2$, show that there will be initial currents $-NQ/C(RN+SL)$ and $MQ/C(RN+SL)$, and find the currents at any time.

292. Two insulated wires A, B of the same resistance have the same coefficient of self-induction L, while that of mutual induction is slightly less than L. The ends of B are connected by a wire of small resistance, and those of A by a battery of small resistance, and at the end of a time t a current J is passing through A. Prove that except when t is very small

$$J=\tfrac{1}{2}(J_0+J'),$$

approximately, where J_0 is the permanent current in A, and J' is the current in each after a time t, when the ends of both are connected in multiple arc by the battery.

293. The ends of a coil forming a long straight uniform solenoid of m turns per unit length are connected with a short solenoidal coil of n turns and cross section A, situated inside the solenoid, so that the whole forms a single complete circuit. The latter coil can rotate freely about an axis at right angles to the length of the solenoid. Show that in free motion without any external field, the current J and the angle θ between the cross sections of the coils are determined by the equations

$$RJ=-\frac{d}{dt}(L_1J+L_2J+8\pi mnAJ\cos\theta),$$

$$I\frac{d^2\theta}{dt^2}+4\pi mnAJ^2\sin\theta=0,$$

where L_1, L_2 are the coefficients of self-induction of the two coils, I is the moment of inertia of the rotating coil, R is the resistance of the whole circuit, and the effect of the ends of the long solenoid is neglected.

294. Two electrified conductors whose coefficients of electrostatic capacity are γ_1, γ_2, Γ are connected through a coil of resistance R and large inductance L. Verify that the frequency of the electric oscillations thus established is

$$\frac{1}{2\pi}\left(\frac{2\Gamma+\gamma_1+\gamma_2}{\gamma_1\gamma_2-\Gamma^2}\cdot\frac{1}{L}-\frac{R^2}{4L^2}\right)^{\frac{1}{2}}.$$

295. An electric circuit contains an impressed electromotive force which alternates in an arbitrary manner and also an inductance. Is it possible, by connecting the extremities of the inductance to the poles of a condenser, to arrange so that the current in the circuit shall always be in step with the electromotive force and proportional to it?

296. Two coils (resistances R, S; coefficients of induction L, M, N) are arranged in parallel in such positions that when a steady current is divided between the two, the resultant magnetic force vanishes at a certain suspended galvanometer needle. Prove that if the currents are suddenly started by completing a circuit including the coils, then the initial magnetic force on the needle will not in general vanish, but that there will be a 'throw' of the needle, equal to that which would be produced by the steady (final) current in the first wire flowing through that wire for a time interval

$$\frac{M-L}{R} - \frac{M-N}{S}.$$

297. A condenser of capacity C is discharged through two circuits, one of resistance R and self-induction L, and the other of resistance R' and containing a condenser of capacity C'. Prove that if Q is the charge on the condenser at any time

$$LR'\frac{d^3Q}{dt^3} + \left(\frac{L}{C} + \frac{L}{C'} + RR'\right)\frac{d^2Q}{dt^2} + \left(\frac{R}{C} + \frac{R}{C'} + \frac{R'}{C}\right)\frac{dQ}{dt} + \frac{Q}{CC'} = 0.$$

298. A condenser of capacity C is connected by leads of resistance r, so as to be in parallel with a coil of self-induction L, the resistance of the coil and its leads being R. If this arrangement forms part of a circuit in which there is an electromotive force of period $2\pi/p$, show that it can be replaced by a wire without self-induction if

$$(R^2 - L/C) = p^2LC\,(r^2 - L/C),$$

and that the resistance of this equivalent wire must be

$$(Rr + L/C)/(R + r).$$

299. Two coils, of which the coefficients of self and mutual induction are L_1, L_2, M, and resistances R_1, R_2, carry steady currents C_1, C_2 produced by constant electromotive forces inserted in them. Show how to calculate the total extra currents produced in the coils by inserting a given resistance in one of them, and thus also increasing its coefficients of induction by given amounts.

In the primary coil, supposed open, there is an electromotive force which would produce a steady current C, and in the secondary coil there is no electromotive force. Prove that the current induced in the secondary by closing the primary is the same, as regards its effects on a galvanometer and an electrodynamometer, and also with regard to the heat produced by it, as a steady current of magnitude

$$-\frac{1}{2}\frac{CMR_1}{R_1L_2 + R_2L_1}$$

lasting for a time

$$\frac{R_1L_2 + R_2L_1}{\frac{1}{2}R_1R_2},$$

while the current induced in the secondary by breaking the primary circuit may be represented in the same respects by a steady current of magnitude $CM/2L_2$ lasting for a time $2L_2/R_2$.

300. Four points A, B, C, D are connected up as follows: A, B are joined through a coil of self-induction L and resistance P; A, D through a resistance Q; B, C through a resistance R; C, D through a resistance S and through a condenser of capacity K, the

resistance and condenser being in parallel; B, D through a galvanometer; A, C through a source of current of period $2\pi/p$. Show that if no current passes through the galvanometer

$$PS = QR \quad \text{and} \quad L = QRK.$$

[The resistances of the connecting wires may be neglected.]

301. Two conductors ABD, ACD are arranged in multiple arc. Their resistances are R, S and their coefficients of self and mutual induction L, N, and M. Prove that when placed in series with leads conveying a current of frequency p, the two circuits produce the same effect as a single circuit whose coefficient of self-induction is

$$\frac{NR^2 + LS^2 + 2MRS + p^2(LN - M^2)(L + N - 2M)}{(L + N - 2M)^2 p^2 + (R + S)^2},$$

and whose resistance is

$$\frac{RS(S + R) + p^2\{R(N - M)^2 + S(L - M)^2\}}{(L + N - 2M)^2 p^2 + (R + S)^2}.$$

302. A condenser of capacity C containing a charge Q is discharged round a circuit in the neighbourhood of a second circuit. The resistances of the circuits are R, S and their coefficients of induction are L, M, N. Obtain equations to determine the currents at any moment.

If $\dot{x}$ is the current in the primary and the disturbance be over in a time less than τ, prove that

$$\left\{\frac{N^2R}{C} + S\left(NR^2 + \frac{M^2}{C}\right) + S^2LR\right\}\int_0^\tau \ddot{x}^2\,dt = \frac{1}{2}\frac{Q^2}{C^2}\left\{S^2 + \frac{N^2RS}{LN - M^2} + \frac{N^3}{C(LN - M^2)}\right\},$$

and that

$$\left\{\frac{N^2R}{C} + S\left(NR^2 + \frac{M^2}{C}\right) + S^2LR\right\}\int_0^\tau \dot{x}^2\,dt = \frac{1}{2}\frac{Q^2}{C^2}\{CS^2L + CSNR + N^2\}.$$

Examine how $\int_0^\tau \ddot{x}^2\,dt$ varies with S.

303. Prove that the currents induced in a solid with an infinite plane face, owing to magnetic changes near the face, circulate parallel to it, and may be regarded as due to the diffusion into the solid of currents induced at each instant on the surface so as to screen off the magnetic changes from the interior.

Show that for periodic changes the current penetrates to a depth proportional to the square root of the period.

304. A magnetic system is moving towards an infinite plane conducting sheet with velocity w. Show that the magnetic potential on the other side of the sheet is the same as it would be if the sheet were away, and the strengths of all the elements of the magnetic system were changed in the ratio $R/(R + w)$, where $2\pi R$ is the specific resistance of the [illegible] unit area. Show that this result is unaltered if the system is moving away from the sheet, and examine the case where $R = -w$.

If this system is a magnetic particle of mass m and moment μ, with its axis perpendicular to the sheet, prove that if the particle has been projected at right angles to the sheet, then when at a distance z from the sheet its velocity is given by

$$\tfrac{1}{2}m(\dot{z} - R)^2 = C - \frac{\mu^2}{8z^3}.$$

305. A small magnet horizontally magnetised is moving with a velocity u parallel to a thin horizontal plate of metal. Show that the retarding force on the magnet due to the currents induced in the plate is

$$\frac{\mu^2}{(2c)^4}\frac{uR}{Q(Q+R)},$$

where μ is the moment of the magnet, c its distance above the plate, $2\pi R$ the resistance of a unit area of the plate, and $Q^2 = u^2 + R^2$.

306. A uniform line of magnetic poles is suddenly generated parallel to the axis of a long cylindrical conducting shell (mean radius a) of small thickness δ. Prove that the field inside the shell is equivalent to that of a line of opposite poles of the same strength starting from the original position of the generated line and moving radially away from the axis so that its distance from this axis at any subsequent time is $be^{\frac{4\pi\sigma\delta t}{a}}$.

Show also that the external field is equivalent to that of two lines of poles of the same uniform strength, a negative one along the axis of the cylinder and a positive one starting from the position of the inverse of the given initial line and moving towards the axis so that its distance at the time t is $\frac{a^2 e^{-\frac{4\pi\sigma\delta t}{a}}}{b}$.

307. Examine the effect of the sudden generation of a line of magnetic poles inside a thin cylindrical conducting shell and parallel to the axis, obtaining the complete image system for both the internal and external fields.

308. If a linear current of strength J be suddenly generated parallel to the axis of a conducting cylindrical shell of mean radius a and thickness δ and at a distance b from the axis ($b > a$), show that the image representing the induced currents for points inside the sphere will be a current of strength $-J$ in a symmetrical line at a distance $be^{\frac{4\pi\sigma t}{\delta a}}$ from the axis. Find also the images representing the effect of these currents at external points.

309. A thin spherical shell of radius a is introduced into an oscillating magnetic field whose potential in the immediate neighbourhood of the sphere can be written in the form

$$\psi = \psi_0 + A_0 \cos pt \left(\frac{r}{a}\right)^n Y_n,$$

where Y_n is a surface harmonic of order n. Show that the field inside the shell has a potential

$$\psi = \psi_0 + A_0 \cos\chi \cos(pt+\chi)\left(\frac{r}{a}\right)^n Y_n,$$

where

$$\tan\chi = \frac{(2n+1)\,\sigma\delta}{4\pi a p},$$

δ being the thickness of the shell and σ its specific inductive capacity.

Find also the external field.

310. A uniform solid conducting sphere of radius a and conductivity σ is introduced into an oscillating magnetic field whose potential in its neighbourhood can be written in the form

$$\psi = \psi_0 + A_0 r^n Y_n \cos pt.$$

Show that to a first approximation when the period of oscillation is not too small the external field is derived from the potential

$$\psi = \psi_0 + A_0 Y_n \cos pt\left[r^n - \frac{4\pi p\sigma}{(2n+1)(2n+3)}\cdot\frac{a^{2n+3}}{r^{n+1}}\right],$$

and examine the internal field.

311. A magnet of moment m is suddenly generated at a distance b from the centre of a thin spherical conducting shell of mean radius a, and thickness δ, with its axis directed towards the centre.

Prove that at any time t after the generation of the magnet, the potential of the field due to the induced currents is, at points inside the sphere, equivalent to that in the field of a magnet of moment $-me^{3kt/2a}$ at a distance from the centre $be^{kt/a}$ and along the same axis.

Prove also that at points outside the sphere the potential is equivalent to that of a magnet of moment $-me^{-3kt/2a}\frac{a^3}{b^3}$ at a distance from the centre $\frac{a^2}{b}e^{-kt/a}$.

312. A circular current is suddenly generated symmetrically outside a thin spherical conducting shell. Examine the magnetic effect of the currents induced in the shell.

Examine also the mechanical reaction between the shell and the current.

313. Examine the effect of the sudden generation of a small symmetrical magnet at a point inside a thin spherical shell and obtain the magnetic image system of the currents induced in the shell.

314. A copper disc of radius a, with an axle hole, is rigidly and symmetrically attached to a thin bar magnet of length $2h$ and moment m, so that the axis of the magnet is at right angles to the plane of the disc, and the system is spun about the axis of the magnet with angular velocity ω. A wire in a fixed position has one end in contact with one end of the axis of the magnet, and the other makes a sliding contact with the edge of the disc. Prove that the electromotive force generating a current in the wire is

$$m\omega\,\{h^{-1} - (h^2 + a^2)^{-\frac{1}{2}}\}.$$

315. A circular conducting ring is fixed in space. A fixed conducting wire joins the centre of the circle to a fixed point of the ring and another straight wire rotates round the centre making sliding contact at the centre with the fixed wire and with the ring at its circumference. The whole arrangement is placed in a uniform magnetic field so that the plane of the circle is perpendicular to the lines of force. Show that the induced electromotive force in the closed circuit formed by the wires is

$$-\frac{H\omega a^2}{2c},$$

H being the strength of the field, ω the angular velocity and a the length of the rotating wire.

Determine the couple necessary to keep up the rotation.

316. Prove that the rotation with angular velocity ω of a solid uniform spherical conductor in a field whose magnetic potential is

$$\psi = \psi_0 + A_0 r^n Y_n{}^s \cos s\phi,$$

where $Y_n{}^s$ is a surface harmonic of order n and class s, generates a system of currents in the conductor which gives a magnetic field outside the conductor derived from a potential

$$\psi = A\,\frac{a^{2n+3}}{r^{n+1}}\,Y_n{}^s \sin s\phi,$$

where
$$A = -A_0\,\frac{4\pi\sigma\omega s n}{(n+1)(2n+1)(2n+3)}.$$

Examine also the internal field and current distribution and prove that the retarding couple exerted by the field on the sphere is approximately for slow rotations

$$A_0{}^2\,\frac{2\,(n+s)!}{(n-s)!}\cdot\frac{n\pi\sigma\omega s^2 a^{2n+2}}{(n+1)(2n+3)(2n+1)}.$$

317. A solid sphere can execute torsional vibrations about an axis of period $2\pi/p$. Prove that if a uniform magnetic field is imposed perpendicular to the axis of vibration the oscillations will decay at a logarithmic rate which is approximately equal to

$$\frac{\pi a^4 \sigma H^2}{15I},$$

where H is the strength of the applied field, σ the conductivity of the material of which the sphere is made, and I the moment of inertia of the sphere about its axis of vibration.

318. Investigate the field of a rigid electrified system moving with a uniform screwing motion comprising a linear translation parallel to the axis of x with velocity $(c\beta)$ and angular rotations $(\omega_x, \omega_y, \omega_z)$ about the three axes.

Show that if the system is a uniformly charged sphere the force function of the electromagnetic reactions is of the general form

$$W = \frac{e^2}{4a}\frac{1-\beta^2}{\beta}\log\frac{1+\beta}{1-\beta} + \frac{e^2 a\omega_x^2}{6c^2}\left(\frac{1}{\beta^2} - \frac{1-\beta^2}{2\beta^3}\log\frac{1+\beta}{1-\beta}\right)$$
$$+\frac{e^2 a(\omega_y^2+\omega_z^2)}{12c^2}\left(-\frac{1}{\beta^2}+\frac{1+\beta^2}{2\beta^3}\log\frac{1+\beta}{1-\beta}\right),$$

where a is the radius of the sphere and e the charge on it.

319. Prove that the electric and magnetic energies in the field of a charged conducting sphere in uniform motion in a straight line are

$$U = \frac{e^2}{4a}\left\{\frac{3-\beta^2}{2\beta}\log\frac{1+\beta}{1-\beta} - 1\right\},$$

$$T = \frac{e^2}{4a}\left\{\frac{1+\beta^2}{2\beta}\log\frac{1+\beta}{1-\beta} - 1\right\}.$$

320. If W is the total energy of a rigid electrical system in quasi-stationary motion show that the longitudinal electromagnetic mass of the system is

$$\frac{1}{|v|}\frac{dW}{d|v|}.$$

Show also that
$$W = |v|\frac{dL}{d|v|} - L.$$

Explain why the above formula for the electromagnetic mass is inapplicable to the case of a system which experiences the Fitzgerald-Lorentz contraction on account of its motion.

321. Show that in the general case of a system of charges moving through the aether in any manner the effective forces of reaction of electrodynamic origin are derivable in the usual way from the Lagrangian function

$$L = \frac{1}{2}\int\rho\left(\phi - \frac{1}{c}(\mathbf{Av})\right)dv + \frac{d}{dt}\int\frac{(\mathbf{AE})}{8\pi}\,dv,$$

both integrals being taken throughout the volume of the field, ρ being the charge density in the position dv and $\mathbf{v}$ the velocity of this element through the aether.

Does this form imply any restriction as to the vector potential function to be used?

322. Find the force exerted between two point charges moving in any manner and show that the rectangular components of the force on the first electron are of the type

$$\frac{\partial L}{\partial x_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1},$$

(x, y, z) being the coordinates of the position of the electron and

$$L = -\frac{q_1 q_2}{\left[r\left(1 - \frac{\mathbf{v}_{2r}}{c}\right)\right]}\left\{1 - \frac{1}{c^2}(\mathbf{v}_1[\mathbf{v}_2])\right\},$$

the square brackets indicating that the functions are taken at the time $t - \frac{r}{c}$, r being the distance apart of the electrons and $\mathbf{v}_1$, $\mathbf{v}_2$ their velocities.

323. A perfectly conducting sphere is initially at rest and carries a charge Q. Show that if a uniform acceleration s is imparted to the sphere the effective force of electrodynamic origin resisting the initial accelerated motion at the time t is

$$\frac{2}{3}\frac{Q^2 s}{ac^2}\left\{1 - e^{-\frac{ct}{2a}}\cos\frac{ct\sqrt{3}}{2a} - \frac{e^{-\frac{ct}{2a}}}{\sqrt{3}}\sin\frac{ct\sqrt{3}}{2a}\right\},$$

a being the radius of the sphere, c the velocity of radiation and $\frac{sa}{c}$ is small compared with 1.

324. Investigate the initial accelerated motion of a uniformly rigidly charged dielectric sphere and shew that the force necessary to maintain the uniform acceleration s is of the form

$$\frac{2}{3}\frac{Q}{a}\left(\frac{es}{c^2} + \Sigma\frac{\lambda_r}{a^2}B_r e^{\frac{\lambda_r ct}{a}}\right),$$

where the B's are constants and the different values of λ are the roots of the equation

$$\tanh(\epsilon^{\frac{1}{2}}\lambda) = \epsilon^{\frac{1}{2}}\lambda\left[1 + \frac{\epsilon\lambda^2(1+\lambda)}{(\epsilon - 1)(1+\lambda) - \epsilon\lambda^3}\right],$$

ϵ being the dielectric constant of the material of the sphere.

325. Show that the small linear oscillations in a period $(2\pi/n)$ of a perfectly conducting charged sphere under the periodic force P may be obtained as a solution of the equation

$$m\ddot{x} + k\dot{\xi} = P,$$

where

$$m = \frac{2}{3}\frac{Q^2}{ac^2}\frac{1 - \frac{a^2n^2}{c^2}}{\left(1 - \frac{a^2n^2}{c^2}\right)^2 + \frac{a^2n^2}{c^2}},$$

$$k = \frac{2}{3}\frac{Q^2}{ac^2}\frac{\frac{an}{c}\cdot\frac{1}{n}}{\left(1 - \frac{a^2n^2}{c^2}\right)^2 + \frac{a^2n^2}{c^2}},$$

and the sphere is presumed to have no material mass of the ordinary kind.

326. Three linear Hertzian vibrators placed at a point in three directions mutually at right angles have the same period of vibration but are not necessarily in the same phase; prove that the rate at which energy is radiated from them is the sum of the rates at which energy is radiated from each separately.

Hence obtain the rate at which energy is radiated from any number of Hertzian vibrators of the same period placed at a point.

327. Electrical oscillations take place on two parallel rectangular conducting plates so that the wave surfaces in the dielectric disturbance between the plates are normal to

these plates. Prove that if a, b are the lengths of the edges of the plates the possible oscillations have wave lengths

$$\frac{2}{\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}},$$

m, n being any positive integers.

[It can be assumed that the current flux outwards at the edges of the plates is zero.

328. Electrical oscillations take place on two coaxal cylindrical conducting sheets of length l and nearly equal radii a: if the electric force in the thin shell of field is everywhere radial show that the wave lengths of the possible oscillations are of the type

$$\frac{2\pi}{\sqrt{\frac{m^2}{a^2}+\frac{n^2\pi^2}{l^2}}},$$

m, n being positive integers.

329. Electric oscillations take place in the shell of dielectric between two perfectly conducting spherical surfaces of radii a, b. Show that if the conditions in the shell are symmetrical round the centre and the wave surfaces are concentric spheres the period equation is

$$\tan p(b-a)=\frac{p(b-a)}{1+p^2ab}.$$

If the wave surfaces are perpendicular to the conducting surfaces shew that the period equation for the first order oscillations is

$$\tan p(b-a)=\frac{p(b-a)(1+p^2ab)}{(1-p^2a^2)(1-p^2b^2)+p^2ab}.$$

330. Prove that the components of the electric force in any electromagnetic radiation field referred to spherical polar coordinates r, θ, ϕ can be written in the form

$$E_r=\frac{\partial^2 W_1}{\partial r^2}-\frac{1}{c^2}\frac{\partial^2 W_1}{\partial t^2},\qquad E_\theta=\frac{1}{r}\frac{\partial^2 W_1}{\partial r\partial\theta}+\frac{1}{r\sin\theta}\cdot\frac{\partial W_2}{\partial\phi},$$

$$E_\phi=\frac{1}{r\sin\theta}\frac{\partial^2 W_1}{\partial r\partial\phi}-\frac{1}{r}\frac{\partial W_2}{\partial\theta},$$

where W_1 and W_2 satisfy the equation

$$\frac{\partial^2 W}{\partial r^2}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial W}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 W}{\partial\phi^2}=\frac{1}{c^2}\frac{\partial^2 W}{\partial t^2},$$

and c is the velocity of radiation.

Find the corresponding expressions in terms of W_1 and W_2 for the components of the magnetic induction in the same directions.

331. Develope a theory of the Hall effect when the metal molecules are regarded as point centres of force repelling the electrons with a force which varies as the inverse nth power of the distance between them, and prove that in this case the gradient of the cross potential which gives rise to the effect is

$$\frac{3\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+\frac{4}{n-1}\right)}{4Nec\left[\Gamma\left(2+\frac{2}{n-1}\right)\right]^2}HJ_y,$$

H being the strength of the field and J_y that of the current directed in perpendicular directions.

332. The electric current in Hall's experiment is replaced by a thermal current driven down a uniform temperature gradient $\frac{d\theta}{dy}$ parallel to the y-axis of a coordinate system. The magnetic field is directed parallel to the axis of x: show that there is a potential gradient parallel to the axis of z of amount

$$-\left(\frac{n-5}{2(n-1)}\right)\frac{\Gamma\left(\frac{3}{2}+\frac{4}{n-1}\right)}{\Gamma\left(2+\frac{2}{n-1}\right)}\frac{Rl_m q^{\frac{1}{2}-\frac{2}{n-1}}}{mc}H\frac{d\theta}{dy},$$

the notation being as before in the text and the questions above.

[The Nernst and von Ettinghausen effect.]

333. Electric waves are travelling along the plane interface separating a dielectric medium from a moderately bad conducting medium. Show that the electric force is nearly normal to the interface if the conduction is bad but that under other conditions it has a considerable slant in the direction of propagation.

Discuss the bearing of this result on the disposition of the aerial wires at a wireless telegraph receiving station.

334. Electric waves are incident on a thin metallic plate. Show that the coefficient of absorption (i.e. percentage of incident energy absorbed) is proportional to the product of the conductivity and the thickness of the plate.

335. Light is incident on a metallic or other densely opaque thin prism of small angle a; prove that when the angle of incidence of the light is small, the real part of the complex index of refraction is given by $1+\delta/a$, where δ is the deviation of the light.

Prove further that if δ_1 is the deviation for the same prism when the light is incident at an angle γ, the imaginary part of the refractive index is given by

$$(1+\delta_1/a)\cos\gamma\left\{\frac{\sin^2\gamma}{\left(1+\frac{\delta}{a}\right)^2-\left(1+\frac{\delta_1}{a}\right)^2\cos^2\gamma}-1\right\}^{\frac{1}{2}}.$$

336. Prove that in the passage of a beam of light normally through a very thin film of metal the phase is accelerated by an amount approximately equal to

$$\tfrac{1}{2}\delta(1-\mu^2-\kappa^2),$$

where δ is the thickness of the film and μ and κ are defined by the relation

$$\frac{\epsilon'}{\epsilon}-\frac{4\pi\sigma i}{p\epsilon}=(\mu-i\kappa)^2,$$

ϵ', σ referring to the metal and ϵ to the surrounding dielectric and p is the frequency of the light.

337. Waves of light are incident perpendicularly on a transparent plate of thickness d; prove that the ratio of the intensities of the reflected and incident light is

$$\frac{(1-\mu^2)\sin^2\theta}{(1-\mu^2)\sin^2\theta+4\mu^2},$$

where μ is the index of refraction of the plate and $2\pi d/\theta$ the wave length of the light on the plate. Find also the intensity of the transmitted light.

338. Show that the equations of the electromagnetic field in free aether can be satisfied by functions of the type

$$H_x+iE_x=\frac{\partial(a,\beta)}{\partial(y,z)}=\pm\frac{i}{c}\frac{\partial(a,\beta)}{\partial(x,t)},$$

where a, β are arbitrary functions which satisfy the conditions implied in these forms.

Show further that suitable pairs of functions α, β are

$$\text{(i)} \quad \alpha = x\cos\theta + y\sin\theta \mp iz,$$
$$\beta = x\sin\theta - y\cos\theta - ct.$$

$$\text{(ii)} \quad \alpha = \frac{x \mp iy}{z + r}, \qquad \beta = r - ct.$$

339. Plane electromagnetic waves fall on the convex surface of an infinite paraboloid of revolution $x = a - r$, whose surface is a perfect reflector. If the incident waves are given by expressions of the type

$$H_x + iE_x = f(\alpha, \beta)\frac{\partial(\alpha, \beta)}{\partial(y, z)} = \frac{i}{c} f(\alpha, \beta)\frac{\partial(\alpha, \beta)}{\partial(x, t)};$$

where $\alpha = y + iz$, $\beta = x + ct$, prove that the boundary conditions at the surface of the paraboloid may be satisfied by subtracting from the primary field a secondary field represented by expressions of a similar type with

$$\alpha = a\frac{y - iz}{x + r}, \qquad \beta = a - r + ct.$$

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